Master M1 Physics - Probability Exam 6/11/23 (3 hours)

Documents, computers and calculators are **NOT** allowed

Question 1. *State the weak and the strong law of large numbers. Then :*

- *Demonstrate the weak law of large numbers.*
- *Demonstrate the strong law of large numbers under the hypothesis of finiteness of the fourth moment.*

Exercise 1. *Paul has 30 red balls, he marks (does not mark) with a black dot each ball (independently) with probability 1/3 (2/3, respectively).*
Alice has 20 blue balls, she marks (does not mark) with a black dot each ball (independently) with probability 1/2 (1/2, respectively).
Then they both put the balls in the same box and James picks a ball at random in the box.

- *What is the probability that the ball chosen by James is marked by a black dot?*
- *Knowing that the chosen ball is marked, what is the probability that it is blue?*

Exercise 2. *Let X be a positive square integrable variable. Prove that for any θ with $0 \leqslant \theta \leqslant 1$ it holds*

$$P\left(X \geqslant \theta E(X)\right) \geqslant (1-\theta)^2 \frac{E(X)^2}{E(X^2)}$$

Hint 0.1. *Start by writing $X = X\mathbb{1}_{X \geqslant \theta E(X)} + X\mathbb{1}_{X<\theta E(X)}$, compute $E(X)$ with this decomposition, upperbound the two terms, using Cauchy-Schwarz inequality for one of them.*

Exercise 3. *Fix $p \in (0,1)$ and consider a Galton Watson process, with*

$$Z_0 = 1, \;\; Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i} \qquad \forall n \geqslant 0$$

with $\xi_{n,i}$ i.i.d. geometrically distributed, $P(\xi = k) = (1-p)^k p$ for $k = 0, 1, \ldots$.

1. *Determine the behavior of the extinction probability as a function of p.*

2. *For an integer $n > 1$ evaluate $E(Z_1 Z_n)$.*

3. *Consider the following modification of the process: at each generation (starting from $n = 1$) some immigrants arrive and become indistinguishable from the other variables, namely each immigrant will give birth (independently from all other individuals) at the next generation to a random number of descendants distributed as ξ. The number of immigrants at each generation are i.i.d. integer variables (and independent from the $\xi_{n,i}$) uniformly distributed in $[1, 10]$. Write the generating function for the number of individuals at generation n for this modified process.*

Exercise 4. *A supermarket is open 7 days per week and 24h per day. The number of people passing daily in front of the supermarket follows a Poisson law of parameter λ_w on working days (Monday to Friday) and of parameter λ_h on Saturday and Sunday. Every person decides, independently from the others, to enter or not the shop with probability p or $1 - p$ respectively. Call N the number of clients per week entering the supermarket.*

- *Determine the mean and the variance of N (as a function of p, λ_w and λ_h)*
- *The amount of money spent by the i-th person entering the supermarket is a random variable. All these random variables are i.i.d (and independent from all other variables) and have mean μ and variance σ^2. Determine the mean and the variance of the total amount of money earned by the supermarket during one week (as a function of p, λ_w, λ_h, μ and σ).*

Exercise 5. *Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. integer variables with $P(X_1 = 1) = 2/3$, $P(X_1 = -1) = x$ and $P(X_1 = -2) = 1/3 - x$. Let $S_n = \sum_{i=1}^n X_i$. There exists a unique value of x s.t.*

$$\lim_{n\to\infty} P(S_n > n^{1/3}) = \lim_{n\to\infty} P(S_n < -n^{1/3})$$

1. *Determine this value of x and prove that for this choice*

$$\lim_{n\to\infty} P(S_n > n^{\alpha}) = \lim_{n\to\infty} P(S_n < -n^{\alpha}) = 1/2$$

   for all $\alpha \in (0, 1/2)$.

2. *For the same value of x, determine, for $a, b \in \mathbb{R}$ with $a < b$*

$$P_{[a,b]} := \lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} \in [a, b]\right)$$

   and provide a choice of $[a, b]$ for which $P_{[a,b]} \sim 0.95$.

3. *Discuss the behavior of*

$$\lim_{n\to\infty}\frac{1}{n}\log P\left(\frac{S_n}{n}\in\left[\frac{1}{6},1\right]\right)$$

when x varies in $[0,1/3]$.

Exercise 6. *Suppose you have an infinite set of balls in your pocket, a pen in your hands and a box in front of you. At time 1 pick a ball in your procket, write the number 1 on the ball and put it in the box; at time 2 pick a ball in your pocket write the number 2 on the ball and put it in the box and so on. Now add the following operation at each time: just after inserting the ball shake the box and extract one ball at random from the box and note its number, then reinsert it inside the box. (Note that at time one we extract with probability 1 ball number 1; at time 2 we extract with probability $1/2$ ball number 1 and probability $1/2$ ball number 2 and so on...).*
Let now X_n be the number of times we have extracted ball number 1 for times $\leqslant n$.

1. *Write the expression of $E(X_n)$ and determine its behavior for large n.*
2. *Prove that for all $\epsilon > 0$*

$$P(X_n \in (\log n - \epsilon^{-1}\sqrt{\log n}, \log n + 1 + \epsilon^{-1}\sqrt{\log n}) \geqslant 1 - \epsilon^2 \ .$$

Hint 0.2. *For point 2 you should use Chebychev inequality and recall that $\frac{1}{k+1} \leqslant \int_k^{k+1}\frac{dx}{x} \leqslant \frac{1}{k}$.*