Differential geometry, mid-term

March 11 2025, 2 hours

Solution

Exercise 1

The map $g \circ f$ is C^1, as it is the composition of two C^1 maps. For any $h \in \mathbb{R}^{n_1}$,

$$d(g \circ f)(a)(h) = dg(f(a)) \circ df(a)(h) = dg(f(a))\,(df(a)(h))\,.$$

For any h, since $dg(f(a))$ is injective (as g is an immersion at $f(a)$), $d(g \circ f)(a)(h)$ can only be zero if $df(a)(h) = 0$. As $df(a)$ is also injective (f is an immersion at a), this is equivalent to $h = 0$. Therefore, $d(g \circ f)(a)$ is injective, meaning that $g \circ f$ is an immersion at a.

Exercise 2

For any $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} f(a,b) &= \frac{1}{2}\sum_{i=1}^{d}(a \odot b - y)_i^2 \\ &= \frac{1}{2}\sum_{i=1}^{d}(a_i b_i - y_i)^2. \end{aligned}$$

From this expression, we see that f is polynomial. In particular, it is C^∞.
For any $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d$, for any $k \in \{1, \dots, d\}$,

$$\begin{aligned} \frac{\partial f}{\partial a_k}(a,b) &= \frac{1}{2}\frac{\partial\left[(a_k b_k - y_k)^2\right]}{\partial a_k}(a,b) \\ &= b_k(a_k b_k - y_k). \end{aligned}$$

For the same reason,

$$\frac{\partial f}{\partial b_k}(a,b) = a_k(a_k b_k - y_k).$$

For any $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d$, the gradient is thus

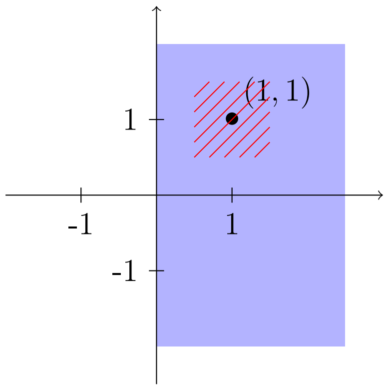
$$\begin{aligned} \nabla f(a,b) &= \begin{pmatrix} \frac{\partial f}{\partial a_1}(a,b) \\ \vdots \\ \frac{\partial f}{\partial a_d}(a,b) \\ \frac{\partial f}{\partial b_1}(a,b) \\ \vdots \\ \frac{\partial f}{\partial b_d}(a,b) \end{pmatrix} \\ &= \begin{pmatrix} b_1(a_1 b_1 - y_1) \\ \vdots \\ b_d(a_d b_d - y_d) \\ a_1(a_1 b_1 - y_1) \\ \vdots \\ a_d(a_d b_d - y_d) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} b\odot(a\odot b-y) \\ a\odot(a\odot b-y) \end{pmatrix}.$$

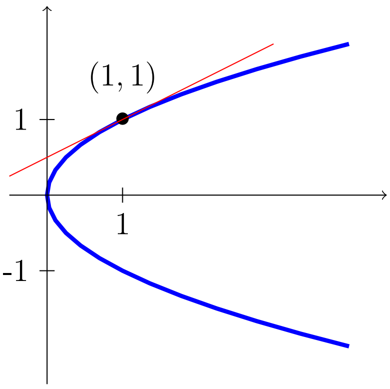
Exercise 3

1. The set A_1 is not a submanifold. Explanation sketch : if it is a submanifold, then, since it contains a whole neighborhood of $(1,1)$, it must have dimension 2. But submanifolds of $\mathbb{R}^2$ with dimension 2 are open sets, and A_1 is not open.
   The set A_2 is a submanifold : since it is an open set of $\mathbb{R}^2$, it is a submanifold with dimension 2.
   The set A_3 is a submanifold. Explanation sketch : it is the image of $x \in]-2;2[\to (x^2,x)$, which is an immersion, and a homeomorphism onto its image (with reciprocal $(x,y) \to y$).
   The set A_4 is not a submanifold, because it has a non-regular point at $(2,2)$. This can be rigorously proved in a similar manner as for the graph of the absolute value.

2.



$A_2 = \mathbb{R}_*^+ \times \mathbb{R}$
(The tangent space is $\mathbb{R}^2$.)



$A_3 = \{(x^2,x), x \in]-2;2[\}$
(The tangent space is $(2,1)\mathbb{R}$.)

Exercise 4

1. We define
$$\begin{array}{rccc} f & : & \mathbb{R}^3 & \to & \mathbb{R}, \\ & & (x,y,z) & \to & x^2+y^2-z^2-1. \end{array}$$
   The map f is C^∞. We have $E_1 = f^{-1}(\{0\})$. Therefore, if we can show that f is a submersion at p for any $p \in E_1$, then E_1 is a submanifold of class C^∞ and dimension $3-1=2$.
   Let $p=(x,y,z)$ be any element of E_1. The differential of f at p is the linear map
$$\begin{array}{rccc} df(p) & : & \mathbb{R}^3 & \to & \mathbb{R}, \\ & & (h,k,l) & \to & 2(xh+yk-zl). \end{array}$$
   This map is not the null map, because x,y,z cannot be all zero (otherwise $p=(0,0,0)$, which is impossible since $(0,0,0) \notin E_1$). Therefore, its image is a subspace of $\mathbb{R}$, which is not $\{0\}$: it is $\mathbb{R}$. Consequently, $df(p)$ is surjective, so f is a submersion at p.

2. For any $(x, y, z) \in E_1$,

$$\begin{aligned} T_{(x,y,z)}E_1 &= \operatorname{Ker}(df(x,y,z)) \\ &= \{(h,k,l) \in \mathbb{R}^3 \text{ s.t. } 2(xh + yk - zl) = 0\} \\ &= \{(x, y, -z)\}^{\perp}. \end{aligned}$$

Exercise 5

1. For any $t \in \mathbb{R}$, $g \circ f(t)$ is well defined, because $(f(t))_2 = \frac{1}{t^2+1} \in \mathbb{R}^*$. Let us fix any $t \in \mathbb{R}$ and show that $g \circ f(t) = t$.

$$\begin{aligned} g \circ f(t) &= g\left(t^3 - t, \frac{1}{t^2+1}, (t-1)^2\right) \\ &= \frac{1}{\frac{2}{t^2+1}} - \frac{(t-1)^2}{2} \\ &= \frac{t^2 + 1 - (t-1)^2}{2} \\ &= \frac{2t}{2} \\ &= t. \end{aligned}$$

2. Since E_2 is the image of f, it holds that E_2 is a C^∞ submanifold of $\mathbb{R}^3$, with dimension 1, provided that we can prove the following two properties :
   1. f is an immersion at any point ;
   2. f is a homeomorphism between $\mathbb{R}$ and $f(\mathbb{R})$.

   For the first point, we fix an arbitrary $t \in \mathbb{R}$ and show that $f'(t) \neq 0$. It holds

$$f'(t) = \left(3t^2 - 1, -\frac{2t}{(t^2+1)^2}, 2(t-1)\right).$$

   The third coordinate is zero if and only if $t = 1$. But, when $t = 1$, the first coordinate is $2 \neq 0$. Therefore, at least one of the coordinates of $f'(t)$ is non-zero, meaning that $f'(t) \neq 0$.
   For the second point, we use the fact that $g \circ f = \mathrm{Id}_{\mathbb{R}}$. Since $\mathrm{Id}_{\mathbb{R}}$ is injective, f must be injective as well. In addition, f is surjective onto its image, so that f is a bijection from $\mathbb{R}$ to $f(\mathbb{R})$. Its inverse is $g_{|f(\mathbb{R})}$, which is a continuous map. Therefore, f is a homeomorphism between $\mathbb{R}$ and $f(\mathbb{R})$.
3. For any $t \in \mathbb{R}$,

$$\begin{aligned} T_{\left(t^3-t, \frac{1}{t^2+1}, (t-1)^2\right)}E_2 &= \operatorname{Im}(df(t)) \\ &= \operatorname{Vect}\{f'(t)\} \\ &= \operatorname{Vect}\left\{\left(3t^2 - 1, -\frac{2t}{(t^2+1)^2}, 2(t-1)\right)\right\}. \end{aligned}$$

Exercise 6

1. For any $p \in G$, the set $]0;1[\times\mathbb{R}$ is a neighborhood of p, and

$$G \cap (]0;1[\times\mathbb{R}) = \mathrm{graph}(f).$$

   Therefore, from the definition "by graph" of submanifolds, G is a submanifold of $\mathbb{R}^2$ with dimension 1 ; it is a curve.

2. The set G is not compact. To check it, we can notice, for instance, that the sequence $(2^{-n}, f(2^{-n}))_{n\in\mathbb{N}}$ has no converging subsequence with limit point in G (if it had, the limit point should be of the form $(0,a)$ for some $a \in \mathbb{R}$, but G contains no such point).
   Therefore, G is a connected non-compact curve. It is diffeomorphic to $\mathbb{R}$.

3. We define

$$\begin{array}{rccc} \phi & : &]0;1[& \to & \mathbb{R}^2 \\ & & x & \to & (x, f(x)). \end{array}$$

   It is a global parametrization :
   - its domain, $]0;1[$, is an open interval ;
   - $\phi(]0;1[) = G$, from the definition of G ;
   - it is a diffeomorphism between $]0;1[$ and G : indeed, it is a bijection, and its reciprocal is $(a,b) \in G \to a$, which is a C^1 map, from the lecture (it is the projection onto the first coordinate).

4.

$$\begin{aligned} \ell(G) &= \int_0^1 ||\phi'(x)||_2 dx \\ &= \int_0^1 ||(1, f'(x))||_2 dx \\ &= \int_0^1 \sqrt{1 + (f'(x))^2} dx. \end{aligned}$$

Exercise 7

1. The maps $(x,y,z) \to x \in \mathbb{R}$ and $(x,y,z) \to z \in \mathbb{R}$ are C^∞ on $\mathbb{S}^2 \setminus \{(0,0,1)\}$, from the class. Therefore, $(x,y,z) \to 1 - z \in \mathbb{R}$ is also C^∞ and $(x,y,z) \to \frac{x}{1-z} \in \mathbb{R}$ is C^∞ on $\mathbb{S}^2 \setminus \{(0,0,1)\}$, since it is the quotient of two C^∞ maps such that the denominator does not vanish (it does not vanish because $(0,0,1)$ is the only point in $\mathbb{S}^2$ whose last coordinate is 1).
   Similarly, $(x,y,z) \to \frac{y}{1-z}$ is C^∞.
   Since its two components are C^∞, ϕ is C^∞ on $\mathbb{S}^2 \setminus \{(0,0,1)\}$.

2. a)

$$a^2 + b^2 + 1 = \frac{x^2 + y^2 + (1-z)^2}{(1-z)^2}$$

$$
\begin{aligned}
&= \frac{1-z^2+(1-z)^2}{(1-z)^2} \\
&= \frac{(1+z)+(1-z)}{1-z} \\
&= \frac{2}{1-z}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{2a}{a^2+b^2+1} &= \frac{2\frac{x}{1-z}}{\frac{2}{1-z}} \\
&= x.
\end{aligned}
$$

Similarly, it holds $\frac{2b}{a^2+b^2+1} = y$. For the last term,

$$
\begin{aligned}
\frac{a^2+b^2-1}{a^2+b^2+1} &= 1 - \frac{2}{a^2+b^2+1} \\
&= 1 - \frac{2}{\frac{2}{1-z}} \\
&= z.
\end{aligned}
$$

b) First, we show that ϕ is injective. Let $(x,y,z),(x',y',z') \in \mathbb{S}^2 \setminus \{(0,0,1)\}$ be such that $\phi(x,y,z) = \phi(x',y',z')$. We define $(a,b) = \phi(x,y,z) = \phi(x',y',z')$. From the previous subquestion,

$$
(x,y,z) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right) = (x',y',z').
$$

This shows the injectivity.
Let us show that ϕ is surjective. Let $(a,b) \in \mathbb{R}^2$ be arbitrary. We show that there exists $(x,y,z) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$ such that $\phi(x,y,z) = (a,b)$. Let us define (x,y,z) by the formula from the previous subquestion :

$$
(x,y,z) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right).
$$

This is a point in $\mathbb{S}^2 \setminus \{(0,0,1)\}$. Indeed,

$$
\begin{aligned}
x^2+y^2+z^2 &= \frac{4a^2+4b^2+(a^2+b^2-1)^2}{(a^2+b^2+1)^2} \\
&= \frac{(a^2+b^2+1)^2}{(a^2+b^2+1)^2} \\
&= 1,
\end{aligned}
$$

so that (x,y,z) belongs to $\mathbb{S}^2$. Moreover, if $x=y=0$, it means that $a=b=0$ (since x,y are the quotient of a and b with a non-zero term). In this case, $z = \frac{-1}{+1} = -1$. Therefore, $(x,y,z) \neq (0,0,1)$.

As expected, it holds

$$\begin{aligned}\phi(x,y,z) &= \left(\frac{\frac{2a}{a^2+b^2+1}}{1-\frac{a^2+b^2-1}{a^2+b^2+1}}, \frac{\frac{2b}{a^2+b^2+1}}{1-\frac{a^2+b^2-1}{a^2+b^2+1}}\right)\\ &= (a,b).\end{aligned}$$

This concludes the proof of the surjectivity, hence of the bijectivity.
For an arbitrary $(a,b) \in \mathbb{R}^2$, let us denote $(x,y,z) = \phi^{-1}(a,b) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$. By definition of the inverse, it must hold $\phi(x,y,z) = (a,b)$. Therefore, from the previous question,

$$\phi^{-1}(a,b) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right).$$

3. The map ϕ^{-1} is C^∞, when seen as a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ (each component is a quotient of polynomial maps, whose denominator never vanishes). Therefore, it is also C^∞ as a map from $\mathbb{R}^2$ to $\mathbb{S}^2 \setminus \{(0,0,1)\}$.
   We have shown that ϕ is a C^∞ map, which is a bijection between $\mathbb{S}^2 \setminus \{(0,0,1)\}$ and $\mathbb{R}^2$, and whose reciprocal is also C^∞. Consequently, ϕ is a C^∞-diffeomorphism.