Fredrickson Andersen 2-spin facilitated model: sharp threshold

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Fredrickson Andersen 2 spin facilitated model (FA-2f)

An interacting particle system on $\{0,1\}^{\mathbb{Z}^d}$, $d \geq 2$.

0=empty, 1=occupied.

Dynamics: birth and death of particles

- Fix a parameter $q \in [0,1]$
- at rate 1 each site gets a proposal to update its state to empty at rate q and to occupied at rate $1-q$.
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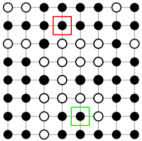

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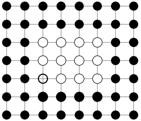

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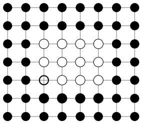

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Several IPS tools fail $\rightarrow$ new tools needed!

Motivations from physics

Introduced in the '80's to model the liquid/glass transition

- major open problem in condensed matter physics;
- sharp divergence of timescales;
- no significant structural changes.

$\Rightarrow$ kinetic constraints mimic *cage effect* :
if temperature is lowered free volume shrinks ($q \leftrightarrow e^{-1/T}$)

$\Rightarrow$ changing the constraint: KCM

$\Rightarrow$ trivial equilibrium and yet sharp divergence of timescales when $q \downarrow 0$, aging, heterogeneities, ... $\rightarrow$ glassy dynamics

Motivations from physics

- Key question: how do KCM time-scales diverge for $q \downarrow 0$?
- Sharp divergence $\rightarrow$ numerical simulations do not give clear-cut answers, some of the conjectures were wrong!

2-neighbour Bootstrap Percolation

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If the initial configuration is distributed with μ_q

$\rightarrow$ $\forall q > 0$ the stable configuration is a.s. empty [Van Enter '88]

$\rightarrow$ τ_0^{BP} = first time at which the origin is emptied

$$\text{for } q \downarrow 0 \text{ w.h.p. } \tau_0^{\text{BP}} = \exp\left(\frac{\lambda(d)}{q^{1/(d-1)}}(1 - o(1))\right)$$

- $\lambda(2) = \pi^2/18$ [Holroyd '08]
- $\lambda(d) = \ldots \ \forall d > 2$ [Balogh Bollobas Duminil-Copin Morris '12]

Back to FA2f: our results

Theorem [Hartarsky, Martinelli, C.T. '20]

As $q \downarrow 0$, w.h.p. for the stationary FA-2f model on $\mathbb{Z}^d$ it holds

$$\tau_0 = \exp\left(\frac{d \times \lambda(d)}{q^{1/(d-1)}}(1 - o(1))\right), \quad d \geq 2$$

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Remark

- This is not a corollary of the BP result:
  the emptying/occupying mechanism of FA-2f has no counterpart in BP!
- We settle contrasting conjectures in physics literature

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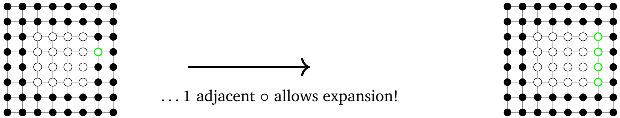

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- Motion requires few additional empty sites $\rightarrow$ this *good environment* is very likely for large droplets ($q \downarrow 0$)

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$$\to \tau_0 \sim 1/\rho_D$$

- $\tau_0^{\mathrm{BP}} \sim$ distance of droplet to origin

$$\to \tau_0^{\mathrm{BP}} \sim 1/\rho_D^{1/d} \sim {\tau_0}^{1/d}$$

How do optimal droplets look like? the d=2 case

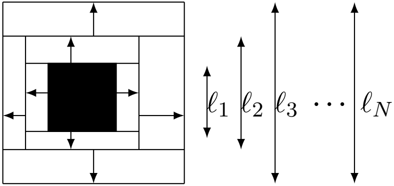
Two key steps:

- identify optimal droplets (...what does optimal means?)
- study the droplet motion and identify its time-scale

Optimal droplets are regions of size poly$(1/q)$ that contain:

1. a segment of $\sim 1/\sqrt{q}$ empty sites $\Rightarrow$ core
2. additional empty sites allowing the core to move inside the droplet without creating a larger empty core
$\Rightarrow$ super-good dust

Super-good dust: multi-scale construction



$$\ell_n := e^{n\sqrt{q}}/\sqrt{q}, \quad N = 8|\log q|/\sqrt{q} \quad \to \quad \ell_N = L_D = q^{-17/2+o(1)}$$

- black square = no double raws fully occupied + one raw with no consecutive filled sites (the core)
- vertical arrow = no double raws fully occupied
- horizontal arrow = no double columns fully occupied

More precisely...

A multi-scale definition

- $\ell_n := e^{n\sqrt{q}}/\sqrt{q}, \quad N = 8|\log q|/\sqrt{q}$
  $\rightarrow \quad \ell_N = L_D = (1/q)^{17/2+o(1)}$
- a rectangle R is *of class* n if
  - R is a single site for $n = 0$;
  - $R = \ell_m \times h$ with $h \in (\ell_{m-1}, \ell_m]$ for $n = 2m$;
  - $R = w \times \ell_m$ with $w \in (\ell_m, \ell_{m+1}]$ for $n = 2m + 1$
- Super-good (SG) rectangles:
  - a rectangle of class 0 is SG if it is empty;
  - a rectangle of class n is SG if it contains a SG rectangle R' of class $n - 1$ (the *core*) AND it satisfies *traversability conditions* elsewhere, i.e. no double column/raw fully occupied.

Droplets are defined as $\ell_N \times \ell_N$ SG rectangles

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$\Longrightarrow$ A generalised CBSEP motion

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$$\to \tau_0 \leq T_{\text{rel}}^{\text{FA-2f,D}}\, T_{\text{rel}}^{\text{g-CBSEP}}$$

$T_{\text{rel}}^{\text{FA-2f, D}}$ = relaxation time of the FA-2f chain inside a droplet
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3. establish the following Poincaré inequalities

$$\rightarrow \; T_{\text{rel}}^{\text{FA-2f,D}} \leq e^{O(\log q)/q^{1/(2d-2)}} \qquad T_{\text{rel}}^{\text{g-CBSEP}} \leq \rho_D^{-1} \log \rho_D$$

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- relaxation is always driven by large rare droplets but their motion can be very different from CBSEP!

Ex. Duarte-KCM:
$d = 2$, constraint = at least 2 empty among N,W,S neighbours

$$\tau_0 = e^{\Theta\left(\frac{(\log q)^4}{q^2}\right)} \gg \tau_0^{\mathrm{BP}} = e^{\Theta\left(\frac{(\log q)^2}{q}\right)}$$

$$\rightarrow \tau_0 \gg (\tau_0^{\mathrm{BP}})^c \quad \forall c$$

[Marêché, Martinelli, C.T. '20]

Thanks for your attention!