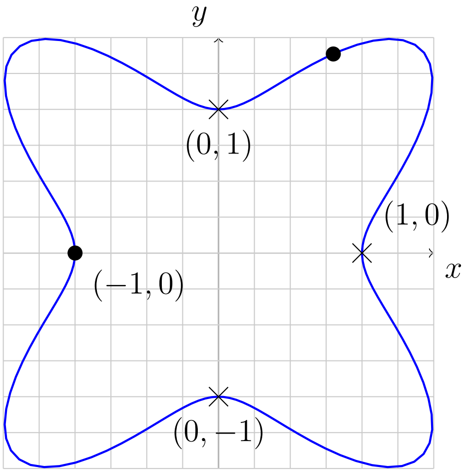
Geometry and differential equations : exam

May 21 2024, 2 hours

You can use any written or printed material.
For each exercise, the number of points is an indication ; it may change.
Except for Exercise 1, you must justify all your answers, clearly and rigorously. Don't forget quantifiers !



Exercise 1

The picture on the left represents a 1-dimensional submanifold of $\mathbb{R}^2$.
Draw a plausible affine tangent space at each of the two points marked by black dots.
[2 points]

Exercise 2

We define

$$\begin{array}{rcccl} f & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \to & -x^3 e^{-\frac{1}{x^2}} \quad \text{if } x \neq 0, \\ & & & & 0 \quad\quad \text{if } x = 0. \end{array}$$

Find all maximal solutions of the equation

$$u' = f(u).$$

[Note : if you need it, you can admit without proof that f is C^∞ over $\mathbb{R}$.]
[4 points]

Exercise 3

We consider the following differential equation

$$\begin{cases} x' &= xe^{xy}, \\ y' &= (y - x^2)e^{xy}. \end{cases}$$

1. a) Show that the only equilibrium is $(0, 0)$.
   b) Is it unstable, stable, asymptotically stable ?
2. Let $(x, y) : I \to \mathbb{R}^2$ be a maximal solution such that, for some $t_0 \in I$, $x(t_0) = 0$. Show that $I = \mathbb{R}$ and, for all $t \in \mathbb{R}$,

$$x(t) = 0,$$
$$y(t) = y(t_0)e^{t-t_0}.$$

3. a) Let $(x, y) : I \to \mathbb{R}^2$ be a maximal solution such that $x(t_0) \neq 0$ for all $t_0 \in I$. We define

$$F \ : \ I \to \mathbb{R}, \quad t \to \frac{y(t)+x(t)^2}{x(t)}.$$

Show that F is constant.

b) For any $\alpha \in \mathbb{R}$, we define

$$f_\alpha \ : \ \mathbb{R} \to \mathbb{R}, \quad x \to \alpha x - x^2.$$

For each $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0 \neq 0$, show that the orbit of (x_0, y_0) under the flow of the differential equation is a subset of the graph of f_α, for some $\alpha \in \mathbb{R}$.

4. Draw the phase portrait.

[9 points]

Exercise 4

Let $n, k \in \mathbb{N}^*$ be fixed, with $k \geq 2$. Let $M \subset \mathbb{R}^n$ be a submanifold of class C^k. We fix $U \in O_n(\mathbb{R})$ an orthogonal matrix and define

$$M_U = \{Ux, x \in M\}.$$

1. Show that M_U is a submanifold of $\mathbb{R}^n$ of class C^k, of the same dimension as M.
2. From now on, we assume that M is connected. Let x_1, x_2 be two points in M.
   a) For some $A \in \mathbb{R}^+$, let $\gamma : [0; A] \to M$ be a path connecting x_1 and x_2. We define

   $$\gamma_U \ : \ [0; A] \to \mathbb{R}^n, \quad t \to U\gamma(t).$$

   Show that γ_U is a path in M_U, connecting Ux_1 and Ux_2.

   b) Show that $\ell(\gamma_U) = \ell(\gamma)$.

   c) Show that $\mathrm{dist}_M(x_1, x_2) = \mathrm{dist}_{M_U}(Ux_1, Ux_2)$.
3. Let $I \subset \mathbb{R}$ be an interval, and $\gamma : I \to M$ a geodesic. We recall that geodesics are locally minimizing, that is, for every $t \in I$,

   $$\ell\left(\gamma_{|[t;t']}\right) = \mathrm{dist}_M(\gamma(t), \gamma(t')) \quad \text{for all } t' \text{ close enough to } t.$$

   a) We define γ_U as previously. Show that γ_U is locally minimizing.

   b) Show that γ_U has constant speed.

   c) Show that γ_U satisfies the geodesic equation, hence is a geodesic.

[9 points]