Random Walks in Random Environment
MASTER 2 MATH
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François Simenhaus
Bureau B 640
simenhaus@ceremade.dauphine.fr

Contents

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1 Introduction. Outils

Code teams : zuvzwai

1. Some examples (Sinaï RW, RWRE $d \geq 2$, random conductances, ant in the labyrinth,...).
2. Difficult problem...but easy to formulate. At the end of this course you will know about many open problems ! ($0-1$ law, characterisation of ballistic RW when $d \geq 2$, dynamics RWRE and Sinaï,...)
3. Modelisation : a general paradigm for system with random environments (DNA depinning modelisation,..)
4. Nice mathematical objects that exhibit strange behaviours as slowdown due to traps produced by the random environment.
5. In this course we try to introduce interesting tools to study RWRE : potential function, EVFP, renewal structure,...There are many other ones (link with reinforced random walk for example) but only five lectures !

Many parts of this course are inspired (sometimes almost copy) from [2], [12], [1] [COMPLETE] and from notes of the course by N. Enriquez at Paris 6 along time ago.

1.1 Reminders about Markov chains

Definition. A Markov chain $(X_n)_{n\geq 1}$ is a family of random variables built on the same probability and with values in a countable space E that satisfy the Markov property:

for all $n \geq 0$ and all $(x_0, \cdots, x_{n-1}, x, y) \in E^{n+2}$ such that
$\mathrm{P}(X_0 = x_0, \cdots, X_{n-1} = x_{n-1}, X_n = x) > 0$,

$$\mathrm{P}(X_{n+1} = y | X_n = x, \cdots, X_0 = x_0) = \mathrm{P}(X_{n+1} = y | X_n = x)$$

If moreover there exist $p : E \times E \to [0,1]$ such that

$$\mathrm{P}(X_{n+1} = y | X_n = x) = p(x, y)$$

we say that the chain is **time homogenous**. In the following we only consider homogeneous Markov chain so we will not precise that each time.

An alternative definition of the Makov chains is to define them as *random iteration.*

Proposition 1. *Let $(\xi_n)_{n\geq 1}$ a sequence of i.i.d. random variables with values in a set I (this sequence is called* ***sequence of innovations****), X_0 a random variable with value in E independent of the sequence of innovations, and finally a function $f : E \times I \to E$. Then the random iteration defined by*

$$X_{n+1} = f(X_n, \xi_{n+1}), \quad \text{for all } n \geq 0,$$

is an homogeneous Markov chain with transitions :

$$p(x, y) = \mathrm{P}(f(x, \xi) = y), \quad x, y \in E.$$

As a direct consequence of Markov property we obtain Chapman Kolmogorov equation that is: if μ is the law of X_0 then for all integer $n \geq 0$ and all $x_0, \cdots, x_n \in E$,

$$\mathrm{P}(X_n = x_n, \cdots, X_0 = x_0) = \mu(x_0) \prod_{i=0}^{n-1} p(x_i, x_{i+1}).$$

In particular the law of a Markov chain is fully characterised by the initial law and the transition matrix p.

Canonical process. It is often convenient to work with the *canonical process* (for example to vary the initial condition without changing the process itself). We thus consider as measurable space the space of trajectories

$$E^{\mathbb{N}}$$

endowed with the product $\sigma-$algbra $\mathcal{G}$. On $(E^{\mathbb{N}}, \mathcal{G})$ one defines for all $n \geq 0$, X_n as the $n-$th projection:

$$\begin{aligned} X_n :& (E^{\mathbb{N}}, \mathcal{G}) && \to (E, \mathcal{E}) \\ & \omega = (\omega_i)_{i\geq 0} && \to \omega_n \end{aligned}$$

Using Daniell Kolmogorov theorem one can prove that given a law μ on E and a transition matrix p there exists a unique law P_μ on $(E^{\mathbb{N}}, \mathcal{G})$ so that under P_μ the canonical process $(X_n)_{n\geq 0}$ is markovian with initial law μ and transition p. On the space $(E^{\mathbb{N}}, \mathcal{G})$ one defines the time shift

$$\begin{aligned} \theta :& (E^{\mathbb{N}}, \mathcal{G}) && \to (E^{\mathbb{N}}, \mathcal{G}) \\ & (\omega_i)_{i\geq 0} && \to (\omega_{i+1})_{i\geq 0} \end{aligned}$$

and for all $n \geq 0$, $\theta_n = \theta^{\circ n}$. We endow the canonical space with the filtration generated by the canonical process: for all $k \geq 0$

$$\mathcal{F}_k^X = \sigma(X_0, \cdots, X_k),$$

(or simply $\mathcal{F}_k$ when no confusion is possible) and one can reformulate the simple Markov property:

Proposition 2. *For all bounded random variable Z on the canonical space, all $k \in \mathbb{N}$, all probability measure μ on E, it holds that P_μ-a.s.*

$$\mathrm{E}\left(Z \circ \theta_k | \mathcal{F}_k^X\right) = \mathrm{E}_{X_k}(Z).$$

This property actually implies the strong Markov property where the deterministic time of previous proposition is replaced by a random time. We remind that a random variable $T : (\Omega, \mathcal{F}) \to \mathbb{N} \cup \{+\infty\}$ is a *stopping time* for the filtration $\mathcal{F}^X$ if for all $n \geq 0$

$$\{T = n\} \in \mathcal{F}_n^X.$$

One can then defined the $\sigma-$algebra associated to this stopping time by

$$\mathcal{F}_T = \left\{A \text{ s.t. } A \cap \{T \leq n\} \in \mathcal{F}_n^X, \text{ for all } n \geq 0\right\}.$$

Proposition 3 (Strong Markov property)**.** *For all bounded random variable Z on the canonical space, all stopping time T, and all probability measure μ on E, it holds that P_μ-a.s.*

$$\mathrm{E}\left(Z \circ \theta_T \ 1_{\{T<+\infty\}} | \mathcal{F}_T\right) = \mathrm{E}_{X_T}(Z) 1_{\{T<+\infty\}}.$$

Kernel. To a transition matrix one can associates a useful operator kown as the kernel and define on $\mathcal{B}(E, \mathbb{R})$ the set of all bounded function from E with values in $\mathbb{R}$:

$$\begin{array}{rll} Q: & \mathcal{B}(E,\mathbb{R}) & \to \mathcal{B}(E,\mathbb{R}) \\ & f & \mapsto Qf \end{array}$$

where Qf is defined for all $x \in E$ by

$$Qf(x) = \sum_{y \in E} p(x,y) f(y) = \mathrm{E}_x(f(X_1)) \qquad \text{for all } x \in \mathbb{Z}^d. \tag{1}$$

This operator characterises the matrix transition p. One can check that for all $n \geq 1$

$$\mathrm{E}_x(f(X_n)) = Q^n f(x).$$

If Q acts to the right on $\mathcal{B}(E, \mathbb{R})$, it also acts on the left on the space of (probability) measures where it is defined by

$$\mu Q(x) = \sum_{y \in E} \mu(y) p(y,x) = \mathrm{P}_\mu(X_1 = x). \tag{2}$$

$$\tag{3}$$

One can easily check that if μ_0 is the law of X_0 then for all $n \geq 0$

$$\mu_n = \mu_{n-1} Q = \mu_0 Q^n$$

In particular for all x, y in E,

$$\mathrm{P}_x(X_n = y) = \delta_x Q^n(\{y\}) =: Q^n(x, y)$$

where this last notation is justified by the fact that Q may be understood as an infinite matrix and Chapman-Kolmogorov equation states

$$Q^n(x, y) = \sum_{z_1, \cdots, z_{n-1} \in E} \prod_{i=0}^{n-1} p(z_i, z_{i+1}),$$

with the convention $z_0 = x$ and $z_n = y$.

Recurrence and Transience. A Markov chain $(X_n)_{n \geq 0}$ is **irreducible** if for all $x, y \in E$ there exists $n \geq 0$ such that $\mathrm{P}_x(X_n = y) > 0$.

For all $z \in E$, we define the **hitting time** of z by

$$T_z := \inf\{n \geq 1,\ X_n = z\},$$

with the convention $\inf \emptyset = +\infty$. We also introduce the **number of visits at** z

$$N_z = \sum_{k=0}^{+\infty} 1_{\{X_k = z\}},$$

and the **number of visits at** z **before time** n

$$N_z(n) = \sum_{k=0}^{n} 1_{\{X_k = z\}}.$$

From Fubini's theorem, for all x, y in E,

$$\mathrm{E}_x(N_y) = \mathrm{E}_x(\sum_{k=0}^{+\infty} 1_{\{X_k = y\}}) = \sum_{k=0}^{+\infty} \mathrm{P}_x(X_k = y) = \sum_{k=0}^{+\infty} Q^k(x, y).$$

Définition 1. *The state $x \in E$ is said to be*

1. ***recurrent*** *if* $\mathrm{P}_x(T_x < +\infty) = 1$
2. ***transient*** *if* $\mathrm{P}_x(T_x < +\infty) < 1$.

Point 2. is equivalent to $\mathrm{P}_x(T_x = +\infty) > 0$, that is the **escape probability** is positive..

Theorem 1. *For all $x \in E$,*

1. *x **is recurrent** $\Leftrightarrow$ $N_x = +\infty \quad \mathrm{P}_x - p.s.$*
2. *x **is transient** $\Leftrightarrow$ $N_x \rightsquigarrow Geo(\mathrm{P}_x(T_x = +\infty))$ under P_x*

Proposition 4. 1. *When the Markov chain is **irreducible**, all states are of same type and we say the chain to be recurrent or transient.*

2. *When the Markov chain is (irreducible) **recurrent**, for all $x, y \in E$*

$$\mathrm{P}_x(N_y = +\infty) = 1 \text{ in particular } \mathrm{P}_x(T_y < +\infty) = 1.$$

3. *When the Markov chain is (irreducible) **transient**, for $x, y \in E$,*

$$\mathrm{P}_x(N_y < +\infty) = 1.$$

For all $k \geq 0$ one defines $T_z^{(0)} = 0$, $T_z^{(1)} = T_z$ and for all $k \geq 2$, one defines the $k-$th visits at z by

$$T_z^{(k)} = \inf\{n \geq T_z^{(k-1)} + 1,\ X_n = z\}.$$

Remark that

$$1 \leq T_z \leq T_z^{(2)} \leq \cdots \leq T_z^{(k)} \leq T_z^{(k+1)} \leq +\infty,$$

and $\{T_z^{(k)} = T_z^{(k+1)}\} = \{T_z^{(k)} = +\infty\}$.

Proposition 5. *For all $k \geq 0$ and all $x \in E$, $T_z^{(k)}$ is a stopping time for the filtration $(\mathcal{F}_n^X)_{n\geq 0}$. Moreover if z is recurrent then for all $k \geq 1$, $T_z^{(k)} < +\infty$ $\mathrm{P}_z-a.s.$ and the excursions from z to z*

$$\left(X_{\left(T_z^{(k)}+n\right)\wedge T_z^{(k+1)}}\right)_{n\geq 0} \qquad k \geq 0,$$

are i.i.d.

Harmonic functions and martingales. A function $f : E \to \mathbb{R}$ is said to be harmonic if for all $x \in E$,

$$f(x) = \sum_{y\in E} p(x,y) f(y).$$

Proposition 6. *The random process $(f(X_n))_{n\geq 0}$ is a martingale (with respect to its own filtration) if and only if f is harmonic.*

More on Markov processes. [COMPLETE if necessary] reversible measures, invariant measures

1.2 Tools from ergodic theory

Définition 2. *A measure preserving transformation (mpt) is a map* $T : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F})$ *that is*

1. *measurable*

2. *preserves the probability measure:* $T(\mathbb{P}) = \mathbb{P}$

Définition 3 (Ergodicity)**.** *Given some mpt* T*, the system* $(\Omega, \mathcal{F}, \mathbb{P}, T)$ *is said to be* ***ergodic*** *if the invariant* $\sigma-$*algebra* $\mathcal{I}$ *is trivial that is for any* $A \in \mathcal{F}$ *such that* $\{T \in A\} = A$, $\mathbb{P}(A) \in \{0, 1\}$.

Proposition 7. *The system* $(\Omega, \mathcal{F}, \mathbb{P}, T)$ *is ergodic if and only if any function* f *a.s. invariant by* T *(that is* $f = f \circ T$ *a.s.) is a.s. constant.*

Définition 4 (Strongly mixing)**.** *Given some mpt* T*, the system* $(\Omega, \mathcal{F}, \mathbb{P}, T)$ *is said to be* ***strongly mixing*** *if for all* $A, B \in \mathcal{F}$

$$\lim_{k \to +\infty} \mathbb{P}(A \cap \ \{T^k \in B\}) = \mathbb{P}(A)\mathbb{P}(B)$$

Proposition 8. *Strong mixing implies ergodicity.*

Theorem 2 (Birkhoff's ergodic theorem)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P}, T)$ *be a measure preserving system and* f *in* L^1*. Then a.s.*

$$\frac{1}{n} \sum_{i=1 \cdots n} f \circ T^i \to \mathrm{E}(f \ |\mathcal{I})$$

as $n \to +\infty$*. In particular when the system is ergodic it satisfies the law of large numbers*

$$\frac{1}{n} \sum_{i=1 \cdots n} f \circ T^i \to \mathrm{E}(f).$$

Utile ?

Theorem 3 (Tempelman)**.** *Let* $T_1, \cdots, T_d$ *be* d *pairwise commuting probability preserving maps on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and suppose* $(B_r)_{r \geq 1}$ *is a sequence of increasing boxes of* $\mathbb{Z}_+^d$ *tending to* $\mathbb{Z}_+^d$ *as* r *goes to* $+\infty$*. Then for all* $f \in \mathrm{L}^1(\Omega)$*, a.s.*

$$\frac{1}{|B_r|} \sum_{x \in B_r} f \circ T^x \to \mathrm{E}(f | \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_d) \quad \text{as } r \text{ goes to } +\infty,$$

where for all $j = 1 \cdots, d$ $\mathcal{I}_j$ *is the invariant* $\sigma-$*algebra of* T_j *and for all* $x = (x_1, \cdots, x_d) \in \mathbb{Z}_+^d$

$$T^x = T_1^{x_1} \circ \cdots \circ T_d^{x_d}.$$

[ADD a version with T_i invertible] We say that the system is **ergodic** when $\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_d$ is a trivial $\sigma-$algebra.

[ADD] Ergodic process. An example : irreducible recurrent positive MC are ergodic.

1.3 Reminders about SRW in Z^d

Given $d \geq 1$ the simple random walk (SRW) $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$ is the Markov chain with value in $\mathbb{Z}^d$ and transitions given by

$$p(x,y) = \frac{1}{2d} \quad \text{for all } x \sim y,$$

where $x \sim y$ if they are neighbours on the lattice $\mathbb{Z}^d$ that is $|x-y|_1 = 1$. Off course this Markov chain is irreducible and we remind the famous

Theorem 4 (Polyà)**.** *The SRW on $\mathbb{Z}^d$ is transient if and only if $d \geq 3$.*

Proof. There are many proofs. Most of them are not robust and will not be very usefull once dealing with RWRE

Green function. We compute

$$G(0,0) = \mathrm{E}(\sum_{n\geq 0} 1_{S_{2n}} = 0) = \sum_{n\geq 0} \mathrm{P}(S_{2n} = 0).$$

In dimension $d = 1$ an easy combinatorial argument and Stirling estimates leads to

$$\mathrm{P}(S_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}}$$

that implies that $G(0,0) = +\infty$ and the SRW is recurrent. We may adapt the same argument for $d = 2$ and obtain

$$\mathrm{P}(S_{2n} = 0) \sim \frac{1}{\pi n}$$

and this estimate leads to the same conclusion. For $d = 3$ we still use the same argument to obtain the following estimate

$$\mathrm{P}(S_{2n} = 0) \leq \frac{C}{n^{3/2}}$$

where $C > 0$ is a suitable constant. This time we obtain that $G(0,0) < +\infty$ and conclude that the SRW is transient. This implies that the SRW is actually transient for all $d \geq 3$.

For $\mathbf{d = 2}$ we may also deduce the result from the one dimensional case. Remark that the projection $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ on the vector line directed by $e_1 + e_2$ and $e_1 - e_2$ are i.i.d. one dimensional random walk. It implies that

$$\mathrm{P}(S_{2n} = 0) = \mathrm{P}(X_{2n} = 0, Y_{2n} = 0) = \mathrm{P}(X_{2n} = 0)\mathrm{P}(Y_{2n} = 0) \sim \frac{1}{\pi n}$$

Martingale argument ($\mathbf{d = 1}$). Remind the definition of T_1 the hitting time of 1. It is a stopping time and $(S_{n\wedge T_1})_{n\geq 0}$ is thus a martingale bounded from above by 1. It implies that it converges a.s. and the only possible limit is 1 that is $T_1 < +\infty$ $\mathrm{P}_0 - a.s.$. The same holds of course for T_{-1} the hitting time of -1. And one deduces easily that the escape probability $\mathrm{P}_0(T_0 = +\infty) = 0$ that is the SRW is recurrent.

Using the characteristic function [ADD] A more robust method: Chung-Fuchs Theorem □

As a sum of bounded i.i.d. increments the SRW satisfies (ordered from the weakest to the most general)

1. the law of large numbers

$$\frac{S_n}{n} \overset{a.s.}{\to} 0 \quad \text{as } n \to +\infty$$

2. the central limit theorem

$$\frac{S_n}{\sqrt{n}} \overset{\text{under } \mathbb{P}}{\Longrightarrow} \mathcal{N}(0,1) \quad \text{as } n \to +\infty$$

3. the invariance principle

$$\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{t\geq 0} \Longrightarrow (B_t)_{t\geq 0}$$

   where the convergence is the weak convergence in the space of continuous function from $\mathbb{R}$ in $\mathbb{R}$ endowed with the topology of the uniform convergence on compact sets.

We typically try to check if these properties are still true or not for RWRE!

1.4 Definition of RWRE

Let us first define the set of **environments**. Fix $d \geq 1$. We say that x and y in $\mathbb{Z}^d$ are neighbours and write $x \sim y$ if $|x-y|_1 = 1$. We denote by E the set of $2d$ neighbours of 0. We introduce the set of all transitions vectors

$$S_{2d-1} = \{(p_e)_{e\in E} \in [0,1]^{2d},\ \sum_{e\in E} p_e = 1\}.$$

Finally the set of environments is a collection indexed by $\mathbb{Z}^d$ of transition vectors that is $\Omega = S_{2d-1}^{\mathbb{Z}^d}$.

Given some environment $\omega = (\omega_x(\cdot))_{x\in\mathbb{Z}^d} \in \Omega$ and $x \in \mathbb{Z}^d$ we consider the canonical discrete time nearest neighbour Markov chain $(X_n)_{n\geq 0}$ with law $\mathrm{P}_{x,\omega}$ such that for all $n \geq 1$ and all $(x_0, \cdots, x_n) \in \mathbb{Z}^d$ such that $\mathrm{P}_{x,\omega}(X_n = x_n, \cdots, X_1 = x_1, X_0 = x_0) > 0$ and all $e \in E$

$$\mathrm{P}_{x,\omega}(X_0 = x) = 1$$
$$\mathrm{P}_{x,\omega}(X_{n+1} = x_n + e \mid X_n = x_n, \cdots, X_1 = x_1, X_0 = x_0) = \omega_{x_n}(e).$$

We will often write P_ω for $\mathrm{P}_{0,\omega}$. We will need the space shift or translation on the space of environments: define for all $z \in \mathbb{Z}^d$ t_z that associates to all $\omega \in \Omega$

$$(t_z\omega)_x = \omega_{z+x}.$$

In the following we want to consider ω as a random object: this is what we will call a **random environment**. We thus consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is the product $\sigma-$algebra and $\mathbb{P}$ is a probability measure. Here are first examples of such random environments :

1. **Product measure.** Consider a probability measure μ on S_{2d-1} endowed with its Borel $\sigma-$algebra and define $\mathbb{P} = \mu^{\otimes\mathbb{Z}^d}$. Remark thats such product probability $\mathbb{P}$ is invariant under any space translation t_z, $z \in \mathbb{Z}^d$.

2. **Random conductances.** We first associate a random conductance to each non oriented edge of $\mathbb{Z}^d$: consider a probability measure μ on $[0,+\infty)$ endowed with its Borel $\sigma-$algebra and define $\mathbb{S} = \mu^{\otimes\mathcal{E}^d}$ where $\mathcal{E}^d$ is the set of edges of $\mathbb{Z}^d$. Finally define $\mathbb{P}$ as the image law of $\mathbb{S}$ by the application that associates to a collection $c = (c_{x,y})_{x,y\in\mathbb{Z}^d}$ of conductances the set of transitions defined for all neighbours x, y in $\mathbb{Z}^d$ by
$$\omega_x(y-x) = \frac{c_{x,y}}{\sum_{z\sim x} c_{x,z}}$$
if $\sum_{z\sim x} c_{x,z} > 0$ and $\omega_x(0) = 1$ otherwise.

3. **The ant in the labyrinth** (P.G. De Gennes). It is the simple random walk on the infinite percolation cluster: in the previous example pick $\mu = p\delta_1 + (1-p)\delta_0$. The probability measure $\mathbb{Q}$ defines on $\mathcal{E}$ is known as Bernoulli i.i.d. percolation. One can prove (see [4]) that for all $d \geq 2$ there exists some critical $p_c(d) \in (0,1)$ such that

   - for all $p < p_c$ $\mathbb{Q}$-a.s. all clusters (=connected components of the weighted graph $(\mathbb{Z}^d, c)$) are finite
   - for all $p > p_c$ $\mathbb{Q}$-a.s. there exists a unique infinite cluster C

   Consider the case $p > p_c(d)$ in dimension $d \geq 3$. In this case the environment is not irreducible and 0 is possibly not in C. We may work with $\mathbb{Q}(\cdot|0 \in C)$ and the environment defines then the simple random walk on C. Does such ant come back to the origin infinitely often [5]? Does it exhibits a annealed or quenched central limit theorem ? Invariance principle ?

Assumptions : we say that the environment satisfies the **ellipticity assumption (E)** if

$$\mathbb{P}\text{-almost surely for all } x \in \mathbb{Z}^d \text{ and all } e \in E,\ \omega_x(e) > 0. \tag{4}$$

This condition assures that $\mathbb{P}-$a.s. $\mathrm{P}_{x,\omega}$ is irreducible. We say that the environment satisfies the **uniform ellipticity assumption (UE)** if there exists some $\varepsilon_0 > 0$ such that

$$\mathbb{P}\text{-almost surely for all } x \in \mathbb{Z}^d \text{ and all } y \sim x,\ \omega_x(e) > \varepsilon_0. \tag{5}$$

Quenched properties / Annealed probability measure. We can now define the *annealed law* P_x as the semi product of $\mathbb{P}$ and $\mathrm{P}_{x,\omega}$ that intuitively corresponds to, first, pick randomly an environment in Ω according to $\mathbb{P}$ and then a markovian path in $\mathbb{Z}^d$ according to $\mathrm{P}_{x,\omega}$. More formally for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$

$$\mathrm{P}_x(A \times B) = \int_A P_{x,\omega}(B)\ \mathrm{d}\mathbb{P}.$$

We will mostly consider only the second marginal of P_x that is

$$\mathrm{P}_x(B) = \int_\Omega P_{x,\omega}(B)\ \mathrm{d}\mathbb{P}.$$

The annealed law is in general not markovian. Suppose the environment is i.i.d. with common law μ and denote for all $x \in \mathbb{Z}^d$, $e \in E$ and $n \geq 0$ by

$N^n(x,y)$ the number of visits of the edge $(x,x+e)$ by the walker before time n

$$N^n_x(e) = \sum_{i=0}^{n-1} 1_{(X_i,X_{i+1})=(x,x+e)}$$

Now for all $n \geq 1$ and all $(x_0,\cdots,x_n) \in \mathbb{Z}^d$ and all $e \in E$

$$\begin{aligned}
\mathrm{P}_x(X_{n+1} = x_n + e \,|X_n = x_n, \cdots, X_1 = x_1, X_0 = x_0) \\
&= \frac{\int \prod_{i=0}^{n} \omega_{x_i}(x_{i+1} - x_i) \; \mathrm{d}\mathbb{P}}{\int \prod_{i=0}^{n-1} \omega_{x_i}(x_{i+1} - x_i) \; \mathrm{d}\mathbb{P}} \\
&= \frac{\int \prod_{x\in\mathbb{Z}^d} \prod_{e\in E} \omega_x(e)^{N_x^{n+1}(e)} \; \mathrm{d}\mathbb{P}}{\int \prod_{x\in\mathbb{Z}^d} \prod_{e\in E} \omega_x(e)^{N_x^{n}(e)} \; \mathrm{d}\mathbb{P}} \\
&\overset{i.i.d.}{=} \frac{\int \prod_{e\in E} \omega_{x_n}(e)^{N_{x_n}^{n+1}(e)} \; \mathrm{d}\mu}{\int \prod_{e\in E} \omega_{x_n}(e)^{N_{x_n}^{n}(e)} \; \mathrm{d}\mu} \\
&= \int \frac{\prod_{e\in E} \omega_{x_n}(e)^{N_{x_n}^{n}(e)}}{\int \prod_{y\sim x_n} \omega_{x_n}(e)^{N_{x_n}^{n}(e)} \; \mathrm{d}\mu} \omega_{x_n}(e) \; \mathrm{d}\mu
\end{aligned}$$

One can imagine that the first visit at $x \in \mathbb{Z}^d$ the walker as no information about the value of $\omega_x(\cdot)$ and draw a transition vector according to μ. But for successive visits the walker updates its knowledge about the environment at x and use a transition vector drawn according to

$$\frac{\prod_{e\in E} \omega_{x_n}(e)^{N_{x_n}^{n}(e)}}{\int \prod_{y\sim x_n} \omega_{x_n}(e)^{N_{x_n}^{n}(e)} \; \mathrm{d}\mu}. \tag{6}$$

If you know Bayesian statistics theory you may think that μ is the prior law and at each visit you obtain a new posterio law given by (6). On can see from the previous expression that walker under the annealed law favorises the transitions it has already used. It is thus non markovian even if the intersection with its own past is local (that is the probability to jump from x to y does only depends on previous jumps from x to one of its neighbours).

As a consequence we should keep in mind that the walker under the annealed law is slowdowned. The walkers favorises the transition it has already used ! This can also be understood by the fact that the environment creates traps where the walker can be blocked.

By opposition to the annealed law we will used the (improper at least for the singular use) term **quenched law** that refers to the situation where the environment in fixed. We say that

some property relative to the walker $(X_n)_{n\geq 0}$ holds *quenched* if it is true under $\mathrm{P}_{x,\omega}$ for $\mathbb{P}-$ almost all $\omega \in \Omega$.

For example we say that the *quenched* central limit theorem holds if $\mathbb{P}-$**a.s. under** $\mathrm{P}_{\mathbf{0},\omega}$

$$\frac{X_n}{\sqrt{n}} \implies \mathcal{N}(0,\sigma^2).$$

This is a stronger result than the *annealed* central limit theorem that only states that the same weak convergence holds **under** $\mathrm{P}_{\mathbf{0}}$.

Recurrence / transience. In this paragraph we assume the ellipticity condition (E) so that almost surely P_ω is irreducible and is transient or recurrent. However the annealed law is not markovian and it does not make sense at first view to speak about recurrence or transience. However note that the event (exercise : why is it an event ?) of $\mathcal{F}$,

$$\text{Rec.} = \text{“}(X_n)_{n\geq 0} \text{ is recurrent under } \mathrm{P}_{0,\omega}\text{”}$$

is invariant under any space translation t_z, $z\in\mathbb{Z}^d$. As soon as the environment is **ergodic**, under $\mathbb{P}$ and for the family of space shifts $\{t_e,\ e\in E\}$, we obtain that $\mathbb{P}(\text{Rec.})\in\{0,1\}$. The same holds of course with the event that the walk is transient under $\mathrm{P}_{0,\omega}$. Thus with a slight abuse of language we can say that the walker is transient or recurrent under the (non markovian) annealed law.

2 One dimensional RWRE

Notation : in the one dimensional setting, we use the notation ω_x instead of $\omega_x(1)$. This implies that $\omega_x(-1) = 1-\omega_x$.

In the following we mainly focus on the case where $\mathbb{P}$ is a product probability measure that satisfies the uniform ellipticity assumption (UE), see (5). More precisely, we consider a probability measure μ on $[0,1]$ such that for some ε_0, $\mu(\omega_0>\varepsilon_0)=1$ and define $\mathbb{P}=\mu^{\mathbb{Z}^d}$.

For this part I borrowed heavily from Nathanael Enriquez's notes and from Ofer Zeitouni's notes [2].

2.1 Harmonic function and potential

Given $\omega\in\Omega$ one defines the operator acting on the space of bounded functions on $\mathbb{Z}^d$ $Q_\omega:\mathcal{B}(\mathbb{Z}^d)\to\mathcal{B}(\mathbb{Z}^d)$ by

$$Q_\omega f(x) = \mathrm{E}_{x,\omega}\left(f(X_1)\right) = \omega_x f(x+1) + (1-\omega_x) f(x-1) \qquad \text{for all } x\in\mathbb{Z}.$$

We first identify harmonic functions for the walker under P_ω. We remind that a function $f:\mathbb{Z}\to\mathbb{R}$ is harmonic if $Q_\omega f = f$ that is for all $x\in\mathbb{Z}$

$$f(x) = \omega_x f(x+1) + (1-\omega_x) f(x-1).$$

Using the notation $\rho_x = (1-\omega_x)/\omega_x$ and $\Delta f_x = f(x) - f(x-1)$ for all $x \in \mathbb{Z}$ this last equation leads to

$$\Delta f_{x+1} = \rho_x \Delta f_x \qquad x \in \mathbb{Z}.$$

and thus

$$\Delta f_x = \begin{cases} \Delta f_0 \prod_{i=0}^{x-1} \rho_i & \text{for } x \geq 1 \\ \Delta f_0 \left(\prod_{i=x}^{-1} \rho_i\right)^{-1} & \text{for } x \leq -1 \end{cases}$$

Harmonic functions are thus the two dimensional vectorial space of functions that write,

$$f(x) = \begin{cases} f(0) + \Delta f_0 \sum_{i=1}^{x} \prod_{j=0}^{i-1} \rho_j & x \geq 0 \\ f(0) - \Delta f_0 \sum_{i=x+1}^{0} \left(\prod_{j=i}^{-1} \rho_j\right)^{-1} & x \leq -1 \end{cases}$$

We now focus on the unique harmonic function that satisfies $f(0) = 1$ and $f(-1) = 0$ that is

$$f(x) = \begin{cases} 1 + \sum_{i=1}^{x} \prod_{j=0}^{i-1} \rho_j & x \geq 0 \\ 1 - \sum_{i=x+1}^{0} \left(\prod_{j=i}^{-1} \rho_j\right)^{-1} & x \leq -1 \end{cases}$$

We introduce the so-called **potential** (the reason of this denomination will be clear later, see below Lemma 2 for instance) as the only function $V : \mathbb{Z} \to \mathbb{R}$ defined by $V_0 = 0$ and $\Delta V_i = \ln \rho_{i-1}$ that is

$$\begin{aligned} &V_0 = 0 \\ &V_i = \ln \rho_0 + \cdots + \ln \rho_{i-1} \qquad i \geq 1 \\ &V_i = -\ln \rho_{-1} - \cdots - \ln \rho_i \qquad i \leq 1 \end{aligned}$$

Note that $(V_i)_{i \geq 0}$ and $(V_i)_{i \leq 0}$ are both random walks on $\mathbb{R}$ that is sum of i.i.d random variables. The increments of the second process is the opposite in law of the increment of the first one.

Rewriting f using the potential we obtain

$$f(x) = \begin{cases} e^{V_0} + \cdots + e^{V_x} & \text{if } x \geq 0 \\ 0 & \text{if } x = -1 \\ -e^{V_{-1}} - \cdots - e^{V_{x+1}} & \text{if } x \leq -2 \end{cases} \tag{7}$$

Note that for all $x \in \mathbb{Z}$, $\Delta_x f = e^{V_x}$.

2.2 Transience / recurrence

We now use the harmonic function to study the recurrence / transience of the one dimensional RWRE. Our goal is to prove

Theorem 5. *Assume (E) and that* $\ln \rho_0$ *is in* $\mathrm{L}^1(\mathbb{P})$*. Then*

1. *If* $\mathbb{E}(\ln \rho_0) = 0$ *then* $\mathrm{P}-$*a.s.* $\limsup X_n = -\liminf X_n = +\infty$ *(and* $(X_n)_{n\geq 0}$ *is thus recurrent).*

2. *If* $\mathbb{E}(\ln \rho_0) < 0$ *then* $\mathrm{P}-$*a.s.* $\lim X_n = +\infty$ *(and* $(X_n)_{n\geq 0}$ *is thus transient).*

3. *If* $\mathbb{E}(\ln \rho_0) > 0$ *then* $\mathrm{P}-$*a.s.* $\lim X_n = -\infty$ *(and* $(X_n)_{n\geq 0}$ *is thus transient).*

Proof.

Remark 1. *When we establish an almost sure property A quenched and annealed statements are equivalent as* $\mathrm{P}(A = 1)$ *means* $\mathbb{E}(\mathrm{P}_\omega(A)) = 1$ *that implies that* $\mathbb{P}-$*a.s.* $\mathrm{P}_\omega(A)$ *as the random variable* $\mathrm{P}_\omega(A)$ *is smaller than* 1*.*

This is not the case anymore when we are considering for example convergence in law as in the central limit theorme where the quenched statement implies the annealed one but the converse implication may be false !

We use the harmonic function f defined in (7) that is an increasing function. In the case of 2. the law of large numbers implies that $\mathbb{P}-$a.s. $(V_i)_{i\in\mathbb{Z}}$ is equivalent to $\mathbb{E}(\ln \rho_0)i$ at $+\infty$ and $-\infty$ thus $\mathbb{P}-$a.s.

$$\lim_{x\to+\infty} f(x) < +\infty \qquad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Now from Proposition 6 the random process $(f(X_n))_{n\geq 0}$ is a martingale under P_ω and as it is moreover bounded from above it converges $\mathrm{P}_\omega-$a.s. to a L^1 random variable. The only possible limit is the supremum of f and it implies that $(X_n)_{n\geq 0}$ goes to $+\infty$ as n goes to infinity P_ω-a.s. Item 3. is of course proven in the same way.

For Item 1. : Using Chung-Fuchs Theorem [COMPLETE] $\mathbb{E}(\ln \rho) = 0$ implies that $(V_i)_{i\in\mathbb{Z}}$ is recurrent and that

$$\lim_{x\to+\infty} f(x) = +\infty \qquad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

We use now an argument similar to the one used for proving the recurrence of $1d-$SRW. For any $k \geq 0$ define the hitting time of k by $T_k = \inf\{n \geq 0;\ X_n = k\}$ with the convention that $\inf \emptyset = +\infty$. It is a stopping time and thus $(f(X_{n\wedge T_k}))_{n\geq 0}$ is a martingale under P_ω and it is bounded from above.

Thus again it converges $P_\omega-$a.s. to a L^1 random variable and the only possible limit is $f(k)$ which implies that $(f(X_{n\wedge T_k}))_{n\geq 0}$ is actually eventually equal to $f(k)$. As a consequence $T_k < +\infty$ $P_\omega-$a.s and the same holds if $k \leq 0$ with a similar argument. It implies that P_ω-a.s.

$$\limsup X_n = +\infty \quad \text{and } \liminf X_n = -\infty$$

and thus $(X_n)_{n\geq 0}$ is recurrent. □

Remark that Lemma 2 provides also formulas that lead to the same conclusions in a very classical way.

2.3 Law of large numbers

We now turn to the law of large numbers.

Theorem 6. *(Somomon 1975) Assume (UE). Then* P_0-*a.s.*

$$\frac{X_n}{n} \underset{n\to+\infty}{\longrightarrow} v = \begin{cases} \frac{1-\mathbb{E}\rho}{1+\mathbb{E}\rho} & \text{if } \mathbb{E}\rho < 1 \\ 0 & \text{if } \mathbb{E}\rho \geq 1 \text{ and } \mathbb{E}\rho^{-1} \geq 1 \\ -\frac{1-\mathbb{E}\rho^{-1}}{1+\mathbb{E}\rho^{-1}} & \text{if } \mathbb{E}\rho^{-1} < 1 \end{cases}$$

Proof. We only focus on the case where the RWRE is transient to the right or recurrent (that is $\mathbb{E}(\ln\rho) \leq 0$) as the other case is easily obtained by symmetry.

Define for all $n \geq 1$ $\tau_n = T_n - T_{n-1}$. As the RWRE is transient to the right or recurrent all τ_n $n \geq 1$ are P_0-a.s. finite.

Lemma 1. *Under* P_0*, the sequence* $(\tau_n)_{n\geq 1}$ *is stationary and ergodic.*

Proof. The fact that the sequence is stationary is a direct consequence of the Markov property and the translation invariance property of the environment : for all $k, m \geq 1$ and all $A_1, \cdots, A_k \subset \mathbb{Z}$

$$\begin{aligned} P_{0,\omega}(\tau_{m+1} \in A_1, \cdots, \tau_{m+k} \in A_k) &= P_{m,\omega}(\tau_{m+1} \in A_1, \cdots, \tau_{m+k} \in A_k) \\ &= P_{0,t_m\omega}(\tau_1 \in A_1, \cdots, \tau_k \in A_k) \end{aligned}$$

and its is enough to conclude as $t_m\omega$ as same law as ω under P. We now prove a stronger property than ergodicity (see Proposition 8) that is: the sequence $(\tau_n)_{n\geq 0}$ is strongly mixing. We thus have to prove that for all $k, l \geq 1$ and all $A_1, \cdots, A_k \subset \mathbb{Z}$, $B_1, \cdots, B_l \subset \mathbb{Z}$

$$\begin{aligned} \lim_{m\to+\infty} & P(\tau_1 \in A_1, \cdots, \tau_k \in A_k \ \cap \tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l) \\ &= P(\tau_1 \in A_1, \cdots, \tau_k \in A_k) P(\tau_1 \in B_1, \cdots \tau_l \in B_l). \end{aligned}$$

First assume that $B_1, \cdots, B_l$ are all bounded by some K. It implies that $\mathrm{P}_{m,\omega}(\tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l)$ is $\sigma(\omega_x, x \geq m-K)-$measurable while $\mathrm{P}_{0,\omega}(\tau_1 \in A_1, \cdots, \tau_k \in A_k)$ is $\sigma(\omega_x, x \leq k)-$measurable so for $m > K+k$, $\mathrm{P}_{0,\omega}(\tau_1 \in A_1, \cdots, \tau_k \in A_k)$ and $\mathrm{P}_{m,\omega}(\tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l)$ are independent. For such m

$$
\begin{aligned}
&\mathrm{P}(\tau_1 \in A_1, \cdots, \tau_k \in A_k \ \cap \tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l) \\
&\qquad = \int \mathrm{P}_{0,\omega}(\tau_1 \in A_1, \cdots, \tau_k \in A_k \ \cap \tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l) d\mathbb{P} \\
&\qquad \overset{Markov}{=} \int \mathrm{P}_{0,\omega}(\tau_1 \in A_1, \cdots, \tau_k \in A_k) \mathrm{P}_{m,\omega}(\tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l) d\mathbb{P} \\
&\qquad = \mathrm{P}(\tau_1 \in A_1, \cdots, \tau_k \in A_k) \mathrm{P}(\tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l) \\
&\qquad = \mathrm{P}(\tau_1 \in A_1, \cdots, \tau_k \in A_k) \mathrm{P}(\tau_1 \in B_1, \cdots \tau_l \in B_l)
\end{aligned}
$$

For the general case, define for all $K \geq 1$

$$
(B_1^K, \cdots B_l^K) = (B_1 \cap [0, K], \cdots, B_l \cap [0, K])
$$

and observe (Markov property + invariance of the environment under translation + monotone convergence theorem) that for all $\varepsilon > 0$ one can find K large enough so that for all $m \geq 0$

$$
\left| \mathrm{P}(\tau_{m+1} \in B_1^K, \cdots, \tau_{m+l} \in B_l^K) - \mathrm{P}(\tau_1 \in B_1, \cdots, \tau_l \in B_l) \right| < \varepsilon.
$$

or equivalently

$$
\left| \mathrm{P}(\tau_{m+1} \in B_1^K, \cdots, \tau_{m+l} \in B_l^K) - \mathrm{P}(\tau_{m+1} \in B_1, \cdots, \tau_{m+l} \in B_l) \right| < \varepsilon.
$$

This is enough to conclude that

$$
\begin{aligned}
&\mathrm{P}(\tau_1 \in A_1, \cdots, \tau_k \in A_k) \mathrm{P}(\tau_1 \in B_1, \cdots \tau_l \in B_l) - \varepsilon \\
&\qquad \leq \liminf \mathrm{P}(\tau_1 \in A_1, \cdots, \tau_k \in A_k \ \cap \tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l) \\
&\qquad \leq \limsup \mathrm{P}(\tau_1 \in A_1, \cdots, \tau_k \in A_k \ \cap \tau_{m+1} \in B_1, \cdots \tau_{m+l} \in B_l) \\
&\qquad \leq \mathrm{P}(\tau_1 \in A_1, \cdots, \tau_k \in A_k) \mathrm{P}(\tau_1 \in B_1, \cdots \tau_l \in B_l) - \varepsilon
\end{aligned}
$$

that gives the result as ε can be taken arbitrarily small. □

From Lemma 1, using the ergodic theorem one deduces that there exists some constant $c = \mathrm{E}_0(\tau_1) \in [1, +\infty]$ such that P_0-a.s.

$$
\frac{T_n}{n} \to c \qquad \text{as } n \to +\infty.
$$

It implies that P_0-a.s.

$$\frac{X_n}{n} \to \frac{1}{c} \qquad \text{as } n \to +\infty.$$

Indeed : define for $n \geq 1$, $X_n^* = \max_{0 \leq k \leq n} X_k$, then by definition for all $n \geq 1$

$$T_{X_n^*} \leq n < T_{X_n^*+1}.$$

As X_n^* goes to $+\infty$ with n going to $+\infty$ it implies that P_0-a.s. $X_n^*/n \to 1/c$ when n goes to infinity. Now observe that for all $n \geq 1$

$$X_n^* - (n - T_{X_n^*}) \leq X_n < X_n^* + 1$$

so that

$$\frac{X_n^*}{n} - \frac{n - T_{X_n^*}}{n} \leq \frac{X_n}{n} < \frac{X_n^* + 1}{n}$$

and the result follows easily once noticed that $\frac{T_{X_n^*}}{n} \to 1$ P_0-a.s. when n goes to infinity.

It remains to compute $\mathrm{E}_0(\tau_1)$. For that one can use a nice one to one correspondance that maps to each excursion from 0 to 1 of the walk a tree (picture on blackboard : glue the top face of the path and stick it !). We obtain that

$$\tau_1 = 2|\mathrm{Edges}_{\leq 0}| + 1 = 2|\mathrm{Nodes}_{\leq 0}| - 1 = 2\sum_{i \leq 0} W_i - 1$$

where W_i is the number of nodes at level i. Now each node at level x has a number of children at level $x-1$ that is independent from the tree cut at level x and follows a Geometric law starting at 0 with parameter ω_x. It implies that $W_0 = 1$ and for all $i \leq -1$

$$\mathrm{E}_{0,\omega}(W_i) = \mathrm{E}_{0,\omega}(\mathrm{E}_{0,\omega}(W_i|W_{i+1})) = \mathrm{E}_{0,\omega}(W_{i+1}\frac{1-\omega_{i+1}}{\omega_{i+1}})$$

and by iteration we obtain

$$\mathrm{E}_{0,\omega}(W_i) = \rho_{i+1} \cdots \rho_0$$

and

$$\begin{aligned} \mathrm{E}_{0,\omega}(\tau_1) &= 2\left(\sum_{i \leq -1} \rho_{i+1} \cdots \rho_0 + 1\right) - 1 \\ &= 2\left(\sum_{i \leq 0} \rho_i \cdots \rho_0 + 1\right) - 1 \end{aligned} \tag{8}$$

As the environment is i.i.d. we obtain

$$\mathrm{E}(\tau_1) = 2\sum_{i\leq 0} \mathbb{E}(\rho)^{|i|} - 1 = \begin{cases} \frac{1+\mathbb{E}(\rho)}{1-\mathbb{E}(\rho)} \text{if } \mathbb{E}(\rho) < 1 \\ +\infty \quad\quad \text{if } \mathbb{E}(\rho) \geq 1 \end{cases}$$

□

Remark 2. 1. *The asymptotic velocity* ***is not*** *the expectation of the drift* $\mathbb{E}(2\omega_0 - 1)$ *! It is not surprising and we will see later a way to correct this wrong intuition.*

2. *There exists a regime where the RWRE is* ***transient with null velocity****. In this case one can even have the annealed expectation of the drift that is oriented in the other direction that the transience direction. It is the case for example when*

$$\mu = \frac{9}{10}\delta_{1/4} + \frac{1}{10}\delta_{1-\varepsilon}$$

and $\varepsilon > 0$ *is chosen small enough.*

3. *When the walk is transient and* ***ballistic (that is*** $v \neq 0$***) the annealed expectation of the drift is necessarily oriented in the same direction than the transience direction*** *as from Jensen inequality* $\mathbb{E}(1/\rho) \geq 1/\mathbb{E}(\rho)$ *and thus* $\mathbb{E}(\rho) < 1$ *implies* $\mathbb{E}(2\omega - 1) > 0$.

4. *Jensen's inequality assures that case* 1 *and* 3 *in the theorem do not occur simultaneously.*

5. *In the ballistic case, one can prove a refinement of this result by computing the second order term.*

2.4 Sinai regime

Sinai [11] provides a description of the RWRE in the recurrent case

Theorem 7 (Sinaï 1982)**.** *Assume* $\mathbb{E}(\ln \rho) = 0$ *and (UE). Then* $(X_n)_{n\geq 0}$ *is recurrent and*

$$(X_n/\ln^2 n)_{n\geq 0} \text{ converges weakly under } \mathrm{P}_0.$$

Before entering the proof we need three lemma describing the behaviour of the walker in a valley of the potential (see below for a precise definition).

Lemma 2 (Exit side of a valley and lower bound on the exit time)**.** *Let $a < b < c$ be real numbers. Define $T = T_a \wedge T_c$ and $N = \sum_{i=0}^{T} 1_{X_i=b}$. Then*

$$\mathrm{P}_{b,\omega}(T_a < T_c) \leq |c-b| \frac{e^{\sup_{[b+1,c]} V}}{e^{\sup_{[a+1,c]} V}}$$

and

$$T \geq N$$

where N has a geometric law with parameter

$$\max(e^{-\sup_{[a+1,b]} V_i - V_b} ; e^{-\sup_{[b+1,c]} V_i - V_{b+1}}).$$

Proof. Using Doob's stopping theorem for the martingale $(f(X_n))_{n\geq 0}$ and the stopping time $T = T_a \wedge T_c$ we obtain

$$\mathrm{P}_{b,\omega}(T_a < T_c) f(a) + \mathrm{P}_{b,\omega}(T_c < T_a) f(c) = f(b)$$

so that

$$\mathrm{P}_{b,\omega}(T_a < T_c) = \frac{f(c) - f(b)}{f(c) - f(a)} = \frac{e^{V_{b+1}} + \cdots + e^{V_c}}{e^{V_{a+1}} + \cdots + e^{V_c}}$$

and the first result follows easily. Moreover applying this last estimates to $b-1, b, c$ and $a, b, b+1$ we obtain

$$\mathrm{P}_{b-1,\omega}(T_a < T_b) = \frac{e^{V_b}}{e^{V_{a+1}} + \cdots + e^{V_b}}$$
$$\mathrm{P}_{b+1,\omega}(T_c < T_b) = \frac{e^{V_{b+1}}}{e^{V_{b+1}} + \cdots + e^{V_c}}$$

We obtain the second part of the result as a consequence of strong Markov's property using only that T is larger than the number of visits to b before exiting $[a, c]$. □

Lemma 3 (Upper bound on mean hitting time)**.** *Let $a < b < c$ be real numbers. Given ω we define the environment ω^a that coincides with ω everywhere except at a where $\omega_a = 1$ (that is the walker is reflected at a). Then*

$$\mathrm{E}_{b,\omega^a}(T_c) \leq 2|c-b||c-a| e^{[V]_{a,c}}$$

where $[V]_{a,c} = \max_{a<i<j\leq c} V_j - V_i$

Proof. From strong Markov property

$$\mathrm{E}_{b,\omega^a}(T_c) = \mathrm{E}_{b,\omega^a}(T_{b+1}) + \mathrm{E}_{b+1,\omega^a}(T_{b+2}) + \cdots + \mathrm{E}_{c-1,\omega^a}(T_c).$$

Now as $\rho_a = 0$, using (8), we obtain that for all $j = b, \cdots, c-1$

$$\begin{aligned}
\mathrm{E}_{j,\omega^a}(\tau_{j+1}) &= 2\left(\sum_{i\le j}\rho_i\cdots\rho_j + 1\right) - 1\\
&= 2\left(\sum_{a+1\le i\le j}\rho_i\cdots\rho_j + 1\right) - 1\\
&= 2\left(\sum_{a+1\le i\le j+1}\exp(V_{j+1}-V_i)\right) - 1.
\end{aligned}$$

Finally

$$\begin{aligned}
\mathrm{E}_{b,\omega^a}(T_c) &\le 2\sum_{\substack{j=b,\cdots,c-1\\ a+1\le i\le j+1}}\exp(V_{j+1}-V_i)\\
&\le 2|c-b||c-a|e^{[V]_{a,c}}
\end{aligned}$$

□

Lemma 4 (staying at the bottom of a valley)**.** *Let $a < b < c$ be real numbers such that b is a minimum of V on the segment $[a,c]$. Given ω we define the environment $\omega^{a,c}$ that coincides with ω everywhere except at a where $\omega_a = 1$ and at c where $\omega_c = 0$ (that is the walker is reflected both at a and c). Then for all time $n \ge 0$ and all $x \in [a,c]$*

$$\mathrm{P}_{b,\omega^{a,c}}(X_n = x) \le \frac{\omega_b}{\omega_x}\frac{e^{V_{b+1}}}{e^{V_{x+1}}}.$$

Proof. One can check that

$$\pi(x) = \frac{\omega_b}{\omega_x}\frac{e^{V_{b+1}}}{e^{V_{x+1}}}, \quad x \in [a,c]$$

defines an invariant measure (do it) and that $\pi \ge \delta_b$. It implies that $\pi \ge \mathrm{P}_{b,\omega^{a,c}}(X_n \in \cdot)$. □

Proof of Theorem 7. Intuition: in the recurrent case ($\mathbb{E}\ln\rho = 0$) the potential is a random walk with mean zero increments. Before time n, the walker looks for a valley of the potential of depth at least $\ln n$ where it can stay a time $\exp(\ln n) = n$. It finds such valley at a distance $\ln^2 n$ as the potential is diffusive. We can make precise the law of the bottom of this valley after correct renormalisation using Donsker's theorem : it is the bottom of the smallest valley of B that surrounds 0 and has depth at least 1.

We start the proof by defining for all $n \geq 1$, the renormalised potential

$$V_t^{(n)} = \frac{V_{\lfloor \ln^2 nt \rfloor}}{\ln n}, \qquad t \in \mathbb{R}.$$

As (UE) implies $\sigma^2 = \mathbb{E}\left((\ln \rho)^2\right) < +\infty$, Donsker's theorem gives that V^n converges weakly for the uniform topology (on compact sets) to the two sided Brownian motion $(\sigma B_t)_{t \in \mathbb{R}}$.
For every scale n one want to define a location $\bar{b}_n$ that corresponds to the theoretical area where the walker lies in the the potential after n steps. For this purpose we need the intuitive notion of valley.

We call a triple (a, b, c) with $a < b < c$ a valley if

$$V_b^{(n)} = \min_{a \leq t \leq c} V_t^{(n)},$$

$$V_a^{(n)} = \max_{a \leq t \leq b} V_t^{(n)},$$

$$V_c^{(n)} = \max_{b \leq t \leq c} V_t^{(n)}.$$

The depth of the valley is defined as

$$d(a, b, c) = \min(V_a^{(n)} - V_b^{(n)}, V_c^{(n)} - V_b^{(n)}).$$

If (a, b, c) is a valley, and $a < d < e < b$ are such that

$$V_e^{(n)} - V_d^{(n)} = \max_{a \leq x < y \leq b} V_y^{(n)} - V_x^{(n)},$$

then (a, d, e) and (e, b, c) are again valleys, which are obtained from (a, b, c) by a left refinement. One defines similarly a right refinement. Define

$$c_n^0 = \min\{t \geq 0 : V_t^{(n)} \geq 1\},$$

$$a_n^0 = \max\{t \leq 0 : V_t^{(n)} \geq 1\},$$

$$V_{b_n^0}^{(n)} = \min_{a_n^0 \leq t \leq c_n^0} V_t^{(n)}.$$

We now apply a sequence of refinements till we find the smallest valley $(\bar{a}_n, \bar{b}_n, \bar{c}_n)$ with $\bar{a}_n < 0 < \bar{c}_n$ and $d(\bar{a}_n, \bar{b}_n, \bar{c}_n) > 1$. Finally for technical reasons we also have to define in the same way for all $\delta > 0$, $(\bar{a}_n^\delta, \bar{b}_n^\delta, \bar{c}_n^\delta)$ except we are looking this time for a valley of depth $d(\bar{a}_n, \bar{b}_n, \bar{c}_n) > 1 + \delta$. The bar in the notation refers to positions in the renormalised potential while we use same notations without it to denote the positions at the original scale that is :

$$a_n = \bar{a}_n \ln^2 n \ ; b_n = \bar{b}_n \ln^2 n \ ; c_n = \bar{c}_n \ln^2 n$$

and the same obvious notations for a_n^δ, b_n^δ and c_n^δ.

As a consequence of the weak convergence of V^n to B one obtains the weak convergence of $\bar{b}_n$ to the bottom of the smallest valley of B that surrounds 0 and has depth at least 1. In order to prove the result it is sufficient to prove that for all $\eta > 0$

$$\mathrm{P}_0\left(\left|\frac{X_n}{\ln^2 n} - \bar{b}_n\right| > \eta\right) \to 0 \quad as\ n \to +\infty.$$

To control the above probability we use a standard strategy dealing with random environments : first define a sequence of sets $G_n \subset \Omega$ of *good* environments with enough conditions such that the walker in any environment $\omega \in G_n$ is close to $b_n = \bar{b}_n \ln^2 n$ with large probability uniformly with respect to ω: for all $\delta, J > 0$

$$G_n^{\delta,J} = \Big\{$$

1. $\bar{b}_n = \bar{b}_n^\delta$
2. any refinement (a, b, c) of $(\bar{a}_n^\delta, \bar{b}_n^\delta, \bar{c}_n^\delta)$ satisfies $d(\bar{a}_n^\delta, \bar{b}_n^\delta, \bar{c}_n^\delta) < 1 - \delta$
3. $\bar{c}_n^\delta - \bar{a}_n^\delta < J$
4. $\min\{V_i^n - V_{\bar{b}_n}^n,\ t \in [\bar{a}_n, \bar{c}_n] \setminus [\bar{b}_n - \delta, \bar{b}_n + \delta] > \delta^3\}$

$$\Big\}$$

[Comment these 4 conditions. Picture to summarise]

Using Donsker's theorem and basic properties (COMPLETE : description of brownian motion around local minimum) of the brownian motion one can prove that for all $\delta, J > 0$ $\lim_{n\to+\infty} \mathbb{P}(G_n^{\delta,J})$ exists and

$$\lim_{J\to+\infty,\delta\to 0} \lim_{n\to+\infty} \mathbb{P}(G_n^{\delta,J}) = 1.$$

Given $\eta, \varepsilon > 0$, we fix $J > 0$ large enough, $\delta \leq \eta/2$ small enough and n large enough so that $\mathbb{P}((G_n^{\delta,J})^c) < \varepsilon$. From now on we omit exponents δ and J in the notations for the event G_n. We obtain

$$\begin{aligned}\mathrm{P}_0\left(\left|\frac{X_n}{\ln^2 n} - \bar{b}_n\right| > \eta\right) &\leq \mathbb{P}(G_n^c) + \mathbb{E}\left(1_{G_n}\mathrm{P}_{0,\omega}\left(\left|\frac{X_n}{\ln^2 n} - \bar{b}_n\right| > \eta\right)\right)\\ &\leq \varepsilon + \sup_{\omega\in G_n} \mathrm{P}_{0,\omega}\left(\left|\frac{X_n}{\ln^2 n} - \bar{b}_n\right| > \eta\right)\end{aligned}$$

and we are left with proving that

$$\sup_{\omega\in G_n} \mathrm{P}_{0,\omega}\left(\left|\frac{X_n}{\ln^2 n} - \bar{b}_n\right| > \eta\right) \to 0$$

when n goes to infinity. In the following we assume $b_n \geq 0$ as both cases are similar.

Step 1. We first prove that

$$\sup_{\omega\in G_n} \mathrm{P}_{0,\omega}\left(T_{b_n} > n\right) \to 0$$

Given $\omega \in G_n$, using Lemma 2 with $(a,b,c) = (a_n^\delta, 0, b_n)$ and $1., 3.$, we obtain

$$\mathrm{P}_{0,\omega^{a_n^\delta}}\left(T_{a_n^\delta} < T_{b_n}\right)| \overset{1.\&3.}{\leq} \frac{J\ln^2 n}{n^\delta} \to 0.$$

Using Lemma 3 again with $(a,b,c) = (a_n^\delta, 0, b_n)$ and $1., 2., 3.$, we obtain

$$\mathrm{E}_{0,\omega^{a_n^\delta}}\left(T_{b_n}\right) \leq 2J^2(\ln^4 n)n^{1-\delta}.$$

and using Markov's inequality

$$\mathrm{P}_{0,\omega^{a_n^\delta}}\left(T_{b_n} > n\right) \leq 2J^2\frac{\ln^4 n}{n^\delta} \to 0.$$

Finally

$$\sup_{\omega\in G_n} \mathrm{P}_{0,\omega^{a_n^\delta}}\left(T_{b_n} \leq n, T_{b_n} \leq T_{a_n^\delta}\right) \to 1$$

and as

$$\mathrm{P}_{0,\omega^{a_n^\delta}}\left(T_{b_n} \leq n, T_{b_n} \leq T_{a_n^\delta}\right) = \mathrm{P}_{0,\omega}\left(T_{b_n} \leq n, T_{b_n} \leq T_{a_n^\delta}\right)$$

we obtain

$$\sup_{\omega\in G_n} \mathrm{P}_{0,\omega}\left(T_{b_n} \leq n\right) \to 1$$

Step 2. We now prove

$$\sup_{\omega\in G_n} \sup_{0\leq k\leq n} \mathrm{P}_{b_n,\omega}\left(\left|\frac{X_k}{\ln^2 n} - \bar{b}_n\right| > \eta\right) \to 0$$

Fix $\omega \in G_n$ and $0 \leq k \leq n$. Define T as the hitting time of $\{a_n^\delta, c_n^\delta\}$. Thus

$$\mathrm{P}_{b_n,\omega}\left(\left|\frac{X_k}{\ln^2 n} - \bar{b}_n\right| > \eta\right) \leq \mathrm{P}_{b_n,\omega}\left(T \leq n\right) + \mathrm{P}_{b_n,\omega^{a_n^\delta,c_n^\delta}}\left(\left|\frac{X_k}{\ln^2 n} - \bar{b}_n\right| > \eta\right)$$

For the first term we use the second part of Lemma 2 with $(a,b,c) = (a_n^\delta, b_n, c_n^\delta)$ to obtain

$$\mathrm{P}_{b_n,\omega}(T \le n) \le 1 - \left(1 - \frac{1}{n^{1+\delta}}\right)^n \le \frac{1}{n^\delta} \to 0.$$

For the second one we use Lemma 4 with $(a,b,c) = (a_n^\delta, b_n, c_n^\delta)$ (here it is usefull that δ has been chosen smaller that $\eta/2$) and point $1., 3., 4.$, to obtain that (we assume (UE) with $\varepsilon_0 > 0$)

$$\mathrm{P}_{b_n,\omega^{a_n^\delta,c_n^\delta}}\left(|X_k - b_n| > \eta \ln^2 n\right) \le \frac{1-\varepsilon_0}{\varepsilon_0} J \frac{\ln^2 n}{n^{\delta^3}}.$$

Step 3. Finally for all $\omega \in G_n$ using Markov property,

$$\begin{aligned}\mathrm{P}_{0,\omega}\left(\left|\frac{X_n}{\ln^2 n} - \bar{b}_n\right| > \eta\right) &\le \mathrm{P}_{0,\omega}(T_{b_n} > n) + \sum_{k\le n} \mathrm{P}_{0,\omega}(T_{b_n} = n-k)\, \mathrm{P}_{b_n,\omega}\left(\left|\frac{X_k}{\ln^2 n} - \bar{b}_n\right| > \eta\right)\\ &\le \mathrm{P}_{0,\omega}(T_{b_n} > n) + \sup_{k\le n} \mathrm{P}_{b_n,\omega}\left(\left|\frac{X_k}{\ln^2 n} - \bar{b}_n\right| > \eta\right)\end{aligned}$$

and the proof is complete.

□

2.5 Kesten-Kozlov-Spitzer regime

Theorem 8 (Kesten-Kozlov-Spitzer 1975). *Assume (UE) and* $\mathbb{E}(\ln \rho) < 0$ *and* $\mathbb{E}(\rho) \ge 1$ *(and some other technical assumptions). There exists a unique positive real number* $\kappa > 0$ *such that* $\mathbb{E}(\rho^\kappa) = 1$ *and*

$$(X_n/n^\kappa)_{n\ge 0} \text{ converges weakly under } \mathrm{P}_0.$$

We will not prove (this year !) this theorem. Intuition. Two ways to prove this result : inhomogeneous branching process (the original proof [7]) & using once again the potential (Enriquez Sabot Zindy [3])

3 Multidimensional RWRE

3.1 Kalikow auxiliary Markov chain

A major problem with the annealed law is that is not markovian. Kalikow introduced a familiy of auxiliary Markov chains that exhibit same exit distribution of connexe sets as the walker under P_0.

More precisely, for any connexe set $U \subsetneq \mathbb{Z}^d$ such that $0 \in U$ we define

$$\partial U = \{z \in \mathbb{Z}^d : \text{there exists } x \in U,\ |x - z| = 1\}$$

the boundary of U and T_U the hitting time of this boundary. We define a Markov kernel on $U \cup \partial U$ by: for all $x \in U$, $|e| = 1$

$$\hat{\mathrm{P}}_U(x, x+e) = \frac{\mathrm{E}_0\left(\sum_{k=0}^{T_U} 1_{X_k = x, X_{k+1} = x+e}\right)}{\mathrm{E}_0\left(\sum_{k=0}^{T_U} 1_{X_k = x}\right)}$$

and for all $x \in \partial U$

$$\hat{\mathrm{P}}_U(x, x) = 1.$$

Remark 3. *Check that transition are well defined (that is all expectations are finite) and positive*

We use $\hat{\mathrm{P}}_{x,U}$ to denote the law of the canonical Makov chain starting at $x \in U \cup \partial U$ with transition $\hat{\mathrm{P}}_U$

The main interest of this chain is the following result

Theorem 9 (Kalikow 81)**.** *If* $\hat{\mathrm{P}}_{0,U}(T_U < +\infty) = 1$ *then* $\mathrm{P}_0(T_U < +\infty) = 1$ *and moreover* X_{T_U} *has same law under* $\hat{\mathrm{P}}_{0,U}$ *and* P_0*.*

Proof. Define for all $x \in U \cup \partial U$

$$\hat{G}^U(x) = \hat{\mathrm{E}}_{0,U}\left(\sum_{k=0}^{T_U} 1_{X_k = x}\right)$$

that is (except at the boundary) the green function associated to $\hat{\mathrm{P}}_{0,U}$. The key of the proof (and actually an interesting result by itself) is to prove that for all $x \in U \cup \partial U$

$$\hat{G}^U(x) = \mathrm{E}_0\left(\sum_{k=0}^{T_U} 1_{X_k = x}\right) = \mathbb{E}\mathrm{E}_{0,\omega}\left(\sum_{k=0}^{T_U} 1_{X_k = x}\right). \tag{9}$$

We admit this for the moment. It implies that if $x \in \partial U$

$$\hat{\mathrm{P}}_{0,U}(T_U < +\infty, X_{T_U} = x) = \hat{G}^U(x) = \mathrm{P}_0(T_U < +\infty, X_{T_U} = x).$$

Summing over all $x \in \partial U$ we obtain $\mathrm{P}_0(T_U < +\infty) = \hat{\mathrm{P}}_{0,U}(T_U < +\infty) = 1$ and then also that X_{T_U} has same law under $\hat{\mathrm{P}}_{0,U}$ and P_0.

It remains to prove (9). First note that $\hat{G}^U(\cdot)$ and $\Psi(\cdot) = \mathrm{E}_0\left(\sum_{k=0}^{T_U} 1_{X_k=\cdot}\right)$ solve the equation

$$f(x) = \delta_{0,x} + \sum_{y\in U} \hat{\mathrm{P}}_U(y,x)f(y) \quad x \in U \cup \partial U \tag{10}$$

To prove that first note that from Markov property for all ω and x, y,

$$\mathrm{E}_{0,\omega}\left(\sum_{k=0}^{T_U} 1_{X_k=y,X_{k+1}=x}\right) = \mathrm{E}_{0,\omega}\left(\sum_{k=0}^{T_U} 1_{X_k=y}\right)\omega_y(x) \tag{11}$$

and it implies

$$\mathrm{E}_{0,\omega}\left(\sum_{k=0}^{T_U} 1_{X_k=x}\right) = \delta_{0,x} + \sum_{y\in U} \mathrm{E}_{0,\omega}\left(\sum_{k=0}^{T_U} 1_{X_k=y}\right)\omega_y(x).$$

The fact that $\hat{G}^U$ solve (10) is just the same last equation with kernel $\hat{\mathrm{P}}_U$ instead of ω. For Ψ, we continue the computation by taking the expectation with respect to the environment to obtain

$$\begin{aligned}
\mathrm{E}_0\left(\sum_{k=0}^{T_U} 1_{X_k=x}\right) &= \delta_{0,x} + \sum_{y\in U} \mathbb{E}\left[\mathrm{E}_{0,\omega}\left(\sum_{k=0}^{T_U} 1_{X_k=y}\right)\omega_y(x)\right] \\
&\overset{(11)}{=} \delta_{0,x} + \sum_{y\in U} \mathbb{E}\left[\mathrm{E}_{0,\omega}\left(\sum_{k=0}^{T_U} 1_{X_k=y,X_{k+1}=x}\right)\right] \\
&= \delta_{0,x} + \sum_{y\in U} \hat{\mathrm{P}}_U(y,x)\mathrm{E}_0\left(\sum_{k=0}^{T_U} 1_{X_k=y}\right).
\end{aligned}$$

Now $\hat{G}^U(\cdot)$ is the minimal non-negative solution of (10) : to see that point introduce for all $n \geq 0$ and for all $x \in U \cup \partial U$

$$\hat{G}^U_n(x) = \hat{\mathrm{E}}_{0,U}\left(\sum_{k=0}^{T_U\wedge n} 1_{X_k=x}\right).$$

Using a computation similar to the last one one can prove that for all $n \geq 0$

$$\hat{G}^U_{n+1}(x) = \delta_{0,x} + \sum_{y\in U} \hat{\mathrm{P}}_U(y,x)\hat{G}^U_n(x) \quad x \in U \cup \partial U \tag{12}$$

Consider now a solution f of (10). Clearly for $n = 0$

$$\hat{G}^U_0 = \delta_{0,x} \leq f$$

and using (12) and the fact that f solves (10), by induction, for all $n \geq 0$

$$\hat{G}^U_n \leq f.$$

Monotonous convergence theorem assures that $\hat{G}^U_n \nearrow \hat{G}^U$ pointwise and we obtain the result and in particular

$$G^U \leq \Psi. \tag{13}$$

Next, $G^U = \Psi$ on ∂U as for all $x \in \partial U$

$$\hat{G}^U(x) = \hat{\mathrm{P}}_{0,U}(T_U < +\infty, X_{T_U} = x)$$

so that

$$\sum_{x \in \partial U} \hat{G}^U(x) = \hat{\mathrm{P}}_{0,U}(T_U < +\infty) = 1$$

and

$$\mathrm{P}_0(T_U < +\infty) = \sum_{x \in \partial U} \Psi(x) \geq \sum_{x \in \partial U} \hat{G}^U(x) = 1.$$

It implies that the inequality in the last equation is actually an equality and due to (13) for all $x \in \partial U$

$$\hat{G}^U(x) = \Psi(x).$$

As a consequence $h = \hat{G}^U - \Psi$ is non negative function such that

$$h(x) = \sum_{y \in U} \hat{\mathrm{P}}_U(y, x) h(y) \quad x \in U \cup \partial U \tag{14}$$

and $h = 0$ on ∂U. Assume by contradiction that $h(x) > 0$ for some $x \in U$. As x is connected to any $y \in \partial U$, (14) implies that $h(y) > 0$ too. Prove that $\hat{\mathrm{P}}_U$ is strictly larger than 0 ! □

From the proof above on easily deduce

Corollary 1. *Under same assumptions as in Theorem 9*

$$\hat{\mathrm{E}}^U_0(T_U) = \mathrm{E}_0(T_U)$$

3.2 Directional transience and $0-1$ law

See [1] and [6] for this part. **In this section again we assume that (UE) holds for some $\varepsilon_0 > 0$ and that the environment is i.i.d.** There is unfortunately no criterium for characterising transience / recurrence in dimension $d \geq 2$. There are however some results relative to directional transience. We say that the walker is transient in the direction $\ell \in \mathbb{R}^d$ if $X_n \cdot \ell \to +\infty$ when n goes to $+\infty$ and denote by A_ℓ the corresponding event

$$A_\ell = \{X_n \cdot \ell \to +\infty\}.$$

Theorem 10 (Kalikow [6]). *For all $\ell \in \mathbb{R}^d \setminus \{0\}$ the event $A_\ell \cup A_{-\ell}$ satisfies a $0-1$ law that is*

$$\mathrm{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}.$$

Proof.

Lemma 5. *Assume (UE) and $\ell \neq 0$. Then*

$$\mathrm{P}_0 - a.s. \quad \liminf X_n \cdot \ell \in \{\pm\infty\}$$

and the same holds for $\limsup X_n \cdot \ell$.
As a consequence $\mathrm{P}_0 - a.s.$ $(X_n)_{n\geq 1}$ belongs to (only) one of these three events : A_ℓ, $A_{-\ell}$ or $\{\liminf X_n \cdot \ell = -\infty, \limsup X_n \cdot \ell = +\infty\}$.

Proof. We have to prove that this result holds $\mathrm{P}_{0,\omega}$-a.s. for $\mathbb{P}$-almost all ω. For a given $u \in \mathbb{R}$, we define the following stopping times

$$\tilde{T}_u^{(0)} = \tilde{T}_u = \inf\{n \geq 0,\ X_n \cdot \ell < u\}$$

and by iteration for all $k \geq 1$

$$\tilde{T}_u^{(k)} := \tilde{T}_u \circ \theta_{T_u^{(k-1)}+1}$$

as on the event $\{\liminf X_N \cdot \ell < u\}$, all $\tilde{T}_u^{(k)}$ are finite. Consider now e such that $\alpha = \ell \cdot e > 0$, it follows from Markov property and the (UE) assumption [ADD DETAILS, see [12]] that $\mathrm{P}_{0,\omega}$-a.s.

$$\{\liminf X_N \cdot \ell < u\} \subset \{\liminf X_N \cdot \ell < u - \alpha\}.$$

Finally iterating this argument we obtain that P_ω-a.s.

$$\{\liminf X_N \cdot \ell < u\} \subset \{\liminf X_N \cdot \ell = -\infty\}.$$

□

Introduce the first time the walker goes behind its initial level in direction ℓ

$$D = \inf\{n \geq 0,\ X_n \cdot \ell < X_0 \cdot \ell\}$$

still with the convention $\inf \emptyset = +\infty$.

Proposition 9. *Assume (UE) (even if (E) would be sufficient) and $\mathrm{P}_0(D = +\infty) = 0$. Then $\mathrm{P}_0 - a.s.$*

$$\liminf X_n \cdot \ell = -\infty.$$

Proof. Using the fact that the environment is invariant by translation we obtain from the assumption that for all $x \in \mathbb{Z}^d$, $\mathrm{P}_x(D < +\infty) = 1$ and thus $\mathbb{P}$-a.s. $\mathrm{P}_{x,\omega}(D < +\infty) = 1$. As $\mathbb{Z}^d$ is countable we obtain that $\mathbb{P}-$a.s. for all $x \in \mathbb{Z}^d$, $\mathrm{P}_{x,\omega}(D < +\infty) = 1$.

As a consequence $\mathbb{P}$-a.s., using the strong Markov property, the successive hitting times of the half-space $\{x \cdot \ell < 0\}$ are all finite $\mathrm{P}_{0,\omega}-$a.s., that is $\mathrm{P}_{0,\omega}(\liminf X_n \cdot \ell \leq 0) = 1$ and the claim follows easily from Lemma 5. $\square$

Define for all $u \in \mathbb{R}$ the hitting time

$$T_u = \inf\{n \geq 0,\ X_n \cdot \ell > u\}.$$

and the sequence of random variables

$$S_0 = 0 \quad M_0 = X_0 \cdot \ell$$

then

$$S_1 = T_{M_0} \text{ and } R_1 = D \circ \theta_{S_1} + S_1 \text{ and } M_1 = \max\{X_n \cdot \ell,\ 0 \leq n \leq R_1\}$$

and by induction for all $k \geq 1$

$$S_{k+1} = T_{M_k} \text{ and } R_{k+1} = D \circ \theta_{S_{k+1}} + S_{k+1} \text{ and } M_{k+1} = \max\{X_n \cdot \ell,\ 0 \leq n \leq R_{k+1}\}$$

One can check that all S_k and R_k, $k \geq 1$ are stopping times and that

$$0 = S_0 \leq R_0 \leq S_1 \leq R_1 \leq S_2 \leq \cdots S_k \leq R_k \leq +\infty$$

and that any of these inequalities is strict if the left member is finite. Finally define

$$K = \inf\{k \geq 1,\ S_k < +\infty, R_k = +\infty\}.$$

and the first renewal time (this terminology will be clear later)

$$\tau_1 = S_K.$$

One could alternatively and in a simpler way define τ_1 as

$$\tau_1 = \inf\{n \geq 0, X_k \cdot \ell < X_n \cdot \ell \text{ for all } k < n \text{ and } X_k \cdot \ell \geq X_n \cdot \ell \text{ for all } k > n\}.$$

However the first definition is more convenient to prove the following proposition.

Proposition 10. *Assume* $\mathrm{P}_0(D = +\infty) > 0$*, then* P_0-*a.s.*

$$\{\limsup X_n \cdot \ell = +\infty\} = \{K < +\infty\} = A_\ell = \{\tau_1 < +\infty\}.$$

Moreover all these events have positive probability under P_0.

Proof. Once proven the two first equalities, the last one is trivial.

We first prove $\subset$ of the first equality. For $k \geq 1$ using Markov property

$$\mathrm{P}_0(R_k < +\infty) = \sum_{x \in \mathbb{Z}^d} \mathbb{E}\Big(\mathrm{E}_{0,\omega}(S_k < +\infty,\ X_{S_k} = x)\mathrm{P}_{x,\omega}(D < +\infty)\Big).$$

The point is now that for a given $x \in \mathbb{Z}^d$, $\mathrm{E}_{0,\omega}(S_k < +\infty,\ X_{S_k} = x)$ is $\sigma(\omega_y(\cdot), y \text{ s.t. } y\cdot\ell < x\cdot\ell)$-measurable while $\mathrm{P}_{x,\omega}(D < +\infty)$ is $\sigma(\omega_y(\cdot), y \text{ s.t. } y\cdot\ell \geq x \cdot \ell)$-measurable. As these two $\sigma-$algebra are independent we obtain

$$\begin{aligned}
\mathrm{P}_0(R_k < +\infty) &= \sum_{x \in \mathbb{Z}^d} \mathrm{E}_0(S_k < +\infty,\ X_{S_k} = x)\mathrm{P}_x(D < +\infty) \\
&= \mathrm{P}_0(D < +\infty)\mathrm{P}_0(S_k < +\infty) \\
&\leq \mathrm{P}_0(D < +\infty)\mathrm{P}_0(R_{k-1} < +\infty) \\
&\leq \mathrm{P}_0(D < +\infty)^k.
\end{aligned}$$

where we used to go from the first to the second line that the environment is invariant by translation ; to go from the second to the third that $\{S_k < +\infty\} \subset \{R_{k-1} < +\infty\}$; and a basic iteration to go from the second to the last line. We thus deduce from Borel Cantelli's lemma that P_0-a.s.

$$\inf\{k \geq 0, R_k = +\infty\} < +\infty$$

As on $\{\limsup X_n \cdot \ell = +\infty\}$, $\{R_k < +\infty\} \subset \{S_{k+1} < +\infty\}$, we obtain that

$$\mathrm{P}_0 - a.s. \text{ on } \{\limsup X_n \cdot \ell = +\infty\},\ K < +\infty.$$

We now prove $\subset$ of the second inequality. Note that on $\{K < +\infty\}$, $\{\liminf X_n \cdot \ell > -\infty\}$ and using Lemma 5 it implies that P_0-a.s.

$$\{K < +\infty\} \subset \{\liminf X_n \cdot \ell = +\infty\} = A_\ell.$$

Finally just notice that from Lemma 5, $\{D = +\infty\} \subset \{\limsup X_n \cdot \ell = +\infty\}$ and thus $\mathrm{P}_0(\limsup X_n \cdot \ell = +\infty) > 0$. □

We come back now to the proof of the $0-1$ law (Theorem 10). For that we introduce the random variable $\tilde{D}$, $\tilde{K}$ defined as D and K but with $-\ell$ instead of ℓ.

Case 1: $\mathrm{P}_{\mathbf{0}}(\mathbf{D} = +\infty) > \mathbf{0}$ and $\mathrm{P}_{\mathbf{0}}(\tilde{\mathbf{D}} = +\infty) = \mathbf{0}$.
From Proposition 9, P_0-a.s. $\limsup X_n \cdot \ell = +\infty$. Now from Proposition 10, $K < +\infty$ P_0-a.s. and

$$\mathrm{P}_0(A_\ell) = 1.$$

Case 2: $\mathrm{P}_{\mathbf{0}}(\mathbf{D} = +\infty) = \mathbf{0}$ and $\mathrm{P}_{\mathbf{0}}(\tilde{\mathbf{D}} = +\infty) > \mathbf{0}$.
Using the same arguments

$$\mathrm{P}_0(A_{-\ell}) = 1.$$

Case 3: $\mathrm{P}_{\mathbf{0}}(\mathbf{D} = +\infty) = \mathbf{0}$ and $\mathrm{P}_{\mathbf{0}}(\tilde{\mathbf{D}} = +\infty) = \mathbf{0}$.
From Proposition 9, P_0-a.s. $\limsup X_n \cdot \ell = +\infty$ and $\liminf X_n \cdot \ell = -\infty$. As a consequence, in the direction ℓ the walker oscillates between $-\infty$ and $+\infty$ and $\mathrm{P}_0(A_\ell) = \mathrm{P}_0(A_{-\ell}) = 0$ thus

$$\mathrm{P}_0(A_\ell \cup A_{-\ell}) = 0.$$

Case 4: $\mathrm{P}_{\mathbf{0}}(\mathbf{D} = +\infty) > \mathbf{0}$ and $\mathrm{P}_{\mathbf{0}}(\tilde{\mathbf{D}} = +\infty) > \mathbf{0}$.
Using Lemma 5 we only have to prove that $\mathrm{P}_0(\liminf X_n \cdot \ell = -\infty, \limsup X_n \cdot \ell = +\infty) = 0$. As $\mathrm{P}_0(\tilde{D} = +\infty) > 0$ it is a direct consequence of Proposition 10 that states that P_0-a.s. $\{\limsup X_n \cdot \ell = +\infty\} = A_\ell = \{\liminf X_n \cdot \ell = +\infty\}$. We have thus proven that

$$\mathrm{P}_0(A_\ell \cup A_{-\ell}) = 1.$$

Actually we can say more about this case. On $\{D = +\infty\}$, $\liminf X_n \cdot \ell \geq 0$ and thus from Lemma 5, P_0-a.s. $\liminf X_n \cdot \ell = +\infty$. This implies that $\mathrm{P}_0(A_\ell) > 0$. and the same holds of course for $\mathrm{P}_0(A_{-\ell})$. □

One can wonder after this theorem if we can improve this result to a stonger $0-1$ law that would be

For all $\ell \in \mathbb{R}^d \setminus \{0\}$ the event A_ℓ satisfies a $0-1$ law that is

$$\mathrm{P}(A_\ell) \in \{0, 1\}.$$

After our analysis it appears that this question can be reformulated as : "is it true that Case 4 can not happen ?" This question is also decisive as it is the key to prove the law of large number. Unfortunately even if it is strongly believed to be true it has been proven so far only in dimension $d = 2$ by Merkl and Zerner ([15] and see also [14])

3.3 Renewal structure

In this section we assume that $\mathrm{P}_0(A_\ell) > 0$ and study more in details the P_0-a.s. behaviour of the walker on A_ℓ that is under the probability measure

$$\mathrm{Q}_0 = \mathrm{P}_0(\cdot \mid A_\ell).$$

The idea is to cut the path in pieces that are independent under P_0 even if the walker is not markovian. The time were the path is cut are random but not stopping times. They are called renewal times and this decomposition is known as a renewal structure.

As a warm-up let us prove that τ_1 separates the path into two P_0-independent pieces. Introduce the $\sigma-$algebra :

$$\mathcal{G}_1 = \sigma((X_{n\wedge\tau_1})_{n\geq 0}, (\omega_y(\cdot))_{y\cdot\ell<X_{\tau_1}\cdot\ell})$$

with the convention $X_{\tau_1}\cdot\ell = +\infty$ on $\{\tau_1 = +\infty\}$. The $\sigma-$algebra $\mathcal{G}_1$ corresponds roughly speaking to the information about the walker and the environment collected before the first renewal time τ_1. Remark in particular that τ_1 is $\mathcal{G}_1$-measurable.

Proposition 11. *The following equality in law holds*

$$\begin{aligned}\mathrm{Q}_0((X_{\tau_1+n} - X_{\tau_1})_{n\geq 0} \in \cdot,\ (\omega_{X_{\tau_1}+y}(\cdot))_{y\cdot\ell\geq 0} \in \cdot \mid \mathcal{G}_1)\\ = \mathrm{P}_0((X_n)_{n\geq 0} \in \cdot, (\omega_y(\cdot))_{y\cdot\ell\geq 0} \in \cdot \mid D = +\infty)\end{aligned}$$

Remark that this conditional law does actually not depend on the information $\mathcal{G}_1$.

Proof. We have to prove that for all bounded functions f of the paths, g of the environment in the positive (with respect to ℓ) half space, and h $\mathcal{G}_1-$measurable

$$\begin{aligned}&\mathrm{E}^{\mathrm{Q}_0}\Big[f\Big((X_{\tau_1+n} - X_{\tau_1})_{n\geq 0}\Big)g\Big((\omega_{X_{\tau_1}+y}(\cdot))_{y\cdot\ell\geq 0}\Big)h\Big]\\ &= \mathrm{E}^{\mathrm{Q}_0}\Big[\mathrm{E}^{\mathrm{P}_0}\Big[f\Big((X_n)_{n\geq 0}\Big)g\Big((\omega_y(\cdot))_{y\cdot\ell\geq 0}\Big) \mid D = +\infty\Big]\ h\Big] \\ &= \mathrm{E}^{\mathrm{P}_0}\Big[f\Big((X_n)_{n\geq 0}\Big)g\Big((\omega_y(\cdot))_{y\cdot\ell\geq 0}\Big) \mid D = +\infty\Big]\mathrm{E}^{\mathrm{Q}_0}\Big[h\Big].\end{aligned} \tag{15}$$

Observe first that suming over all possible values $k \geq 1$ such that $\tau_1 = S_k$ and all possible positions for X_{τ_1} we obtain

$$\begin{aligned}&\mathrm{E}^{\mathrm{P}_0}\Big[f\Big((X_{\tau_1+n} - X_{\tau_1})_{n\geq 0}\Big)g\Big((\omega_{X_{\tau_1}+y}(\cdot))_{y\cdot\ell\geq 0}\Big)h,\ \tau_1 < +\infty\Big]\\ &= \sum_{\substack{k\geq 1\\ x\in\mathbb{Z}^d}} \mathrm{E}^{\mathrm{P}_0}\Big[f\Big((X_{S_k+n} - x)_{n\geq 0}\Big)g\Big((\omega_{x+y}(\cdot))_{y\cdot\ell\geq 0}\Big)h,\ S_k < +\infty,\ R_k = +\infty, X_{S_k} = x\Big]\end{aligned}$$

On $\{\tau_1 = S_k\} \cap \{X_{\tau_1} = x\}$ the function h coincides actually with a function $h_{k,x}$ of $(\omega_y(\cdot))_{y\cdot\ell\leq x\cdot\ell}$ and $(X_1, \cdots, X_{S_k})$ thus, using strong Markov property at the stopping time S_k the last term equals

$$\sum_{\substack{k\geq 1\\ x\in\mathbb{Z}^d}} \mathbb{E}\Big\{\mathrm{E}_{0,\omega}\Big[h_{k,x}, S_k < +\infty, X_{S_k} = x\Big]\mathrm{E}_{x,\omega}\Big[f\Big((X_n - x)_{n\geq 0}\Big),\ D = +\infty\Big]g\Big((\omega_{x+y}(\cdot))_{y\cdot\ell\geq 0}\Big)\Big\}$$

For all $k \geq 1$, $x \in \mathbb{Z}^d$, $\mathrm{E}_{0,\omega}\Big[h_{k,x}, S_k < +\infty, X_{S_k} = x\Big]$ is $\sigma(\omega_y(\cdot), y \text{ s.t. } y \cdot \ell < x \cdot \ell)$-measurable while $\mathrm{E}_{x,\omega}\Big[f\Big((X_n - x)_{n\geq 0}\Big),\ D = +\infty,\Big)\Big] g\Big((\omega_{x+y}(\cdot))_{y\cdot\ell\geq 0}\Big)$ is $\sigma(\omega_y(\cdot), y \text{ s.t. } y \cdot \ell \geq x \cdot \ell)$-measurable. As the environment is i.i.d. these two variables are thus independent and the last term is

$$\sum_{\substack{k\geq 1\\ x\in\mathbb{Z}^d}} \mathbb{E}\Big\{\mathrm{E}_{0,\omega}\Big[h_{k,x}, S_k < +\infty, X_{S_k} = x\Big]\Big\}$$

$$\mathbb{E}\Big\{\mathrm{E}_{x,\omega}\Big[f\Big((X_n - x)_{n\geq 0}\Big),\ D = +\infty\Big] g\Big((\omega_{x+y}(\cdot))_{y\cdot\ell\geq 0}\Big)\Big\}.$$

As the environment is translation invariant one can change ω for $t_x\omega$ and

$$\begin{aligned}&\mathbb{E}\Big\{\mathrm{E}_{x,\omega}\Big[f\Big((X_n - x)_{n\geq 0}\Big),\ D = +\infty\Big] g\Big((\omega_{x+y}(\cdot))_{y\cdot\ell\geq 0}\Big)\Big\}\\ &= \mathbb{E}\Big\{\mathrm{E}_{0,\omega}\Big[f\Big((X_n)_{n\geq 0}\Big),\ D = +\infty\Big] g\Big((\omega_{y}(\cdot))_{y\cdot\ell\geq 0}\Big)\Big\}.\end{aligned}$$

We finally obtain

$$\begin{aligned}&\mathrm{E}^{\mathrm{P}_0}\Big[f\Big((X_{\tau_1+n} - X_{\tau_1})_{n\geq 0}\Big) g\Big((\omega_{X_{\tau_1}+y}(\cdot))_{y\cdot\ell\geq 0}\Big) h,\ \tau_1 < +\infty\Big]\\ &= \mathbb{E}\Big\{\mathrm{E}_{0,\omega}\Big[f\Big((X_n)_{n\geq 0}\Big),\ D = +\infty\Big] g\Big((\omega_{y}(\cdot))_{y\cdot\ell\geq 0}\Big)\Big\} \sum_{\substack{k\geq 1\\ x\in\mathbb{Z}^d}} \mathbb{E}\Big\{\mathrm{E}_{0,\omega}\Big[h_{k,x}, S_k < +\infty, X_{S_k} = x\Big]\Big\}.\end{aligned}$$

Picking $f = g = 1$ leads to

$$\sum_{\substack{k\geq 1\\ x\in\mathbb{Z}^d}} \mathbb{E}\Big\{\mathrm{E}_{0,\omega}\Big[h_{k,x}, S_k < +\infty, X_{S_k} = x\Big]\Big\} = \mathrm{E}^{\mathrm{P}_0}\Big[h,\ \tau_1 < +\infty\Big] \mathrm{P}_0(D = +\infty)^{-1}.$$

Gathering everything we obtain (15). □

We want to iterate this construction and define a sequence $(\tau_k)_{k\geq 1}$ of such renewal times. Under $\mathrm{Q}_0(\cdot \mid \mathcal{G}_1)$, the law of the walker (and of the environment) after this first renewal time is $\mathrm{P}_0(\cdot \mid D = +\infty)$ according to Proposition 12. This is the law we want to study in order to iterate instead of $\mathrm{Q}_0(\cdot|\mathcal{G}_1)$ as in Proposition 12. Note first that $\{D = +\infty\} \subset \{\tau_1 < +\infty\}$ P_0-a.s. from Proposition 10, so that τ_1 is finite a.s. under $\mathrm{P}_0(\cdot \mid D = +\infty)$. It also implies that $\mathrm{P}_0(\cdot \mid D = +\infty)$ coincides with $\mathrm{Q}_0(\cdot \mid D = +\infty)$. Under this law as $\{D = +\infty\} \in \mathcal{G}_1$ one can use again Proposition 12 and conclude that under $\mathrm{P}_0(\cdot \mid D = +\infty)$ the law of $(X_{\tau_1+n} - X_{\tau_1})_{n\geq 0},\ (\omega(X_{\tau_1} + y, \cdot))_{y\cdot\ell\geq 0}$ is again given by $\mathrm{P}_0(\cdot \mid D = +\infty)$.

From this analysis one can thus define by iteration for all $k \geq 1$

$$\tau_{k+1} = \tau_1 + \tau_k((X_{\tau_1+n} - X_{\tau_1})_{n\geq 0})$$

that are all finite Q_0−a.s. and prove that P_0-a.s.

$$A_\ell = \{\tau_1 < +\infty\} = \cap_{k\geq 1}\{\tau_k < +\infty\}.$$

We now define for all $k \geq 1$ the σ−algebra

$$\mathcal{G}_k = \sigma((X_{n\wedge\tau_k})_{n\geq 0}, (\omega_y(\cdot))_{y\cdot\ell < X_{\tau_k}\cdot\ell})$$

and by iteration from Proposition 12 one can prove (see [12] for more details)

Proposition 12. *The following equality in law holds for all $k \geq 1$*

$$\begin{aligned} Q_0((X_{\tau_k+n} - X_{\tau_k})_{n\geq 0} \in \cdot,\ (\omega(X_{\tau_k} + y, \cdot))_{y\cdot\ell\geq 0} \in \cdot \mid \mathcal{G}_k) \\ = P_0((X_n)_{n\geq 0} \in \cdot, (\omega_y(\cdot))_{y\cdot\ell\geq 0} \in \cdot \mid D = +\infty) \end{aligned}$$

As a direct consequence of this proposition we obtain the **renewal structure** of the path under $Q_0 = P_0(\cdot \mid A_\ell)$ that is the decomposition into consecutive pieces that are i.i.d.

Theorem 11. *Under Q_0, the paths and environments*

$$\left((X_{n\wedge\tau_1})_{n\geq 0}, (\omega_y)_{y\cdot\ell < X_{\tau_1}\cdot\ell}\right) \quad \left((X_{(\tau_k+n)\wedge\tau_{k+1}} - X_{\tau_k})_{n\geq 0}, (\omega_y)_{X_{\tau_k}\cdot\ell\leq y\cdot\ell < X_{\tau_{k+1}}\cdot\ell}\right)_{k\geq 1}$$

are independent. Moreover $\left((X_{(\tau_k+n)\wedge\tau_{k+1}} - X_{\tau_k})_{n\geq 0}, (\omega_y)_{X_{\tau_k}\cdot\ell\leq y\cdot\ell < X_{\tau_{k+1}}\cdot\ell}\right)_{k\geq 1}$ *have same law as $(X_{n\wedge\tau_1})_{n\geq 0}$ under $P_0(\cdot \mid D = +\infty)$.*

3.4 "Law of large numbers"

We have now the main tools to derive the law of large numbers. Using the renewal structure one can easily deduce a first result.

Proposition 13. *Assume $P_0(A_\ell) > 0$ and $E_0(\tau_1 \mid D = +\infty) < +\infty$. Then Q_0−a.s.*

$$\frac{X_n}{n} \mapsto v := \frac{E_0(X_{\tau_1} \mid D = +\infty)}{E_0(\tau_1 \mid D = +\infty)} \qquad \textit{as } n \to +\infty$$

and $v \cdot \ell > 0$.

Proof. We first remark that $E_0(\tau_1 \mid D = +\infty) < +\infty$ implies $E_0(|X_{\tau_1}| \mid D = +\infty) < +\infty$ so that v is well-defined. Then the proposition holds for the subsequence $(\tau_k)_{k\geq 1}$ as from the law of large numbers and Theorem 11, Q_0-a.s.

$$\begin{aligned} \frac{\tau_k}{k} &\to E_0(\tau_1 \mid D = +\infty) \quad \text{and} \\ \frac{X_{\tau_k}}{k} &\to E_0(X_{\tau_1} \mid D = +\infty) \end{aligned} \tag{16}$$

as k goes to $+\infty$ which implies that Q_0-a.s.

$$\frac{X_{\tau_k}}{\tau_k} \to \frac{\mathrm{E}_0(X_{\tau_1} \mid D = +\infty)}{\mathrm{E}_0(\tau_1 \mid D = +\infty)} \quad \text{as } k \to +\infty.$$

To control times between the renewal times, introduce for all $n \geq 1$ the integer k_n such that

$$\tau_{k_n} \leq n < \tau_{k_n+1}.$$

From (16), Q_0-a.s. n/k_n goes to $\mathrm{E}_0(\tau_1 \mid D = +\infty)$ when n goes to infinity so that Q_0-a.s.

$$\frac{X_{\tau_{k_n}}}{n} \to \frac{\mathrm{E}_0(X_{\tau_1} \mid D = +\infty)}{\mathrm{E}_0(\tau_1 \mid D = +\infty)} \quad \text{as } n \to +\infty.$$

To control the reminder: Using again (16), Q_0-a.s.

$$\frac{X_n - X_{\tau_{k_n}}}{n} \leq \frac{\tau_{k_n+1} - \tau_{k_n}}{n} \to 0 \quad \text{as } n \to +\infty.$$

Finally $\mathrm{E}_0(X_{\tau_1} \cdot \ell \mid D = +\infty) > 0$ implies that $v \cdot \ell > 0$. □

Some remarks about this first result :

1. The result obtained states that the RW is **ballistic** as $v \cdot \ell > 0$. By ballistic we mean that a.s.

$$\liminf X_n \cdot \ell = +\infty \quad \text{as } n \to +\infty.$$

2. The assumption $\mathrm{E}_0(\tau_1 \mid D = +\infty) < +\infty$ is difficult to check and one would like to find an effective criterium. In [12] the authors make use ok the so-called Kalikow criterium to provide a sufficient condition. The caracterisation of ballistic RWRE is a fundamental open question in this topic ([ADD REFERENCES]). In particular one wonder if

   under assumption (UE) directional transience implies ballisticity as soon as $d \geq 2$.

   It is not the case under (E) and Dirichlet environments provides nice counter-examples (see [10] for an overview about this environments).

Two results help us to improve Proposition 13. If we do not have condition $\mathrm{E}_0(\tau_1 \mid D = +\infty) < +\infty$ one can still prove that $\mathrm{E}_0(X_{\tau_1} \cdot \ell \mid D = +\infty) < +\infty$ using

Lemma 6. *We assume $\ell \in E$ and (UE) and $\mathrm{P}_0(D = +\infty) > 0$. Then*

$$\mathrm{E}_0(X_{\tau_1} \cdot \ell \mid D = +\infty) = \frac{1}{\mathrm{P}_0(D = +\infty)}.$$

Proof. The idea of the proof is to compute in two different ways

$$\lim_{i \to +\infty} \mathrm{Q}_0(\exists k \geq 1,\ X_{\tau_k} \cdot \ell = i)$$

First, using similar arguments as the ones used to define the renewal structure [ADD DETAILS]

$$\mathrm{Q}_0(\exists k \geq 1,\ X_{\tau_k} \cdot \ell = i) = \frac{\mathrm{P}_0(T_i < +\infty)\mathrm{P}_0(D = +\infty)}{\mathrm{P}_0(A_\ell)}$$

From Theorem 10 we know that $\mathrm{P}_0(A_\ell \cup A_{-\ell}) = 1$ so that $A_{-\ell} \bigcap \cap_{i \geq 1}\{T_i < +\infty\} = \emptyset$ implies $\lim_{i \to +\infty} \mathrm{P}_0(T_i < +\infty) = \mathrm{P}_0(A_\ell)$. Finally

$$\lim_{i \to +\infty} \mathrm{Q}_0(\exists k \geq 1,\ X_{\tau_k} \cdot \ell = i) = \mathrm{P}_0(D = +\infty). \tag{17}$$

On the other hand using Theorem 11,

$$\lim_{i \to +\infty} \mathrm{Q}_0(\exists k \geq 1,\ X_{\tau_k} \cdot \ell = i) = \lim_{i \to +\infty} \sum_{n \geq 1} \mathrm{Q}_0(X_{\tau_1} \cdot \ell = n)\mathrm{Q}_0(\exists k \geq 2,\ (X_{\tau_k} - X_{\tau_1}) \cdot \ell = i - n).$$

The renewal theorem [ADD DETAILS] assures that

$$\lim_{i \to +\infty} \mathrm{Q}_0(\exists k \geq 2,\ (X_{\tau_k} - X_{\tau_1}) \cdot \ell = i - n) = \frac{1}{\mathrm{E}_0(X_{\tau_1} | D = +\infty)}$$

and finally by dominated convergence

$$\lim_{i \to +\infty} \mathrm{Q}_0(\exists k \geq 1,\ X_{\tau_k} \cdot \ell = i) = \frac{1}{\mathrm{E}_0(X_{\tau_1} | D = +\infty)}.$$

Comparing to (17) we obtain the result. □

With this result in hands one obtains the following

Proposition 14. *Assume (UE) and $\mathrm{P}_0(A_\ell) > 0$. Then $\mathrm{Q}_0-a.s.$*

$$\frac{X_n \cdot \ell}{n} \mapsto v_\ell \quad as\ n \to +\infty$$

where $v_\ell = \frac{\mathrm{E}_0(X_{\tau_1} | D = +\infty)}{\mathrm{E}_0(\tau_1 | D = +\infty)}$.

We remark that this result get rid of the assumption $\mathrm{E}_0(\tau_1 \mid D = +\infty) < +\infty$ but we loose the conclusion that the velocity is non zero that is the walker ballistic.

Proof. If $\mathrm{E}_0(\tau_1 \mid D = +\infty) < +\infty$ we have already proven what we need (and even more) in Proposition 13. Assume now that $\mathrm{E}_0(\tau_1 \mid D = +\infty) = +\infty$. As $\mathrm{P}_0(A_\ell) > 0$ implies that $\mathrm{P}_0(D = +\infty) > 0$, one can use Lemma 6 to deduce that $\mathrm{E}_0(X_{\tau_1} \cdot \ell \mid D = +\infty) < +\infty$. As Q_0-a.s.

$$\frac{X_{\tau_k} \cdot \ell}{k} \to \mathrm{E}_0(X_{\tau_1} \cdot \ell \mid D = +\infty) < +\infty$$

as k goes to $+\infty$ and

$$\frac{k_n}{n} \to 0$$

as n goes to $+\infty$ we conclude that for all $n \geq 1$

$$\frac{X_n \cdot \ell}{n} \leq \frac{X_{\tau_{k_n+1}} \cdot \ell}{n} = \frac{X_{\tau_{k_n+1}} \cdot \ell}{k_n + 1} \frac{k_n + 1}{n}$$

so that Q_0-a.s. $\limsup \frac{X_n \cdot \ell}{n} = 0$. In this case we thus have $v_\ell = 0$. □

To complete this description we mention without proof the following result by Zerner [13] which get rid of the ''oscillating case"

Proposition 15 (Zerner 2002). *Assume* (UE) *and let* $e \in \mathbb{R}^d$ *such that* $|e| = 1$ *and* $\mathrm{P}_0(A_e \cup A_{-e}) = 0$. *Then* Q_0-*a.s.*

$$\lim_{n \to +\infty} \frac{X_n \cdot \ell}{n} = 0.$$

We gather now all our results. We use $(e_i)_{i=1\cdots,n}$ to denote the canonical basis of $\mathbb{R}^d$ and decompose the position of the walker in this basis. We obtain

Theorem 12 (Zerner 2002). *Assuming* (UE) *it holds that* Q_0-*a.s.*

$$\frac{X_n}{n} = \sum_{i=1}^{n} \frac{X_n \cdot e_i}{n} e_i \to v := \sum_{i=1}^{n} v_i 1_{A_{e_i}} + v_{-i} 1_{A_{-e_i}}(-e_i)$$

as $n \to +\infty$ *with obvious notations. When* i *is such that* $\mathrm{P}(A_i \cup A_i) = 1$ *the expression of* $v_{\pm i}$ *is given by Proposition 14 while Proposition 15 assures that for* i *such that* $\mathrm{P}(A_i \cup A_i) = 0$, $v_{\pm i} = 0$.

Note that this theorem does not establish a true law of large number as v is a random variable that is not necessarily deterministic. To obtain a true law of large numbers one would need an improvement of the $0-1$ law proven in Section 3.2 : For all $\ell \neq 0$,

$$\mathrm{P}_0(A_\ell) \in \{0, 1\}.$$

If this conjecture turns out to be true then it is easy to prove that v in Theorem 12 is deterministic. I remind you that Conjecture **??** has been proven so far only in dimension $d = 2$ ([15, 14]) but that it is believed to be true for all $d \geq 1$.

4 Environment viewed from the particle

4.1 Introduction

In this section we follow Sznitman's Lecture in [1] that refers himself to [8] and [9]. We assume (UE) and that the environment is i.i.d. or in the case of random conductances model that conductances are i.i.d.

One of the principal difficulty with RWRE is that they are not markovian under the annealed law. One way to overcome this difficulty is to work with the environment as seen from the particle (EVFP) : We thus introduce the process

$$\bar{\omega}_n = t_{X_n}\omega, \quad n \geq 0$$

with values in Ω where t_x denotes the space translation of the environment by x:

$$\begin{aligned} t_x : \Omega \qquad &\to \Omega \\ (\omega_u(\cdot))_{u\in\mathbb{Z}^d} &\to (\omega_{x+u}(\cdot))_{u\in\mathbb{Z}^d} \end{aligned}$$

- Main interest : it is Markovian even under the annealed law!
- Main problem : the states space is huge !

Proposition 16. *The process $(\bar{\omega}_n)_{n\geq 0}$ is Markovian both under $\mathrm{P}_{0,\omega}$ (for all $\omega \in \Omega$) and under P_0 with transition kernel R defined for all bounded measurable function f and all $\omega \in \Omega$ by*

$$Rf(\omega) = \sum_{|e|=1} \omega_0(e) f(t_e\omega).$$

and initial law δ_ω under $\mathrm{P}_{0,\omega}$ and $\mathbb{P}$ under P_0.

As the space state is not countable we define the law of the Markov process via its kernel but here as only finite transitions have positive probability from a given state we can also at least informally say that the transitions are given by

$$p(\omega, t_e\omega) = \omega_0(e), \qquad |e| = 1, \omega \in \Omega$$

for all $\omega \in \Omega$ and all $e \in E$.

Proof. Fix $n \geq 1$ and $\phi_0, \cdots, \phi_{n+1}$ measurable functions from Ω in $\mathbb{R}$. We obtain from the definition of $(\bar{\omega}_n)_{n\geq 0}$ and Markov property

$$\begin{aligned}
\mathrm{E}_{0,\omega}\left(\phi_{n+1}(\bar{\omega}_{n+1})\cdots\phi_0(\bar{\omega}_0)\right) &= \mathrm{E}_{0,\omega}\left(\phi_{n+1}(t_{X_{n+1}}\omega)\cdots\phi_0(t_{X_0}\omega)\right) \\
&= \mathrm{E}_{0,\omega}\left(\mathrm{E}_{X_n,\omega}\left(\phi_{n+1}(t_{X_1}\omega)\right)\phi_n(t_{X_n}\omega)\cdots\phi_0(t_{X_0}\omega)\right)
\end{aligned}$$

Now

$$\begin{aligned}
\mathrm{E}_{X_n,\omega}\left(\phi_{n+1}(t_{X_1}\omega)\right) &= \sum_{|e|=1}\omega_{X_n}(e)\phi_{n+1}(t_{X_n+e}\omega) \\
&= \sum_{|e|=1} t_{X_n}\omega(e)\phi_{n+1}(t_e(t_{X_n}\omega)) \\
&= R\phi_{n+1}(t_{X_n}\omega)
\end{aligned}$$

and finally

$$\mathrm{E}_{0,\omega}\left(\phi_{n+1}(\bar{\omega}_{n+1})\cdots\phi_0(\bar{\omega}_0)\right) = \mathrm{E}_{0,\omega}\left(R\phi_{n+1}(\bar{\omega}_n)\phi_n(\bar{\omega}_n)\cdots\phi_0(\bar{\omega}_0)\right)$$

This implies that $(\bar{\omega}_n)_{n\geq 0}$ is markovian under $\mathrm{P}_{0,\omega}$. This property is still true under the annealed law after integrating under $\mathbb{P}$. □

Theorem 13. *Assume there exists a probability measure $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ ($\mathbb{Q} = f\mathbb{P}$ for some density function f) and invariant for R ($\int Rh\mathbb{Q} = \int h\mathbb{Q}$ for all measurable bounded h). Then $\mathbb{Q} \sim \mathbb{P}$ and the Markov chain $(\bar{\omega}_n)_{n\geq 0}$ with initial law $\mathbb{Q}$ is ergodic. Moreover there is at most one such probability measure $\mathbb{Q}$.*

Proof. **Equivalence.** We first prove $\mathbb{Q} \sim \mathbb{P}$. Let $E = \{f = 0\}$. We have to prove that $\mathrm{P}(E) = 0$. It implies that if A is such that $\mathbb{Q}(A) = 0$ then

$$\int f 1_A \mathrm{d}\mathbb{P} = \int f 1_A 1_{E^c} \mathrm{d}\mathbb{P} = 0$$

and thus $f1_A 1_{E^c} = 0$ $\mathbb{P}-$a.s. that implies $1_A = 0$ $\mathbb{P}-$a.s.

As $\mathbb{Q}$ is invariant

$$\int R1_E \; \mathrm{d}\mathbb{Q} = \int 1_E \; \mathrm{d}\mathbb{Q} = \int 1_E \; f \; \mathrm{d}\mathbb{P} = 0.$$

It implies that $(R1_E)f = 0$ $\mathbb{P}-$a.s. and still $\mathbb{P}-$a.s. for all $|e| = 1$

$$1_E \geq R1_E \geq \varepsilon_0 \sum_{|f|=1} 1_E \circ t_f \geq \varepsilon_0 \; 1_E \circ t_e.$$

As 1_E takes values in $\{0, 1\}$, $1_E \geq 1_E \circ t_e$ $\mathbb{P}-$a.s. and both functions are actually equal $\mathbb{P}-$a.s. as they have same expectation under $\mathbb{P}$. Finally after composition we obtain that for all $x \in \mathbb{Z}^d$ and $\mathbb{P} - a.s.$

$$1_E = 1_E \circ t_x.$$

As $\mathbb{P}$ is ergodic it implies that 1_E is constant $\mathbb{P}-a.s.$. As an indicator function it can only be 0 or 1 but as $\int f d\mathbb{P} = 1$ it implies that it is 0 that is $\mathbb{P}(E) = 0$.

Ergodicity. We now prove that $(\bar{\omega}_n)_{n\geq 0}$ with initial law $\mathbb{Q}$ is ergodic. We work with the canonical process $(\tilde{\omega}_n)_{n\geq 0}$ that is just the collection of all projections on $\tilde{\Omega} = \Omega^{\mathbb{N}}$ endowed with the product $\sigma-$algebra $\tilde{\mathcal{B}}$ and the shift $\tilde{\theta}$. We use $\tilde{\mathrm{P}}_\omega$ for the law of the chain with kernel R and initial position ω ; $\tilde{\mathrm{P}}_{\mathbb{Q}}$ for the one with same kernel and initial law $\mathbb{Q}$.

Our goal is to prove that for all **invariant** $A \in \tilde{\mathcal{B}}$ (that is such that $\tilde{\theta}^{-1} A = A$) it holds that

$$\tilde{\mathrm{P}}_{\mathbb{Q}}(A) \in \{0, 1\}.$$

Step 1. We introduce the function ϕ defined for all $\omega \in \Omega$ by

$$\phi(\omega) = \tilde{\mathrm{P}}_\omega(A).$$

The process $(\phi(\tilde{\omega}_n))_{n\geq 0}$ is a martingale under $\tilde{\mathrm{P}}_{\mathbb{Q}}$ closed by 1_A [DETAILS] so that

$$\phi(\tilde{\omega}_n) \overset{\tilde{\mathrm{P}}_{\mathbb{Q}}-a.s.}{\rightarrow} 1_A \tag{18}$$

Step 2. We now deduce from the previous step that there exists some B in the product $\sigma-$algebra $\mathcal{B}$ on Ω such that $\phi = 1_B$ $\mathbb{Q}-$a.s.

If it is not the case $\mathbb{Q}(\phi \in [a, b]) > 0$ for some $a < b$ in $[0, 1]$. Using Birkhoff's ergodic theorem (Theorem 2) we obtain that $\tilde{\mathrm{P}}_{\mathbb{Q}} - a.s.$

$$\frac{1}{n} \sum_{i=0}^{n-1} 1_{\phi(\tilde{\omega}_i) \in [a,b]} \rightarrow \mathrm{E}_{\tilde{\mathbb{Q}}}(1_{\phi(\tilde{\omega}_0) \in [a,b]} \; |\mathcal{I}).$$

The expectation of the limit is $\mathbb{Q}(\phi \in [a, b])$ that is positive while it should be 0 according to (18). We note that

$$\tilde{\mathrm{P}}_{\mathbb{Q}}(A) = \int \phi(\omega) \; d\mathbb{Q} = \mathbb{Q}(B)$$

so that to conclude it remains to prove that $\mathbb{Q}(B) \in \{0,1\}$.

Step 3. As a consequence of the martingale property, the event B satisfies $R1_B = 1_B$ $\tilde{\mathrm{P}}_{\mathbb{Q}}-$a.s. It implies that $\mathbb{Q}-$a.s. and thus $\mathbb{P}-$a.s.

$$1_B \geq \varepsilon_0 \sum_{|e|=1} 1_B \circ t_e$$

and we conclude as in the proof of the equivalence between $\mathbb{P}$ and $\mathbb{Q}$.

□

4.2 Law of large numbers for conductances model

We remind the conductances model defined in the introduction: consider a probability measure μ on $[\nu_0, 1/\nu_0]$ (where $0 < \nu_0 < 1$ is some real number) endowed with its Borel $\sigma-$algebra and define the probability measure $\mathbb{P} = \mu^{\otimes \mathcal{E}^d}$ (where $\mathcal{E}^d$ is the set of *non oriented* edges of $\mathbb{Z}^d$) on $\mathrm{C} = \mathbb{R}^{\mathcal{E}^d}$. We remind that that to a collection $c = (c_a)_{a \in \mathcal{E}^d} \in \mathrm{C}$ of conductances we associates the environment $\omega \in \Omega$ defined for all neighbours x, y in $\mathbb{Z}^d$ by

$$\omega_x(y-x) = \frac{c_{x,y}}{\sum_{z \sim x} c_{x,z}}.$$

Observe that under $\mathbb{P}$ ω satisfies (UE) with $\varepsilon_0 = \nu_0^2/2d > 0$. In this context the EVFP is defined as

$$\bar{c}_n = t_{X_n} c, \quad n \geq 0$$

with values in C where (X_n) denotes the walker in $\omega(c)$. An inspection of the proof of Theorem 13 shows that it is still true in this context.

Theorem 14. *The probability measure*

$$\mathbb{Q} := \frac{\sum_{|e|=1} c_{0,e}}{Z} \mathbb{P}$$

(where $Z = \mathbb{E}_{\mathbb{P}}(\sum_{|e|=1} c_{0,e})$ is a normalising constant) is invariant for R and satisfies the assumptions of Theorem 13. Moreover

$$\frac{X_n}{n} \overset{\mathrm{P}_0-a.s.}{\longrightarrow} 0 \qquad \textit{as } n \to +\infty.$$

Proof. We actually prove the stronger result that is R is self-adjoint in $\mathrm{L}^2(\mathbb{Q})$. Let h and g be two bounded measurable functions defined on Ω and observe that

$$\begin{aligned}\int h \ Rg \ \mathrm{d}\mathbb{Q} &= \int h(c) \sum_{|e|=1} \omega_0(e) g(t_e c) \mathrm{d}\mathbb{Q} \\ &= \frac{1}{Z} \int h(c) \sum_{|e|=1} c_{0,e} g(t_e c) \mathrm{d}\mathbb{P}\end{aligned}$$

Now use that c has same law as $t_{-e}c$ under $\mathbb{P}$ so that the last term equals

$$\frac{1}{Z}\sum_{|e|=1}\int (t_{-e}c)_{0,e}\ h(t_{-e}c)\ \ g(c)\mathrm{d}\mathbb{P}$$

and as $(t_{-e}c)_{0,e} = c_{0,-e}(\omega)$ this last term is

$$\begin{aligned}&\frac{1}{Z}\sum_{|e|=1}\int c_{0,-e}\ h(t_{-e}c)\ \ g(c)\mathrm{d}\mathbb{P}\\ &=\int Rh\ g\ \mathrm{d}\mathbb{Q}.\end{aligned}$$

We turn to the law of large numbers. For $c \in \mathrm{C}$ and $x \in \mathbb{Z}^d$ one defines the *drift* at x in the environment $\omega(c)$ by

$$d(c,x) = \sum_{|e|=1}\omega_x(e)e = \mathrm{E}_{x,\omega}(X_1 - X_0).$$

From Theorem 13 and Birkhoff's ergodic theorem one deduces that

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\bar{c}_i,0) \overset{\mathrm{P}_{0,\omega}-a.s.}{\longrightarrow} \mathbb{E}_{\mathbb{Q}}\ \left(d(\cdot,0)\right).$$

for $\mathbb{Q}$ (or $\mathbb{P}$ as both probability measures are equivalent) almost all c.

This last proposition is the key of the proof as one can remark that for all $\omega \in \Omega$ and $x \in \mathbb{Z}^d$

$$d(c,x) = d(t_x c, 0)$$

so that

$$\sum_{i=0}^{n-1} d(\bar{c}_i,0) = \sum_{i=0}^{n-1} d(c,X_i).$$

From Markov's property for all $c \in \mathrm{C}$

$$M_n := X_n - \sum_{i=0}^{n-1} d(c,X_i), \quad n \geq 1$$

is under $\mathrm{P}_{0,\omega}$ a martingale with *bounded* increments. One can thus prove (use Azuma's inequality : if (Z_n) is a real martingale with increments bounded by 1 then for all $\lambda > 0$ and all $n \geq 1$

$$\mathrm{P}(Z_n \geq \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$$

) that $\mathrm{P}_{0,\omega}$-a.s.
$$\frac{M_n}{n} \to 0 \qquad \text{as } n \text{ goes to } +\infty.$$
We finally obtain
$$\frac{X_n}{n} \overset{\mathrm{P}_0-a.s.}{\longrightarrow} E_{\mathbb{Q}} \ (d(\cdot, 0)) .$$
It remains to check that the expectation of the drift at 0 under $\mathbb{Q}$ is 0:
$$\begin{aligned} E_{\mathbb{Q}} \ d(\cdot, 0) &= \int \sum_{|e|=1} \omega_0(e) e \ \mathrm{d}\mathbb{Q} \\ &= \frac{1}{Z} \int \sum_{|e|=1} c_{0,e} \ e \ \mathrm{d}\mathbb{P} \\ &= \frac{1}{Z} \int \sum_{i=1}^{d} (c_{0,e_i} - c_{0,-e_i}) \ e_i \ \mathrm{d}\mathbb{P} \\ &= 0. \end{aligned}$$
□

4.3 Back to the $1D$- ballistic law of large number

In this section we find back the law of large numbers in the ballistic regime when $d = 1$. For $\omega \in \Omega$ and $x \in \mathbb{Z}^d$ the *drift* at x in the environment ω writes when $d = 1$
$$d(\omega, x) = 2\omega_x - 1$$

Proposition 17. *Assume that $\mathbb{E}(\rho) < 1$. Then the probability measure $\mathbb{Q} = f\mathbb{P}$ where f is defined for all ω by*
$$f(\omega) = \frac{1 - \mathbb{E}(\rho)}{1 + \mathbb{E}(\rho)} (1 + \rho_0) \left(1 + \sum_{i \geq 1} \prod_{j=1}^{i} \rho_j \right)$$
satisfies the assumptions of Theorem 13 and
$$\frac{X_n}{n} \overset{\mathrm{P}_0-a.s.}{\to} \int d(\omega, 0) \mathrm{d}\mathbb{Q} = \frac{1 - \mathbb{E}(\rho)}{1 + \mathbb{E}(\rho)} \qquad \text{as } n \to +\infty.$$

Remark 4. *It is easy to check that f is a density in the case of an i.i.d. environment as*
$$\begin{aligned} \mathbb{E} \left((1 + \rho_0) \left(1 + \sum_{i \geq 1} \prod_{j=1}^{i} \rho_j \right) \right) &= (1 + \mathbb{E}(\rho)) \left(\sum_{i \geq 1} \mathbb{E}(\rho)^i \right) \\ &= \frac{1 + \mathbb{E}(\rho)}{1 - \mathbb{E}(\rho)} \end{aligned}$$

This a way to make correct the wrong intuition that the asymptotic speed is $\int d(\omega,0)\mathrm{d}\mathbb{P}$. The walker certainly does not see $\mathbb{P}$ as an invariant measure as it prefers to stay at the bottom of valleys of the potential ! Very roughly here we can say that f is large when the environment has large $(\rho_i)_{i\geq 0}$ on the positive half line. That is the walker has in front of itself small $(\omega_i)_{i\geq 0}$'s that is a barrier of the potential. It thus need many tries before it is able to overcome it.

Proof. We only have to prove that $\mathbb{Q}$ is invariant that is $\int Rh\mathbb{Q} = \int h\mathbb{Q}$ for all measurable bounded h. Now using that both $t_1\omega$ and $t_{-1}\omega$ have law $\mathbb{P}$ when ω has law $\mathbb{P}$ we obtain

$$\begin{aligned}\int Rh\mathbb{Q} &= \int \Big(\omega_0 h(t_1\omega) + (1-\omega_0)h(t_{-1}\omega)\Big) f(\omega)\mathbb{P} \\ &= \int h(\omega)\omega_{-1} f(t_{-1}\omega)\mathbb{P} + + \int h(\omega)(1-\omega_1) f(t_1\omega)\mathbb{P} \\ &= \int h(\omega)\Big(\omega_{-1} f(t_{-1}\omega) + (1-\omega_1) f(t_1\omega)\Big)\mathbb{P}.\end{aligned}$$

Now for all ω (notice that $1+\rho = 1/\omega$) using $C := \frac{1+\mathbb{E}(\rho)}{1-\mathbb{E}(\rho)}$

$$\begin{aligned}C\{\omega_{-1} f(t_{-1}\omega) + (1-\omega_1) f(t_1\omega)\} &= 1 + \sum_{i\geq 1}\prod_{j=1}^{i} \rho_j(t_{-1}\omega) + \rho_1\left(1 + \sum_{i\geq 1}\prod_{j=1}^{i}\rho_j(t_1\omega)\right) \\ &= 1 + \sum_{i\geq 1}\prod_{j=1}^{i}\rho_{j-1} + \rho_1\left(1 + \sum_{i\geq 1}\prod_{j=1}^{i}\rho_{j+1}\right) \\ &= 1 + \frac{\rho_0}{1+\rho_0} Cf(\omega) + \left(\frac{Cf(\omega)}{1+\rho_0} - 1\right) \\ &= Cf(\omega).\end{aligned}$$

We finally obtain

$$\int Rh\mathbb{Q} = \int hf\mathbb{P} = \int h\mathbb{Q}.$$

Arguing exactly in the same way as in the previous sequence we obtain that

$$\frac{X_n}{n} \overset{\mathrm{P}_0-a.s.}{\to} \int d(\omega,0)\mathrm{d}\mathbb{Q} \qquad \text{as } n\to+\infty$$

and it only remains to prove that

$$\int d(\omega,0)\mathrm{d}\mathbb{Q} = \frac{1-\mathbb{E}(\rho)}{1+\mathbb{E}(\rho)}.$$

For that we use the definition of $\mathbb{Q}$ together with $d(\cdot,0)(1+\rho_0) = 1-\rho_0$ and the assumption that the environment is i.i.d.

$$\begin{aligned}\frac{1+\mathbb{E}(\rho)}{1-\mathbb{E}(\rho)}\int d(\omega,0)\mathrm{d}\mathbb{Q} &= \mathbb{E}\left(d(\cdot,0)(1+\rho_0)\left(1+\sum_{i\geq 1}\prod_{j=1}^{i}\rho_j\right)\right)\\ &= \mathbb{E}\left((1-\rho_0)\left(1+\sum_{i\geq 1}\prod_{j=1}^{i}\rho_j\right)\right)\\ &= (1-\mathbb{E}\rho)\left(\sum_{i\geq 0}(\mathbb{E}\rho)^i\right)\\ &= 1.\end{aligned}$$

Remark 5. *A way to find the probability measure $\mathbb{Q}$ in Proposition 17 is to guess and prove that the measure $\tilde{\mathbb{Q}}$ define for all event B by*

$$\tilde{\mathbb{Q}}(B) = \mathrm{E}_0\left(\sum_{i=0^{T_1-1}} 1_{\bar{\omega}_i\in B}\right)$$

is invariant for R. It has finite mass $\mathrm{E}_0(T_1)$ we have already computed so that we can define $\mathbb{Q} = \tilde{\mathbb{Q}}/\mathrm{E}_0(T_1)$ that is an invariant probability for R. It remains to prove that it has density f with respect to $\mathbb{P}$ using again the one to one map between the excursion and the tree to comput the expected number of visits at a given level.

□

References

[1] E. Bolthausen and A.-S. Sznitman. *Ten lectures on random media*, volume 32 of *DMV Seminar*. Birkhäuser Verlag, Basel, 2002.

[2] F. den Hollander, S. A. Molchanov, and O. Zeitouni. *Random media at Saint-Flour*. Probability at Saint-Flour. Springer, Heidelberg, 2012. Reprints of lectures from the Annual Saint-Flour Probability Summer School held in Saint-Flour.

[3] N. Enriquez, C. Sabot, and O. Zindy. Limit laws for transient random walks in random environment on $\mathbb{Z}$. *Ann. Inst. Fourier (Grenoble)*, 59(6):2469–2508, 2009.

[4] G. Grimmett. *Percolation*, volume 321 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[5] G. R. Grimmett, H. Kesten, and Y. Zhang. Random walk on the infinite cluster of the percolation model. *Probab. Theory Related Fields*, 96(1):33–44, 1993.

[6] S. A. Kalikow. Generalized random walk in a random environment. *Ann. Probab.*, 9(5):753–768, 1981.

[7] H. Kesten, M. V. Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Compositio Math.*, 30:145–168, 1975.

[8] S. M. Kozlov. The averaging method and walks in inhomogeneous environments. *Uspekhi Mat. Nauk*, 40(2(242)):61–120, 238, 1985.

[9] G. C. Papanicolaou and S. R. S. Varadhan. Diffusions with random coefficients. In *Statistics and probability: essays in honor of C. R. Rao*, pages 547–552. North-Holland, Amsterdam, 1982.

[10] C. Sabot and L. Tournier. Random walks in dirichlet environment: an overview. *Annales de la Faculté des Sciences de Toulouse*, 6 (26) 2, 2016.

[11] Y. G. Sinaĭ. The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatnost. i Primenen.*, 27(2):247–258, 1982.

[12] A.-S. Sznitman and M. Zerner. A law of large numbers for random walks in random environment. *Ann. Probab.*, 27(4):1851–1869, 1999.

[13] M. P. W. Zerner. A non-ballistic law of large numbers for random walks in i.i.d. random environment. *Electron. Comm. Probab.*, 7:191–197, 2002.

[14] M. P. W. Zerner. The zero-one law for planar random walks in i.i.d. random environments revisited. *Electron. Comm. Probab.*, 12:326–335, 2007.

[15] M. P. W. Zerner and F. Merkl. A zero-one law for planar random walks in random environment. *Ann. Probab.*, 29(4):1716–1732, 2001.