Tutorial. Solving PDEs with the finite element method using FreeFEM

FreeFEM is an open-source partial differential equation solver using the finite element method. It is freely available for all platforms here and its documentation is found here.

Exercise 1 (a Poisson problem). We first consider a boundary-value problem for the Laplace operator on the unit domain $\Omega = (0,1)^2$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. Write the weak form of this problem.

2. Write a FreeFEM script that computes an approximation to the solution of the problem using P_1 Lagrange finite elements, setting $f \equiv 1$ for instance.

To provide an assesment of the convergence of the method, one needs to consider a case for which the analytical solution is known and smooth. To this end, take for example the solution $u(x_1,x_2) = x_1(1-x_1)x_2(1-x_2)$, associated with the right-hand side function $f(x_1,x_2) = 2x_1(1-x_1) + 2x_2(1-x_2)$.

3. Compute the numerical approximation u_h to the solution of the problem on various meshes corresponding to different levels of refinement and verify that the norms of the error $\|u-u_h\|_{L_2(\Omega)}$ and $\|u-u_h\|_{H^1(\Omega)}$ are decreasing with respect to the mesh size.

4. Answer the preceding question using P_2 Lagrange elements.

Exercise 2 (temperature evolution in a cooling fin). We consider a rectangular cooling fin represented by the domain $\Omega = (0,L)\times(-H,H)$, its third dimension being negligible with respect to the other two so that the problem is two-dimensional.

On the boundary $\Gamma_4 = \{0\}\times[-H,H]$, the fin is in contact with a heat source with constant temperature T_c (in kelvins). The temperature evolution in the fin is governed by the heat equation

$$\frac{\partial \theta}{\partial t}(t,x) - \Delta\theta(t,x) = 0 \text{ in } (0,+\infty)\times\Omega,$$

where the function $\theta = \frac{T-T_a}{T_a}$ is the dimensionless temperature profile, T_a being the ambiant temperature. The equation is completed by an initial condition at time $t = 0$:

$$\theta(0,\cdot) = 0 \text{ in } \Omega,$$

and the following boundary conditions, of Dirichlet's type (fixed temperature)

$$\theta(t,\cdot) = \theta_c \text{ on } \Gamma_4,$$

and of Robin's type (convective heat transfer)

$$\frac{\partial \theta}{\partial n}(t,\cdot) + \alpha\theta(t,\cdot) = 0 \text{ on } \Gamma_1\cup\Gamma_2\cup\Gamma_3,$$

where $\Gamma_1 = [0,L]\times -H$, $\Gamma_2 = \{L\}\times[-H,H]$, $\Gamma_3 = [0,L]\times H$, $\frac{\partial\theta}{\partial n}$ being the normal derivative of θ at the boundary, and α is a constant coefficient.

This problem will be numerically solved using the method of lines with a finite element discretisation in space based on P_1 Lagrange elements.

1. Write a weak formulation of this parabolic problem.

2. An implicit Euler scheme is used for the time discretisation, with step length $\Delta t = 0.01$. Compute up to time $t = 0.5$ and plot an approximation to the dimensionless temperature profile at each time step using the following set of values: $L = 1$ m, $H = 0.25$ m, $T_a = 293$ K, $T_c = 319$ K, $\alpha = 10$ m^{-1}.

3. Compare the approximation at the final time with the corresponding approximation to the stationary solution. Comment.