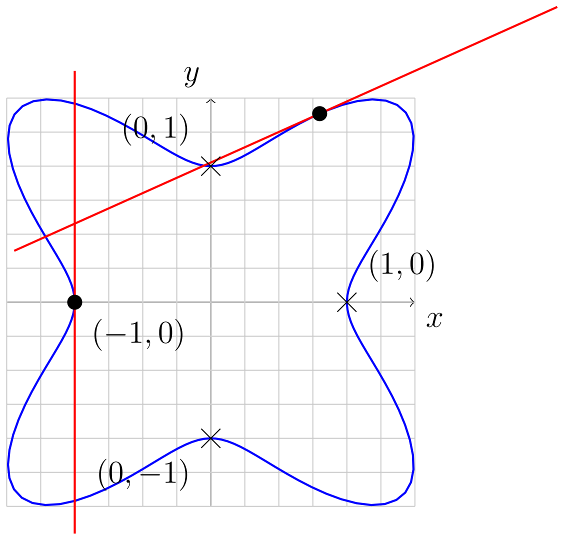
Geometry and differential equations : solution

May 21 2024

Answer of exercise 1



Answer of exercise 2

The map f is locally Lipschitz (since it is C^∞), hence we can follow the method described in class.

We observe that the only point where f vanishes is 0. Therefore, $\big(t \in \mathbb{R} \to 0\big)$ is a maximal solution, and it is the only constant solution.

The two maximal intervals over which f does not vanish are $\mathbb{R}_-^*$ and $\mathbb{R}_+^*$. Therefore, any non-constant solution stays in one of these intervals.

<u>First case</u> : we look for maximal solutions with values in $\mathbb{R}_+^*$. Let $u : I \to \mathbb{R}$ be such a solution. Following the strategy described in class, we use the equality

$$\frac{u'}{f(u)} = 1.$$

For any $x \in \mathbb{R}_+^*$, we have

$$\frac{1}{f(x)} = -\frac{1}{x^3} e^{\frac{1}{x^2}} = g'(x),$$

where g is defined by

$$\begin{array}{rccc} g & : & \mathbb{R}_+^* & \to & \mathbb{R} \\ & & x & \to & \frac{1}{2} e^{\frac{1}{x^2}}. \end{array}$$

Therefore, g is a primitive of $\frac{1}{f}$ over $\mathbb{R}_+^*$ and, over I,

$$(g \circ u)' = 1.$$

This implies that there exists a constant $D \in \mathbb{R}$ such that, for all $t \in I$,

$$g(u(t)) = t - D.$$

We observe that g is a bijection between $\mathbb{R}_+^*$ and $\left]\frac{1}{2}; +\infty\right[$: it is a continuous, strictly decreasing map, which goes to $+\infty$ at 0 and $\frac{1}{2}$ at $+\infty$. Its reciprocal is

$$\begin{array}{rccl} g^{-1} & : & \left]\frac{1}{2}; +\infty\right[& \rightarrow \mathbb{R}_+^* \\ & & x & \rightarrow \frac{1}{\sqrt{\ln(2x)}}. \end{array}$$

Consequently, there exists a constant $D \in \mathbb{R}$ such that, for all $t \in I$,

$$\begin{aligned} u(t) &= g^{-1}(t - D) \\ &= \frac{1}{\sqrt{\ln(2(t-D))}}. \end{aligned}$$

Following the course, the domain I of u is the set of all t such that $g^{-1}(t - D)$ is well-defined, that is

$$I = \left]\frac{1}{2} + D; +\infty\right[.$$

To summarize, the solution u is

$$\begin{array}{rccc} u & : & \left]\frac{1}{2} + D; +\infty\right[& \rightarrow & \mathbb{R}_+^* \\ & & t & \rightarrow & \frac{1}{\sqrt{\ln(2(t-D))}}. \end{array} \tag{1}$$

<u>Second case</u> : we look for maximal solutions with values in $\mathbb{R}_-^*$. The reasoning is the same as in the first case except that, this time, we must consider a primitive of $\frac{1}{f}$ on $\mathbb{R}_-^*$. Therefore, we set

$$\begin{array}{rccc} g & : & \mathbb{R}_-^* & \rightarrow & \mathbb{R} \\ & & x & \rightarrow & \frac{1}{2}e^{\frac{1}{x^2}}. \end{array}$$

This map is a bijection between $\mathbb{R}_-^*$ and $\left]\frac{1}{2}; +\infty\right[$, with reciprocal

$$\begin{array}{rccl} g^{-1} & : & \left]\frac{1}{2}; +\infty\right[& \rightarrow \mathbb{R}_-^* \\ & & x & \rightarrow -\frac{1}{\sqrt{\ln(2x)}}. \end{array}$$

This implies that, for some constant $D \in \mathbb{R}$, the solution u is

$$\begin{array}{rccc} u & : & \left]\frac{1}{2} + D; +\infty\right[& \rightarrow & \mathbb{R}_-^* \\ & & t & \rightarrow & -\frac{1}{\sqrt{\ln(2(t-D))}}. \end{array} \tag{2}$$

<u>Conclusion</u> : the maximal solutions are all maps of the form (1) or (2), for some $D \in \mathbb{R}$, and the zero constant map.

Answer of exercise 3

1. The equation we consider is an autonomous equation, defined as

$$u' = f(u)$$

for

$$\begin{array}{cccc} f & : & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & & (x,y) & \to & (xe^{xy}, (y-x^2)e^{xy}). \end{array}$$

All the theory we have seen in class about the equilibria of autonomous equations applies, since f is C^1 (actually, C^∞).

a) Since $f(0,0) = (0,0)$, $(0,0)$ is an equilibrium. Conversely, let $(x_0, y_0) \in \mathbb{R}^2$ be an equilibrium. Then $f(x_0, y_0) = 0$, hence

$$\begin{aligned} 0 = x_0 e^{x_0 y_0} &\quad \Rightarrow \quad x_0 = 0; \\ 0 = (y_0 - x_0^2)e^{x_0 y_0} = y_0 e^{x_0 y_0} &\quad \Rightarrow \quad y_0 = 0. \end{aligned}$$

Therefore, $(x_0, y_0) = (0,0)$.

b) We compute the differential of f at $(0,0)$ and apply the theorem seen at the last lecture of the semester. To compute the differential, we use the following Taylor expansion : for all h, l in the neighborhood of 0,

$$\begin{aligned} f(h,l) &= (he^{hl}, (l-h^2)e^{hl}) \\ &= (h(1+o(1)), (l-h^2)(1+o(1))) \\ &= (h,l) + o(||h,l||). \end{aligned}$$

Therefore, the Jacobian of f at $(0,0)$ is

$$Jf(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This matrix has a single eigenvalue, which is 1. Since $\mathrm{Re}(1) = 1 > 0$, the point $(0,0)$ is an unstable equilibrium.

2. Since $(x(t_0), y(t_0)) = (0, y(t_0))$, this solution (x,y) is a maximal solution of the Cauchy problem

$$\left\{ \begin{array}{rl} u' &= f(u), \\ u(t_0) &= (0, y(t_0)). \end{array} \right.$$

As f is C^1, from the Cauchy-Lipschitz theorem, the maximal solution to this problem is unique. Therefore, it suffices to check that the map defined as

$$\begin{array}{cccc} v & : & \mathbb{R} & \to & \mathbb{R}^2 \\ & & t & \to & (0, y(t_0)e^{t-t_0}) \end{array}$$

is a maximal solution : this implies that $I = \mathbb{R}$ and $(x,y) = v$.

The map v (of which we denote v_1, v_2 the components) is indeed a solution : for all $t \in \mathbb{R}$,

$$\begin{aligned} v'(t) &= (0, y(t_0)e^{t-t_0}) \\ &= (v_1(t)e^{v_1(t)v_2(t)}, (v_2(t) - v_1(t)^2)e^{v_1(t)v_2(t)}) \quad \text{since } v_1(t) = 0 \\ &= f(v(t)). \end{aligned}$$

It is maximal because it is defined on $\mathbb{R}$, hence cannot be extended.

3. a) Observe that F is well-defined because x does not vanish on I. It is differentiable (since it is a quotient of differentiable maps, whose denominator does not vanish) and, for all $t \in I$,

$$\begin{aligned} F'(t) &= \frac{(y'(t) + 2x'(t)x(t))x(t) - x'(t)(y(t) + x(t)^2)}{x(t)^2} \\ &= \frac{y'(t)x(t) - x'(t)(y(t) - x(t)^2)}{x(t)^2} \\ &= \frac{(y(t) - x(t)^2)e^{x(t)y(t)}x(t) - x(t)e^{x(t)y(t)}(y(t) - x(t)^2)}{x(t)^2} \\ &= 0. \end{aligned}$$

Therefore, F is constant.

b) Let $(x_0, y_0) \in \mathbb{R}^2$ be such that $x_0 \neq 0$. We denote $(x, y) : I \to \mathbb{R}^2$ the maximal solution of the associated Cauchy problem :

$$\begin{cases} (x, y)' &= f(x, y), \\ (x(0), y(0)) &= (x_0, y_0). \end{cases}$$

For all $t_0 \in I$, $x(t_0) \neq 0$ (otherwise, from Question 2., it would hold $x(t) = 0$ for all $t \in I$, hence $x_0 = 0$). We can therefore apply Question 3.a) : for all $t \in I$,

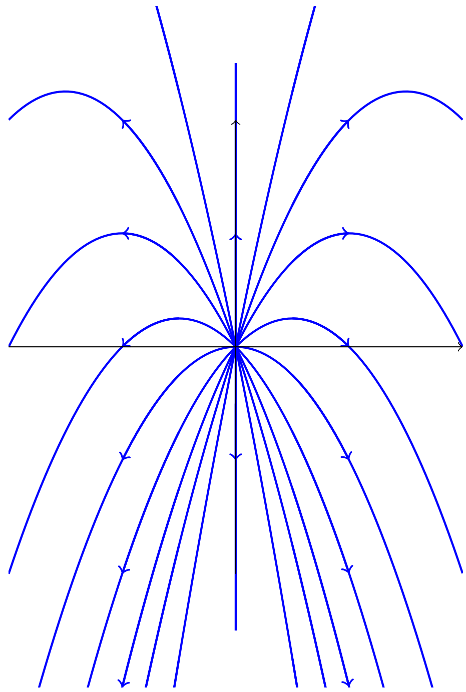
$$\frac{y(t) + x(t)^2}{x(t)} = F(t) = F(0).$$

As a consequence, for all $t \in I$,

$$y(t) = F(0)x(t) - x(t)^2,$$

which means that $(x(t), y(t))$ belongs to the graph of $f_{F(0)}$.
Since the orbit of (x_0, y_0) is $\{(x(t), y(t)), t \in I\}$, the orbit is a subset of the graph of $f_{F(0)}$.

4.



Answer of exercise 4

1. It is possible to use any of the four definitions of a submanifold. Here, for once, we propose to use the definition « by diffeomorphism ».
   Let d be the dimension of M.
   Let x be a point in M_U. Let us show the existence of neighborhoods V_x of x and V_0 of 0 in $\mathbb{R}^n$, and a C^k-diffeomorphism $\phi : V_x \to V_0$ such that

$$\phi(M_U \cap V_x) = \left(\mathbb{R}^d \times \{0\}^{n-d}\right) \cap V_0.$$

   Let $z \in M$ be such that $x = Uz$. As M is a submanifold of class C^k and dimension d, there exist neighborhoods V_z of z and V_0 of 0 in $\mathbb{R}^n$, and a C^k-diffeomorphism $\phi_z : V_z \to V_0$ such that

$$\phi_z(M \cap V_z) = \left(\mathbb{R}^d \times \{0\}^{n-d}\right) \cap V_0.$$

   Let us fix such V_z, V_0, ϕ_z.
   We define $V_x = UV_z = \{Us, s \in V_z\}$. It is a neighborhood of x.[1] Let us define

$$\begin{array}{cccc} \phi & : & V_x & \to & V_0 \\ & & x' & \to & \phi_z(U^{-1}x'). \end{array}$$

   Observe that ϕ is well-defined, and it is a C^k-diffeomorphism. Indeed, ϕ is the composition of ϕ_z, which is a C^k-diffeomorphism between V_z and V_0,

1. Justification : it contains $Uz = x$ and, for all ϵ small enough, $B(x, \epsilon) \subset UB(z, |||U^{-1}|||\epsilon) \subset UV_z = V_x$.

and of the map $\big(x' \to U^{-1}x'\big)$, which is a C^∞-diffeomorphism between V_x and V_z.[2]
Moreover,

$$\begin{aligned}\phi(M_U \cap V_x) &= \phi\left(\{Ux', x' \in M \cap V_z\}\right)\\ &= \left\{\phi_z(U^{-1}Ux'), x' \in M \cap V_z\right\}\\ &= \{\phi_z(x'), x' \in M \cap V_z\}\\ &= \phi_z(M \cap V_z)\\ &= \left(\mathbb{R}^d \times \{0\}^{n-d}\right) \cap V_0.\end{aligned}$$

2. a) The map γ_U is continuous and piecewise C^1, as it is the composition of γ, which is itself continuous and piecewise C^1, and a linear, hence C^∞, map. In addition,

$$\gamma_U(0) = U\gamma(0) = Ux_1 \quad \text{and} \quad \gamma_U(A) = U\gamma(A) = Ux_2.$$

Therefore, γ_U is a path connecting Ux_1 and Ux_2.

b) For all $t \in [0; A]$ such that γ is differentiable at t, the map γ_U is also differentiable at t (by the theorem of composition of differentiable maps) and

$$\gamma_U'(t) = U\gamma'(t).$$

As U is orthogonal, it holds for all such t that $||\gamma_U'(t)||_2 = ||\gamma'(t)||_2$. Consequently,

$$\begin{aligned}\ell(\gamma_U) &= \int_0^A ||\gamma_U'(t)||_2 dt\\ &= \int_0^A ||\gamma'(t)||_2 dt\\ &= \ell(\gamma).\end{aligned}$$

c) From the previous two subquestions,

$$\begin{aligned}\mathrm{dist}_{M_U}(Ux_1, Ux_2) &= \inf\{\ell(\gamma), \gamma \text{ is a path connecting } Ux_1 \text{ and } Ux_2\}\\ &\leq \inf\{\ell(\gamma_U), \gamma \text{ is a path connecting } x_1 \text{ and } x_2\}\\ &= \inf\{\ell(\gamma), \gamma \text{ is a path connecting } x_1 \text{ and } x_2\}\\ &= \mathrm{dist}_M(x_1, x_2).\end{aligned}$$

To show the converse inequality, we observe that, if we replace M with M_U and U with U^{-1} in the previous questions, it holds

$$M = (M_U)_{U^{-1}} \overset{def}{=} \{U^{-1}x, x \in M_U\};$$

2. Remark : for any invertible matrix M and any set $E \subset \mathbb{R}^n$, $\big(x' \to Mx'\big)$ is a C^∞-diffeomorphism between E and ME, with reciprocal $\big(z' \in ME \to M^{-1}z' \in E\big)$.

$$x_1 = U^{-1}(Ux_1);$$
$$x_2 = U^{-1}(Ux_2).$$

Therefore, the inequality we have just shown also implies that

$$\begin{aligned}\mathrm{dist}_M(x_1, x_2) &= \mathrm{dist}_{(M_U)_{U^{-1}}}(U^{-1}(Ux_1), U^{-1}(Ux_2)) \\ &\leq \mathrm{dist}_{M_U}(Ux_1, Ux_2).\end{aligned}$$

The two inequalities, together, imply that

$$\mathrm{dist}_M(x_1, x_2) = \mathrm{dist}_{M_U}(Ux_1, Ux_2).$$

3. a) Let us fix $t \in I$.
Let us for a moment consider a fixed $t' \in I$. Let us set $\tilde{\gamma} = \gamma_{|[t;t']}$ and define, as in Question 2., $\tilde{\gamma}_U : s \in [t;t'] \to U\tilde{\gamma}(s) \in \mathbb{R}^n$. By the same reasoning as in Question 2.b),

$$\ell(\tilde{\gamma}_U) = \ell(\tilde{\gamma}).$$

In addition, we observe that $\tilde{\gamma}_U = \gamma_{U\,|[t;t']}$. Therefore, the above equality is equivalent to

$$\ell(\gamma_{U\,|[t;t']}) = \ell(\gamma_{|[t;t']}).$$

From this we deduce that, for all t' close enough to t,

$$\begin{aligned}\ell(\gamma_{U\,|[t;t']}) &= \ell(\gamma_{|[t;t']}) \\ &= \mathrm{dist}_M(\gamma(t); \gamma(t')) \text{ as } \gamma \text{ is locally minimizing} \\ &= \mathrm{dist}_{M_U}(\gamma_U(t); \gamma_U(t')) \text{ from Question 2.c).}\end{aligned}$$

b) The map γ_U is differentiable, since it is the composition of two differentiable maps. For all $t \in I$, $||\gamma_U'(t)||_2 = ||U\gamma'(t)||_2 = ||\gamma'(t)||_2$. As γ has constant speed (it is a geodesic), γ_U also does.

c) The map γ_U is C^2 (as it is the composition of γ, which is C^2, and a C^∞-map). We must show that it satisfies the geodesic equation. A possibility would be to explicitely compute γ_U'' and the tangent space to M_U at every point. Here, we will rather deduce this result from Questions 3.a) and 3.b).
We must show that γ_U satisfies the geodesic equation at each point of I. Let t belong to I. Let us show that γ_U satisfies the geodesic equation at t :

$$\gamma_U''(t) \in (T_{\gamma_U(t)} M_U)^\perp. \tag{3}$$

Let $t' \in I \setminus \{t\}$ be such that

$$\ell(\gamma_{U\,|[t;t']}) = \mathrm{dist}_{M_U}(\gamma_U(t), \gamma_U(t')).$$

Such a t' exists from Question 3.a).
From a theorem seen in class (labelled as 3.22 in the lecture notes), since $\gamma_{U|[t;t']}$ is a path with constant speed between $\gamma_U(t)$ and $\gamma_U(t')$, whose length is equal to the distance between $\gamma_U(t)$ and $\gamma_U(t')$, it is a geodesic. Consequently, it satisfies the geodesic equation at each point of $[t;t']$. In particular, Equation (3) is true.