Non-convex inverse problems

February 21, 2025, 2 hours

You can use any written or printed material.
The bonus questions are either not directly related to inverse problems or difficult. They will not grant you many points. You are encouraged to skip them, unless you have solved everything else.
Don't forget quantifiers!

Exercise 1 *(2.5 points)*

1. Among the following problems, which ones would you call an *inverse problem*? Give a (short) justification of your answer.
   1. Given a house map, produce an image mimicking as closely as possible a photo of the house.
   2. Estimate the time of day at which you should turn on your heating system so that the temperature reaches 19°C at 7pm.
   3. Knowing the distance of an object to several sensors, and the position of the sensors, determine the position of the object.
2. We consider the following inverse problem :
$$\begin{aligned} &\text{recover } (x,y) \in \mathbb{R}^2 \\ &\text{from } M(x,y) \overset{def}{=} (x^2, x^2 y, x+y). \end{aligned}$$
   Is reconstruction unique? Justify your answer.

Exercise 2 *(3 points + 1 bonus point)*

Fix $d_1, d_2, m \in \mathbb{N}^*$. We define the ℓ^0*-row norm* of some $X \in \mathbb{R}^{d_1 \times d_2}$ as the number of its non-zero rows :

$$||X||_{0,row} = \mathrm{Card}\left\{ i \le d_1 \text{ such that } (X_{i,1}, \dots, X_{i,d_2}) \neq (0, \dots, 0) \right\},$$

and the *mixed* ℓ^1/ℓ^2*-norm* of X as

$$||X||_{1,2} = \sum_{i=1}^{d_1} ||(X_{i,1}, \dots, X_{i,d_2})||_2.$$

Given a linear operator $\mathcal{L} : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^m$ and a vector $b \in \mathbb{R}^m$, we consider the non-convex problem

$$\begin{aligned} &\text{minimize } ||X||_{0,row} \\ &\text{over all } X \in \mathbb{R}^{d_1 \times d_2} \\ &\text{such that } \mathcal{L}(X) = b. \end{aligned} \tag{Row-Sparse}$$

1. [Bonus] Let $B \subset \mathbb{R}^{d_1 \times d_2}$ be the unit ball for the mixed ℓ^1/ℓ^2-norm. Show that its extremal points are the matrices with exactly one non-zero row, and unit Frobenius norm.
2. Which convex relaxation could you propose for Problem (Row-Sparse)?
3. For $k \le d_1$ and $\delta \in [0;1]$, propose a sensible definition of (k, δ)-restricted isometry in this context.

Exercise 3 *(6 points + 1 bonus point)*

Let $m, d \in \mathbb{N}^*$ be fixed. Let $A \in \mathbb{R}^{m \times d}$ be a matrix, and $y \in \mathbb{R}^m$ a vector. We consider the problem

$$\begin{aligned} &\text{minimize } ||x||_0, \\ &\text{over all } x \in \mathbb{R}^d, \\ &\text{such that } Ax = y. \end{aligned} \tag{CS}$$

1. Recall why this optimization problem is non-convex.
2. Write its standard convex relaxation. We call it (Conv-Rel).

3. [Bonus] Let $z \in \mathbb{R}^d$ be any vector. Show that
$$\min_{x \in \mathbb{R}^d} ||x||_1 - \langle x, z \rangle = 0 \text{ if } |z_i| \leq 1, \forall i \leq d,$$
$$= -\infty \text{ otherwise.}$$
In the first case, show that a vector x is a minimizer if and only if
$$z_i = \text{sgn}(x_i), \forall i \leq d \text{ s.t. } x_i \neq 0.$$
4. Show that the dual problem to (Conv-Rel) is
$$\begin{aligned} &\text{maximize } \langle y, b \rangle, \\ &\text{over all } b \in \mathbb{R}^m, \\ &\text{such that } |(A^T b)_i| \leq 1, \forall i \leq d. \end{aligned}$$
5. Show that a pair $(x, b) \in \mathbb{R}^d \times \mathbb{R}^m$ is primal-dual optimal if and only if it satisfies the following properties :
   1. $Ax = y$;
   2. $|(A^T b)_i| \leq 1, \forall i \leq d$;
   3. $(A^T b)_i = \text{sgn}(x_i), \forall i \leq d$ s.t. $x_i \neq 0$.
6. We assume that
   — the solution x_{sol} of Problem (CS) is 1-sparse ;
   — all columns of A have norm 1.
   a) Show that x_{sol} is a minimizer of (Conv-Rel).
   [Hint : consider the dual certificate $b = \text{sgn}((x_{sol})_s) A_{:s}$, where s is the index of the only non-zero coordinate of x_{sol} and $A_{:s}$ is the s-th column of A.]
   b) [Bonus] In addition, we assume that no two columns of A are colinear. Show that x_{sol} is the unique minimizer of (Conv-Rel).
   c) Using the terminology from the class, what is the name of the property proved at Question 6.b) ?
   d) When this property holds true, which algorithm can solve Problem (CS) ?

Exercise 4 *(8.5 points + 1 bonus point)*

Let $d \in \mathbb{N}^*$ be fixed. For any vectors $a, b \in \mathbb{R}^d$, we denote $a \odot b \in \mathbb{R}^d$ their componentwise multiplication, i.e. the vector such that, for all $i \leq d$, $(a \odot b)_i = a_i b_i$.
Let $y \in \mathbb{R}^d$ be fixed. We define
$$\begin{array}{cccc} f & : & \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{R} \\ & & (a, b) & \to & \frac{1}{2} ||a \odot b - y||_2^2. \end{array}$$
1. What are the global minimizers of f ?
2. Compute the gradient and Hessian of f.
3. In this question, we assume there exist m, M such that $0 < m < M$ and
$$m \leq |y_i| \leq M, \forall i \leq d.$$
We fix $\tau \in \left]0; \frac{2}{3M}\right]$ and run gradient descent over f starting from an initial point $(a_0, b_0) \in \mathbb{R}^d \times \mathbb{R}^d$, with constant stepsize τ. We denote $(a_t, b_t)_{t \in \mathbb{N}}$ the sequence of iterates.
   a) For some $t \in \mathbb{N}$, we assume that $||a_t \odot b_t - y||_2 \leq \frac{m}{2}$. Show that, for all $i \leq d$,
$$(a_t)_i^2 + (b_t)_i^2 \geq m \quad \text{and} \quad |((a_t)_i (b_t)_i - y_i)(a_t)_i (b_t)_i| \leq \frac{3}{4} mM.$$
   b) Show that $||a_{t+1} \odot b_{t+1} - y||_2 \leq \left(1 - \frac{\tau m}{2}\right) ||a_t \odot b_t - y||_2$.
   c) Deduce from the previous two questions that, if $||a_0 \odot b_0 - y||_2 \leq \frac{m}{2}$, then
$$f(a_t, b_t) \to 0 \quad \text{when } t \to +\infty.$$
   d) [Bonus] Show that, under the same assumption, $(a_t, b_t)_{t \in \mathbb{N}}$ converges to a global minimizer of f.
   e) Using the terminology from the class, how would you call the result proved at Question 3.d) ?
4. a) Describe the first-order critical points of f.
   b) Describe the second-order critical points of f.
   c) How do you expect gradient descent iterates to behave, if the initial point is chosen uniformly at random in the unit ball of $\mathbb{R}^d$, and the initial stepsize is small enough ?
   [For this question, only a conjecture is required, not a proof attempt.]