Differential geometry, mid-term

March 11 2025, 2 hours

You can use any written or printed material.
For each exercise, the number of points is an indication ; it may change.
You can write in either English or French, but don't forget quantifiers !

Exercise 1 *(1.5 points)*

Let n_1, n_2, n_3 be positive integers, $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, $g : \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$ be C^1 maps, and $a \in \mathbb{R}^{n_1}$ be a point. Show that if f is an immersion at a, and g an immersion at $f(a)$, then $g \circ f$ is an immersion at a.

Exercise 2 *(2 points)*

Let $d \in \mathbb{N}^*$ be fixed. For any $a, b \in \mathbb{R}^d$, we denote $a \odot b$ the vector in $\mathbb{R}^d$ whose coordinates are the multiplications of the coordinates of a and b :

$$\forall i \leq d, (a \odot b)_i = a_i b_i.$$

We fix an arbitrary $y \in \mathbb{R}^d$ and define

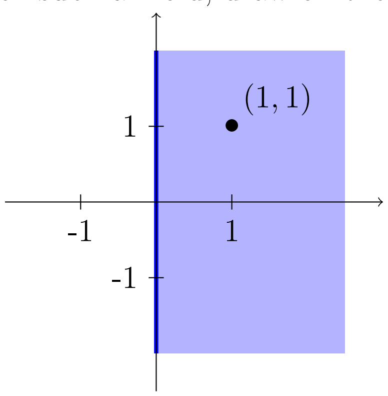
$$\begin{array}{ccccc} f & : & \mathbb{R}^d \times \mathbb{R}^d \quad (= \mathbb{R}^{2d}) & \to & \mathbb{R}, \\ & & (a, b) & \to & \frac{1}{2} \, ||a \odot b - y||_2^2 \, . \end{array}$$

Show that f is C^∞, and compute its gradient.

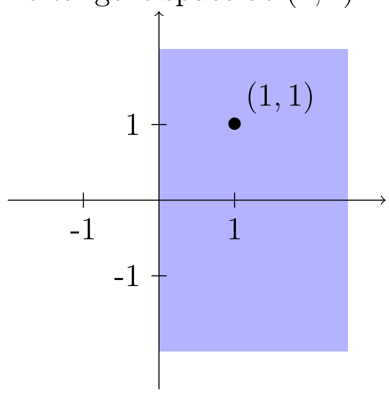
Exercise 3 *(3 points)*

For this exercise, you do not need to justify your answers.

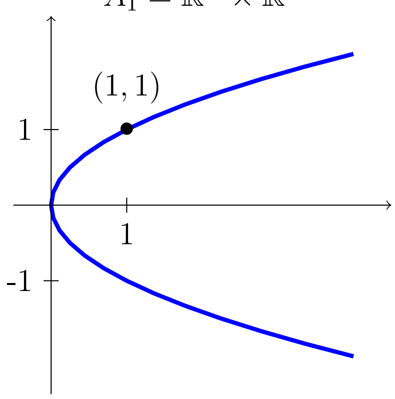
1. For each of the four sets given below, say whether it is a submanifold of $\mathbb{R}^2$.
2. For each submanifold, draw on the figure the affine tangent space at $(1, 1)$.



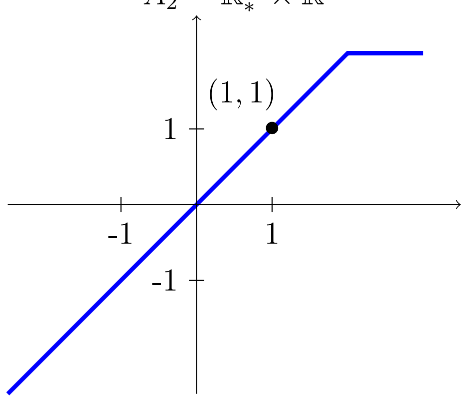
$A_1 = \mathbb{R}^+ \times \mathbb{R}$



$A_2 = \mathbb{R}^+_* \times \mathbb{R}$



$A_3 = \{(x^2, x), x \in]-2; 2[\}$



$A_4 = \{(x, \max(x, 2)), x \in \mathbb{R}\}$

Exercise 4 *(2.5 points)*

We define $E_1 = \{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } x^2 + y^2 = z^2 + 1\}$.

1. Show that E_1 is a C^∞ submanifold of $\mathbb{R}^3$. Specify the dimension.
2. Compute the tangent space to E_1 at any point.

Exercise 5 *(4 points)*

We define

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^3, \\ t & \to & \left(t^3 - t, \frac{1}{t^2+1}, (t-1)^2\right), \end{array} \qquad g : \begin{array}{ccc} \mathbb{R} \times \mathbb{R}^* \times \mathbb{R} & \to & \mathbb{R}, \\ (x,y,z) & \to & \frac{1}{2y} - \frac{z}{2}. \end{array}$$

1. Show that $g \circ f = \mathrm{Id}_{\mathbb{R}}$.
2. Show that $E_2 \stackrel{def}{=} \left\{\left(t^3 - t, \frac{1}{t^2+1}, (t-1)^2\right), t \in \mathbb{R}\right\}$ is a C^∞ submanifold of $\mathbb{R}^3$. Specify the dimension.
3. Compute the tangent space to E_2 at any point.

Exercise 6 *(3 points)*

Let $f :]0;1[\to \mathbb{R}$ be a C^1 map. We consider its graph

$$G = \{(x, f(x)), x \in]0;1[\} \subset \mathbb{R}^2.$$

1. Justify that G is a curve.
2. Which simple curve is it diffeomorphic to ? (You can admit that G is connected.)
3. Give a global parametrization of G.
4. Express the length of G as a function of f'.

Exercise 7 : stereographic projection *(4 points)*

We define

$$\phi : \begin{array}{ccc} \mathbb{S}^2 \setminus \{(0,0,1)\} & \to & \mathbb{R}^2, \\ (x,y,z) & \to & \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \end{array}$$

(You can admit that $\mathbb{S}^2 \setminus \{(0,0,1)\}$ is a C^∞ submanifold of $\mathbb{R}^3$, with dimension 2.)

1. Show that ϕ is C^∞.
2. a) Let $(x,y,z) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$, $(a,b) \in \mathbb{R}^2$ be such that $\phi(x,y,z) = (a,b)$. Show that

$$(x,y,z) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right).$$

   b) Show that ϕ is a bijection between $\mathbb{S}^2 \setminus \{(0,0,1)\}$ and $\mathbb{R}^2$ and give an explicit expression for $\phi^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(0,0,1)\}$.
3. Show that ϕ is a C^∞-diffeomorphism.