Habilitation à diriger la recherche :
Periodic and half-periodic fermionic systems

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Fermions (= **electrons** in this talk)
Pauli principle: «*two identical fermions cannot be in the same quantum state*».
A system of N (uncorrelated) fermions is described by $N-$**orthonormal** functions (the *orbitals*)...
or by the orthogonal projector on these N functions.

In my work, a system of N fermions is described an orthogonal projector of rank N.

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In this habilitation, we are interested in systems with infinitely many fermions
$\equiv$ orthogonal projectors of infinite rank.

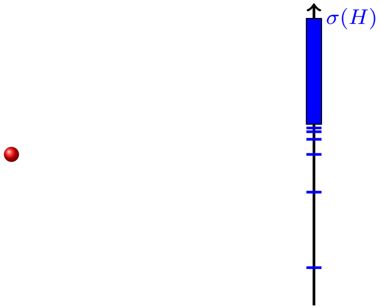
Outline of the manuscript:

- **Low energy spectrum of periodic systems**
  (with É. Cancès, H. Cornean, V. Ehrlacher, A. Levitt, D. Lombardi, D. Monaco, S. Perrin-Roussel, S. Siraj-Dine).
  Wannier functions, homotopy of projectors, Brillouin zone integration, ...
- **Semi-periodic systems**
  Bulk-edge correspondence, edge modes, spectral flows, ...
- **The Hartree–Fock gas (and Peierls model)**
  (with M. Lewin, Ch. Hainzl, A. Kouandé. É. Séré).
  Spin symmetry breaking, spatial symmetry breaking, SSH model for polyacetylene, ...
- **Lieb–Thirring (and related) inequalities**
  (with R.L. Frank, M. Lewin, F.Q. Nazar).
  Lieb–Thirring inequalities, fermionic non-linear Schrödinger, ...

Semi-periodic systems

Start with a single atom in $\mathbb{R}^d$. We study the spectrum of the (one–body) Schrödinger operator

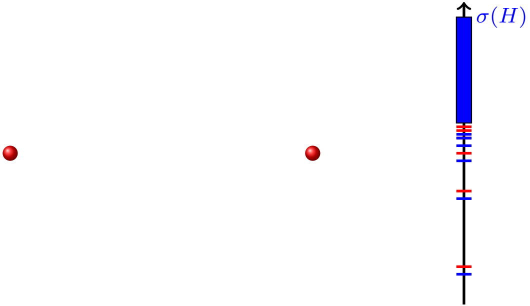
$$H = -\Delta + V(\mathbf{x}), \quad \text{e.g.} \quad V(\mathbf{x}) = \frac{-Z}{|\mathbf{x}|}.$$



- Discrete spectrum (= eigenvalues). The energy levels are *quantized.*
- The N fermions occupies the N first eigenvectors/orbitals (associated to the N lowest eigenvalues).

Then take two atoms in $\mathbb{R}^d$.

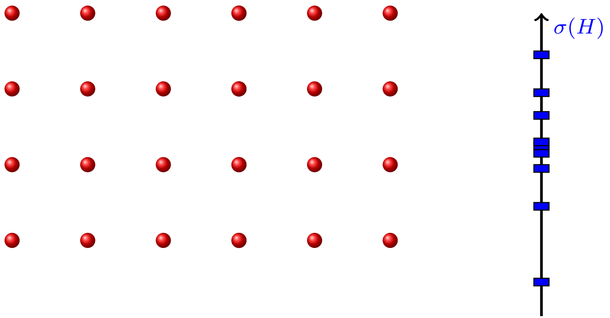
$$H = -\Delta + V\left(\mathbf{x} - \frac{R}{2}\right) + V\left(\mathbf{x} + \frac{R}{2}\right).$$



- When $R = \infty$, the spectrum is copied twice (each eigenvalue doubles its multiplicity);
- When $R \gg 1$, *tunnelling* effect = interaction of eigenvectors $\implies$ splitting of the eigenvalues;
- The eigenvectors are delocalized between the two atoms.

Now take an infinity of atoms in $\mathbb{R}^d$, located along a lattice (= material)

$$H = -\Delta + \sum_{\mathbf{v} \in R\mathbb{Z}^d} V(\mathbf{x} - \mathbf{v})$$



- When $R = \infty$, each eigenvalue is of infinite multiplicity;
- When $R \gg 1$, each eigenvalue becomes a **band of essential spectrum**;
- Each band represents «*one electron per unit cell*»;
- When R decreases, the bands may overlap.

The spectrum of $-\Delta + V$ with V-periodic has a band-gap structure!
One band = one electron per unit cell.

Usual proof with the *Bloch transform* ($\sim$ discrete version of the Fourier transform).

Motivation: Spectral pollution

Let's compute numerically the spectrum of the (simple, one-dimensional) operator

$$H := -\partial_{xx}^2 + V(x), \quad \text{with} \quad V(x) = 50 \cdot \cos(2\pi x) + 10 \cdot \cos(4\pi x).$$

The potential V is 1-periodic. We expect a band-gap structure for the spectrum.
We study H in a box $[t, t+L]$ with Dirichlet boundary conditions, and with finite difference.

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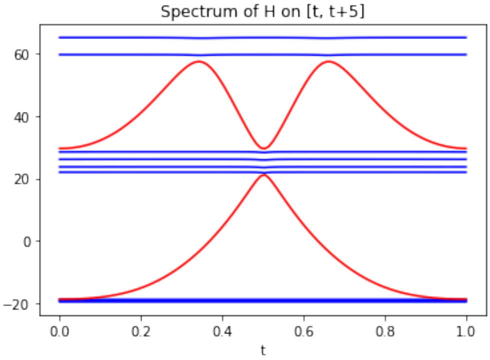


Depending on where we fix the origin t, the spectrum differs...
There are branches of spurious eigenvalues = spectral pollution (they appear for all L).
The corresponding eigenvectors are edge modes: they are localized near the boundaries.

Setting

Let V be a 1-periodic potential, and consider the cut (one-dimensional) Hamiltonian

$$H_t^\sharp = -\partial_{xx}^2 + V(x-t) \quad \text{on} \quad L^2(\mathbb{R}^+),$$

with Dirichlet boundary conditions (with domain $H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$).
Since V is 1-periodic, the map $t \mapsto H_t^\sharp$ is also 1-periodic.

Theorem (Korotyaev 2000, Hempel Kohlmann 2011, DG 2020)

In the n-th essential gap, there is a flow of n eigenvalues going downwards as t goes from 0 to 1.
These eigenvalues are simple, and their associated eigenvectors are exponentially localised (= edge modes).

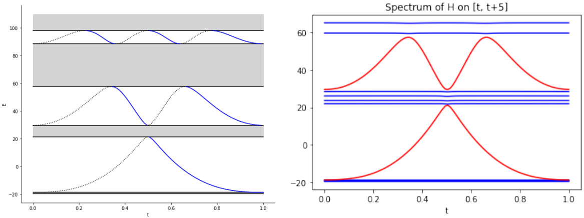


Figure: (Left) Spectrum of $H^\sharp(t)$ for $t \in [0,1]$. (Right) Spectrum of the operator on $[t, t+L]$.

E. Korotyaev, Commun. Math. Phys., 213(2):471–489, 2000.
R. Hempel and M. Kohlmann, J. Math. Anal. Appl., 381(1):166–178, 2011.
D. Gontier, J. Math. Phys. 61, 2020.

Idea of the proof

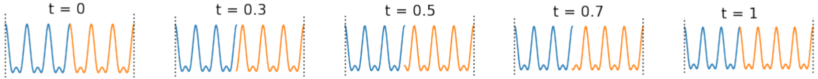
Step 1. Prove the result for *dislocations* (following *Hempel and Kohlmann*).
Introduce the dislocated operator

$$H_t^{\rm disloc} := -\partial_{xx}^2 + \left[V(x) \mathbb{1}(x < 0) + V(x - t) \mathbb{1}(x > 0) \right], \quad \text{on} \quad L^2(\mathbb{R}).$$

Let $L \in \mathbb{N}$ be a (large) integer. Consider the periodic dislocated operator

$$H_{L,t}^{\rm disloc} := -\partial_{xx}^2 + \left[V(x) \mathbb{1}(x < 0) + V(x - t) \mathbb{1}(x > 0) \right], \quad \text{on} \quad L^2([-\tfrac{1}{2}L, \tfrac{1}{2}L + t])$$

with periodic boundary conditions.



Remarks

- The branches of eigenvalues of $t \mapsto H_{L,t}^{\rm disloc}$ are continuous;
- At $t = 0$, the system is 1-periodic, on a box of size L. Each «band» contributes to L eigenvalues;
- At $t = 1$, the system is 1-periodic, on a box of size $L + 1$. Each «band» contributes to $L + 1$ eigenvalues.

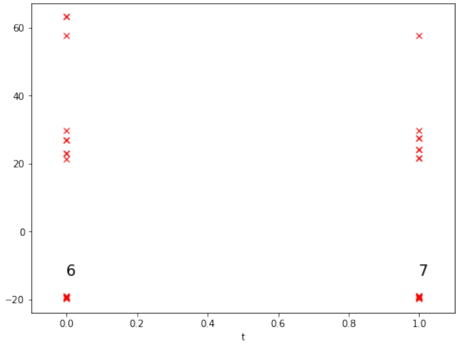


Figure: Spectrum of $H_{L,t}^{\text{disloc}}$ for $L = 6$ at $t = 0$ (6 cells) and $t = 1$ (7 cells).

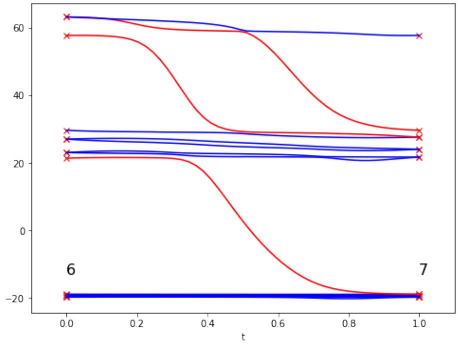


Figure: Spectrum of $H_{L,t}^{\text{disloc}}$ for all $t \in [0, 1]$.

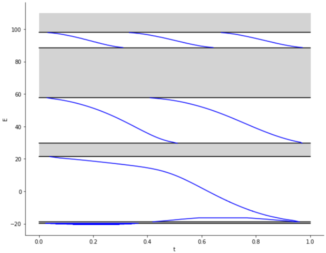
The presence and the number of the red lines are independent of $L \in \mathbb{N}$.
They survive in the limit $L \to \infty$.

This implies that there the result holds for the family of dislocated operators $t \mapsto H_t^{\text{disloc}}$.

The Spectral flow

If $t \mapsto A_t$ is a 1-periodic and *continuous* family of self-adjoint operators, and if $E \notin \sigma_{\rm ess}(A_t)$ for all t, we can define its Spectral flow as

$$\mathrm{Sf}\,(A_t, E) := \text{number of eigenvalues going } \textbf{downwards} \text{ in the essential gap where } E \text{ lies.}$$



The previous result can be formulated as:

$$\mathrm{Sf}\left(H_t^{\rm disloc}, E\right) = \mathcal{N}(E), \quad \mathcal{N}(E) := \text{number of bands below } E.$$

Facts :

- If $t \mapsto K_t$ is a 1-periodic continuous family of **compact** operators, then

$$\mathrm{Sf}\,(A_t, E) = \mathrm{Sf}\,(A_t + K_t, E)\,.$$

- If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then

$$\mathrm{Sf}\,(f\,(A_t)\,, f(E)) = \mathrm{Sf}\,(A_t, E)\,.$$

Step 2. From the dislocated case to the Dirichlet case.

Recall that the dislocated operator is

$$H_t^{\text{disloc}} := -\partial_{xx}^2 + [V(x)\mathbb{1}(x<0) + V(x-t)\mathbb{1}(x>0)] \quad \text{on} \quad L^2(\mathbb{R}).$$

Consider the cut Hamiltonian

$$H_t^{\text{cut}} := -\partial_{xx}^2 + [V(x)\mathbb{1}(x<0) + V(x-t)\mathbb{1}(x>0)] \quad \text{on} \quad L^2(\mathbb{R}) = L^2(\mathbb{R}^-) \cup L^2(\mathbb{R}^+),$$

and with Dirichlet boundary conditions at $x=0$ (only the domain differs).

Fact: For any Σ negative enough (below the essential spectra of all operators), we have

$$K_t := \left(\Sigma - H_t^{\text{cut}}\right)^{-1} - \left(\Sigma - H_t^{\text{disloc}}\right)^{-1} \quad \text{is compact (here, it is finite rank).}$$

So

$$\text{Sf}\left(\left(\Sigma - H_t^{\text{disloc}}\right)^{-1}, (\Sigma - E)^{-1}\right) = \text{Sf}\left(\left(\Sigma - H_t^{\text{cut}}\right)^{-1}, (\Sigma - E)^{-1}\right).$$

Since $f(x) := (\Sigma - x)^{-1}$ is strictly increasing on $x > \Sigma$, we have

$$\mathcal{N}(E) = \text{Sf}\left(H_t^{\text{disloc}}, E\right) = \text{Sf}\left(H_t^{\text{cut}}, E\right) = \text{Sf}\left(H_t^{\sharp,+}, E\right). \quad \square$$

The case of junctions

Take two 1-periodic potentials

$$V_L(x) = 50\cos(2\pi x) + 10\cos(4\pi x), \qquad V_R(x) = 10\cos(2\pi x) + 50\cos(4\pi x)$$

Consider the **junction** Hamiltonian

$$H_t^{\text{junct}} := -\partial_{xx}^2 + \left(V_L(x)\mathbb{1}(x<0) + V_R(x-t)\mathbb{1}(x>0)\right) \quad \text{on} \quad L^2(\mathbb{R}).$$

Reasoning as before (using a cut as a compact perturbation), one can prove that $\mathrm{Sf}\left(H_t^{\text{junct}}, E\right) = \mathcal{N}_R(E)$.

The case of junctions

Take two 1-periodic potentials

$$V_L(x) = 50\cos(2\pi x) + 10\cos(4\pi x), \qquad V_R(x) = 10\cos(2\pi x) + 50\cos(4\pi x)$$

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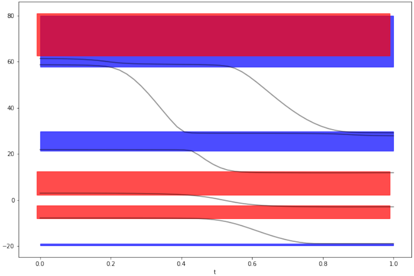


Figure: Spectrum of $H_t^{\rm junc}$ as a function of t.

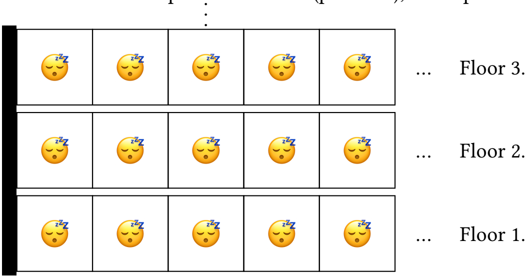
A typical spectrum contains:

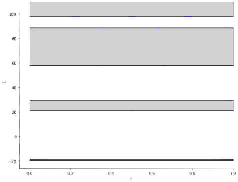
- The essential spectrum of the left and right side.
- Additional edge modes at the junction.

A «fun» analogy

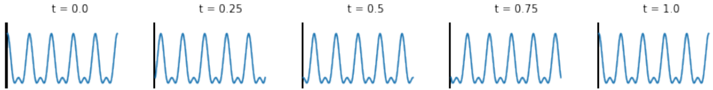
The *«Grand Hilbert Hotel»*
An infinite number of floors, and an infinite number of rooms per floor.

Idea: each unit cell represents 1 room (per floor), each spectral band represents one floor.

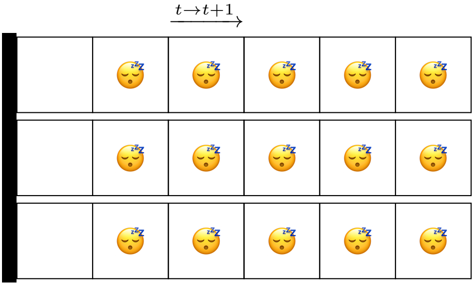




As t moves from 0 to 1...



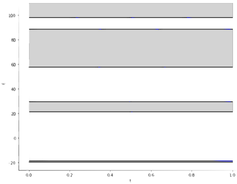
... a new room is created on each floor!



... Floor 3.

... Floor 2.

... Floor 1.



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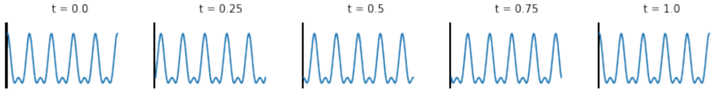

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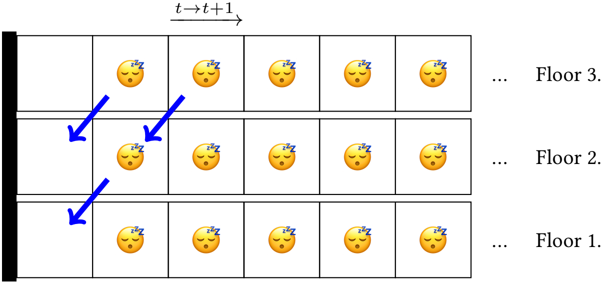
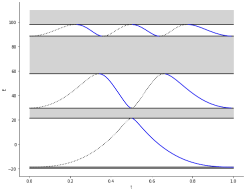




In order to fill the new rooms,

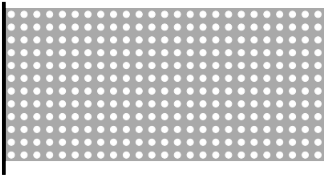
- 1 person from floor 2 must come down to floor 1;
- 2 persons from floor 3 must come down to floor 2;
- and so on.

If we reverse the motion, (we delete rooms, or new guests arrive), then people climb up instead.

The two-dimensional case.

Let V be a $\mathbb{Z}^2$-periodic potential. We study the edge operator

$$H^\sharp(t) = -\Delta + V(x-t, y), \quad \text{on} \quad L^2(\mathbb{R}_+ \times \mathbb{R}), \quad \text{with Dirichlet boundary conditions.}$$



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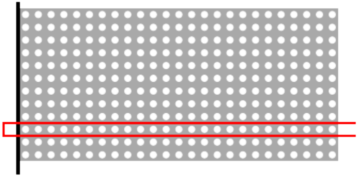


After a Bloch transform in the y-direction, we need to study the **family** of operators

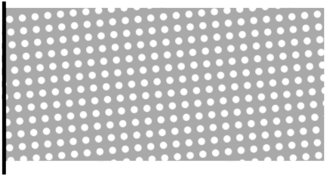
$$H_k^\sharp(t) = -\partial_{xx}^2 + (-\mathrm{i}\partial_y + k)^2 + V(x-t, y), \quad \text{on the \textbf{tube}} \quad L^2(\mathbb{R}_+ \times [0,1]).$$

- Consider again the **«Grand Hilbert Hotel»** (= on a tube).
- For each k, as t moves from 0 to 1, a new room is created on each floor $\Longrightarrow$ spectral flow.
- As k varies, each branch of eigenvalue becomes of branch of essential spectrum.

There is a «spectral flow» of **essential spectrum** appearing in each gap.
The corresponding modes can only propagate along the boundary.

The two-dimensional twisted case.

We rotate V by θ.



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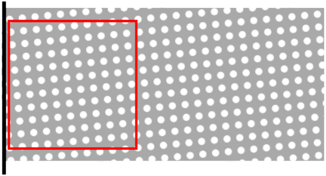


Commensurate case ($\tan\theta = \frac{p}{q}$)

Considering a **Supercell** of size $L = \sqrt{p^2+q^2}$, we recover a $L\mathbb{Z}^2$-periodic potential.
On the tube $\mathbb{R}^+ \times [0, L]$ (at the k-Bloch point $k = 0$ for instance),

« As t moves from 0 to L, L^2 new rooms are created»

Key remark:

- The map $t \mapsto H_\theta^\sharp(t)$ is now $1/L$-periodic (up to some x_2 shifts)
- So the map $t \mapsto \sigma(H_\theta^\sharp(t))$ is $1/L$ periodic.

«As t moves from 0 to $\frac{1}{L}$, 1 new room is created»

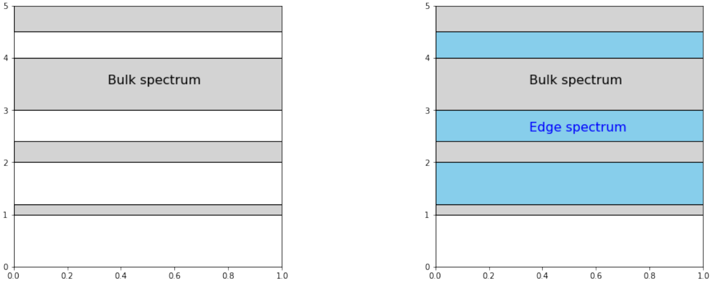
In-commensurate case ($\tan\theta \notin \mathbb{Q}$, corresponds to $L \to \infty$)

Theorem (DG 2021)

If $\tan\theta \notin \mathbb{Q}$, the spectrum of $H^\sharp_\theta$ is of the form $[\Sigma, \infty)$.

Remarks:

- The spectrum of $H^\sharp(t)$ is independent of t (ergodicity);
- All bulk gaps are filled with edge spectrum.



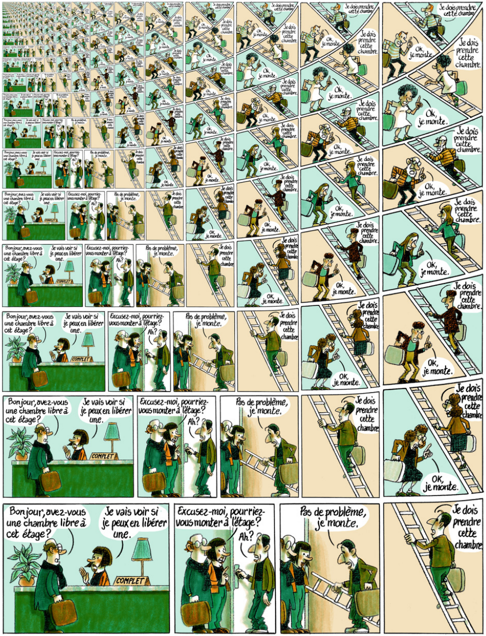
(a) Uncut two-dimensional material

(b) Two-dimensional material with **incommensurate** cut

Open question

Is the edge spectrum pure point ($\sim$ Anderson localization), or absolutely continuous (travelling waves)?

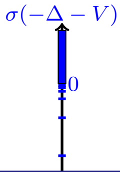
D. Gontier, Comptes Rendus Mathématique, 359(8), 949-958 (2021).



The Grand Hilbert Hotel, by Étienne Lécroart.

Crystallization in Lieb-Thirring inequalities

Joint work with Rupert L. Frank, Mathieu Lewin and Faizan Q. Nazar.



Keller problem (1961).
Among all potentials V with $\int_{\mathbb{R}^d} V^p$ fixed, which one minimizes $\lambda_1(-\Delta - V)$?

- Existence of an optimal potential in all dimensions (explicit in dimension $d = 1$).
- Links with Gagliardo–Niremberg inequality, non-linear Schrödinger equation, ...

Theorem (Keller/Gagliardo–Niremberg inequality)

For all $\gamma > \max(0, 1 - \frac{d}{2})$, there is an optimal (smallest) constant $L^{(1)}_{\gamma,d}$ so that, for all $V \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d, \mathbb{R}^+)$,

$$|\lambda_1(-\Delta - V)|^\gamma \le L^{(1)}_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}. \qquad \textit{(Keller inequality).}$$

J.B. Keller, J. Mathematical Phys. 2 (1961), 262–266.
E.H. Lieb, W.E. Thirring, Phys. Rev. Lett. 35 (1975), 687–689.
E.H. Lieb, W.E. Thirring, Studies in Math. Phys. (1976), 269–303.

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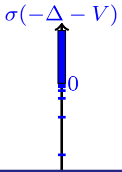

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$$|\lambda_1(-\Delta - V)|^\gamma \le L^{(1)}_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}. \qquad \text{(Keller inequality).}$$

This inequality was then extended by Lieb and Thirring to the *(infinite) sum* of eigenvalues:

Theorem (Lieb-Thirring inequality, '75-76)

For all $\gamma > \max(0, 1 - \frac{d}{2})$, *there is an optimal (smallest) constant* $L_{\gamma,d}$ *so that, for all* $V \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d, \mathbb{R}^+)$,

$$\sum_{n=1}^{\infty} |\lambda_n(-\Delta - V)|^\gamma \le L_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}. \qquad \text{(Lieb-Thirring inequality).}$$

"Open question": is there an optimal potential V for Lieb–Thirring?

J.B. Keller, J. Mathematical Phys. 2 (1961), 262–266.
E.H. Lieb, W.E. Thirring, Phys. Rev. Lett. 35 (1975), 687–689.
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Finite rank Lieb-Thirring.

For all $\gamma > \max(0, 1 - \frac{d}{2})$, there is an optimal (smallest) constant $L_{\gamma,d}^{(N)}$ so that

$$\sum_{n=1}^{N} |\lambda_n(-\Delta - V)|^\gamma \leq L_{\gamma,d}^{(N)} \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}.$$

Basic fact: The sequence $N \mapsto L_{\gamma,d}^{(N)}$ is increasing, and $L_{\gamma,d} = \lim \uparrow L_{\gamma,d}^{(N)}$.

Theorem (R.L. Frank, DG, M. Lewin (2024?))

For all $\gamma > \max(0, 1 - \frac{d}{2})$, and all integer $N > 0$, there is an optimal potential $V_N \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d, \mathbb{R}^+)$.

In addition, if $\gamma > \max(0, 2 - \frac{d}{2})$, then $L_{\gamma,d}^{(2N)} > L_{\gamma,d}^{(N)}$.

In particular, if $\gamma > \max(0, 2 - \frac{d}{2})$, then $L_{\gamma,d} > L_{\gamma,d}^{(N)}$ for all N.

If the problem defining $L_{\gamma,d}$ has an optimal potential V_* (?),
then this one must generate an infinite number of eigenvalues.

R.L. Frank, D. Gontier, M. Lewin, accepted in American Journal of Mathematics.

Finite rank Lieb-Thirring.

For all $\gamma > \max(0, 1 - \frac{d}{2})$, there is an optimal (smallest) constant $L^{(N)}_{\gamma,d}$ so that

$$\sum_{n=1}^{N} |\lambda_n(-\Delta - V)|^\gamma \le L^{(N)}_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}.$$

Basic fact: The sequence $N \mapsto L^{(N)}_{\gamma,d}$ is increasing, and $L_{\gamma,d} = \lim \uparrow L^{(N)}_{\gamma,d}$.

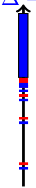
Theorem (R.L. Frank, DG, M. Lewin (2024?))

For all $\gamma > \max(0, 1 - \frac{d}{2})$, and all integer $N > 0$, there is an optimal potential $V_N \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d, \mathbb{R}^+)$.

In addition, if $\gamma > \max(0, 2 - \frac{d}{2})$, then $L^{(2N)}_{\gamma,d} > L^{(N)}_{\gamma,d}$.

In particular, if $\gamma > \max(0, 2 - \frac{d}{2})$, then $L_{\gamma,d} > L^{(N)}_{\gamma,d}$ for all N.

If the problem defining $L_{\gamma,d}$ has an optimal potential V_* (?),
then this one must generate an infinite number of eigenvalues.



Proof of the second part: Consider the test function

$$V_R(x) := \left[V_N^q \left(x - \frac{R}{2} \right) + V_N^q \left(x + \frac{R}{2} \right) \right]^{\frac{1}{q}}, \quad q := \gamma + \frac{d}{2} - 1 \ge 0,$$

and compute the contribution of the *tunnelling* effect. If $q > 1$, we find $L^{(2N)}_{\kappa,d} > L^{(N)}_{\kappa,d}$.

R.L. Frank, D. Gontier, M. Lewin, accepted in American Journal of Mathematics.

Similar phenomenon for the **fermionic non-linear Schrödinger** inequality

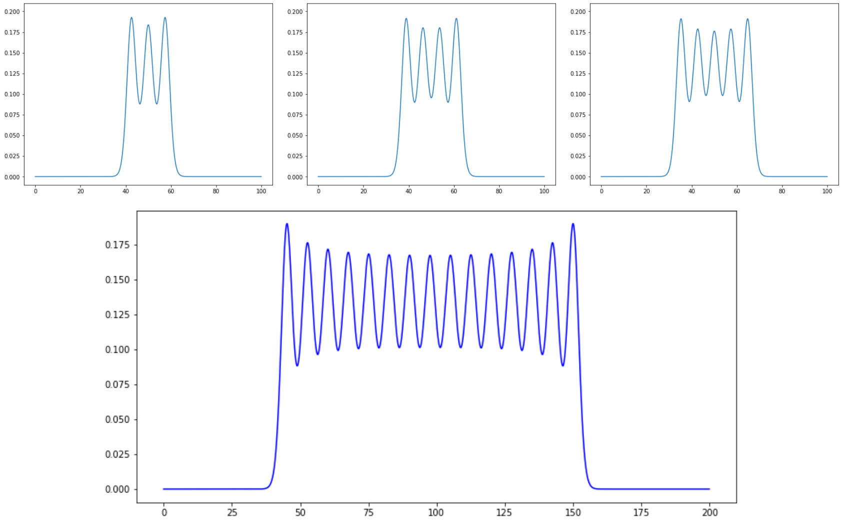


Figure: Optimal density for $J_{p,d}^{\mathrm{NLS}}$ in the case $d = 1$, $p = 1.3$ and $N = 3, 4, 5, 13$.

D. Gontier, M. Lewin, F.Q. Nazar. ARMA 240(3), 1203–1254 (2021).

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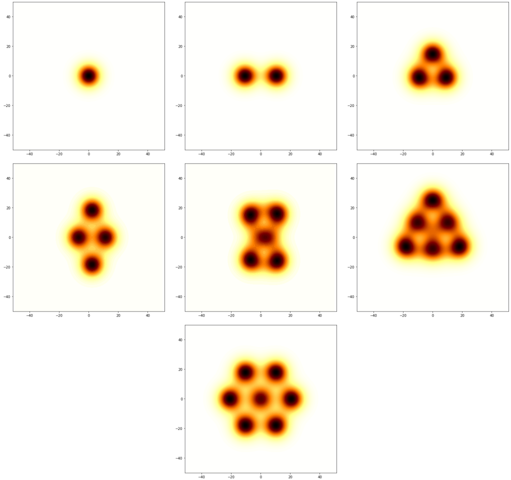

Figure: Optimal density for $J_{p,d}^{\mathrm{NLS}}$ in the case $d = 2$, $p = 1.5$ and for N from 1 to 7.

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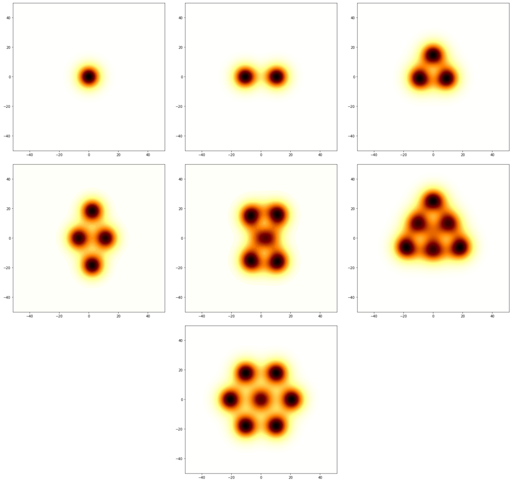

Figure: Optimal density for $J^{\mathrm{NLS}}_{p,d}$ in the case $d = 2$, $p = 1.5$ and for N from 1 to 7.

$\Longrightarrow$ There is a **crystallization phenomenon**!
Conjecture: If $\gamma > \max\{0, 2 - \frac{d}{2}\}$, The sequence $(V_N)_N$ "converges" to a periodic potential.

D. Gontier, M. Lewin, F.Q. Nazar. ARMA 240(3), 1203–1254 (2021).

Periodic Lieb–Thirring inequality

Lemma (R.L. Frank, DG, M. Lewin (2021))

Let $\gamma > \max\{0, 1 - \frac{d}{2}\}$*. Then, for all* **periodic** $V \in L^{\gamma+\frac{d}{2}}_{\rm loc}(\mathbb{R}^d, \mathbb{R}^+)$*, we have*

$$\underline{\mathrm{Tr}}\left((-\Delta - V)_-^\gamma \right) \leq L_{\gamma,d} \fint V^{\gamma+\frac{d}{2}}.$$

with the **same** *best Lieb–Thirring constant* $L_{\gamma,d}$*.*

Remark. Taking the test function $V = cst$ shows that

$$L_{\gamma,d} \geq L^{\rm sc}_{\gamma,d} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|\mathbf{k}|^2 - 1)^\gamma \mathrm{d}\mathbf{k}. \qquad (\textit{semi–classical constant}).$$

Lieb-Thirring "conjecture": $\qquad L_{\gamma,d} \overset{?}{=} \max\{L^{(1)}_{\gamma,d}, L^{\rm sc}_{\gamma,d}\}.$

The optimal scenario is either the one-bound state, or the semi-classical one = fluid phase.

R.L. Frank, D. Gontier, M. Lewin. Partial Differential Equations, Spectral Theory, and Mathematical Physics. The Ari Laptev Anniversary Volume, volume 18. EMS Publishing House (2021).

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Facts.

- In all dimensions d, there is $1 \le \gamma_{\rm sc}(d) \le \frac{3}{2}$ so that

$$\begin{cases} \text{if } \gamma < \gamma_{\rm sc}(d), & L_{\gamma,d} > L^{\rm sc}_{\gamma,d} \\ \text{if } \gamma \ge \gamma_{\rm sc}(d), & L_{\gamma,d} = L^{\rm sc}_{\gamma,d}. \end{cases}$$

- In dimension $d = 2$, we have $\gamma_{\rm sc}(2) \ge 1.165378$.

In dimension $d = 2$, for $\gamma \in (1, \gamma_{\rm sc}(2))$, we expect crystallization.

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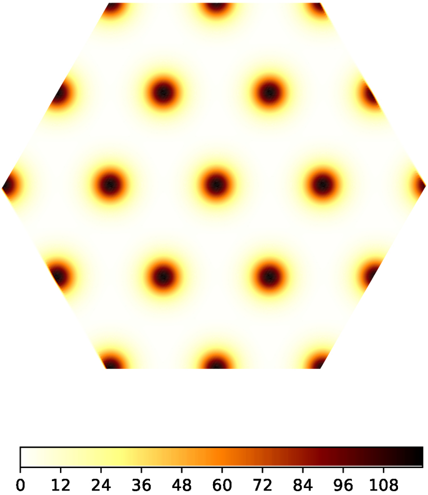

Figure: Numerical computation of the optimal periodic potential in dimension $d = 2$, for $\gamma = 1.165400$.

The integrable case $\gamma = 3/2$ in dimension $d = 1$.

In the original article by Lieb-Thirring 1976, they proved

$$\boxed{L_{3/2,1} = L_{3/2,1}^{(1)} = L_{3/2,1}^{(N)} = L_{3/2,1}^{\rm sc} = \frac{3}{16}.}$$

The optimal potentials for $L_{\gamma,d}^{(N)}$ is the set of N-solitons (links with the Korteweg-de-Vries equation)

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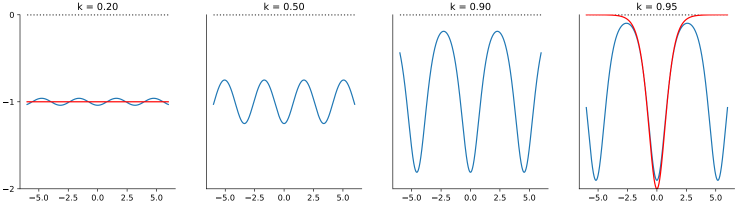
The optimal potentials for $L^{(N)}_{\gamma,d}$ is the set of N-solitons (links with the Korteweg-de-Vries equation)

Theorem (R.L. Frank, DG, M. Lewin (2021))

For all $0 < k < 1$, the potential $\boxed{V_k(x) := 1 + k^2 - 2k^2 {\rm sn}\,(x|k)^2}$ with minimal period $2K(k)$, is an optimizer for the periodic problem at $\gamma = 3/2$ and $d = 1$. Here, ${\rm sn}(\cdot|k)$ is the Jacobi elliptic function, and $K(\cdot)$ is the complete elliptic integral of the first kind. In addition,

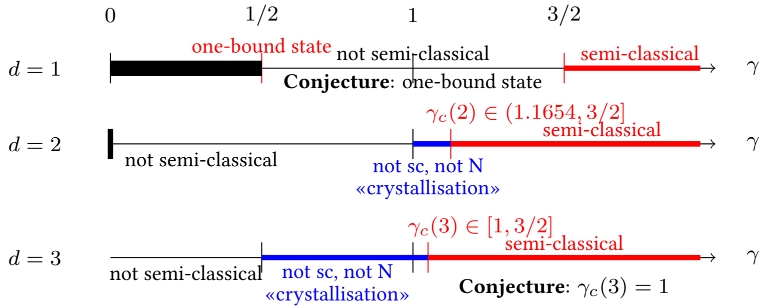
$$\lim_{k\to 0} V_k(x) = 1 \text{ (semi-classical)} \quad \text{and} \quad \lim_{k \to 1} V_k(x) = \frac{2}{\cosh^2(x)} \text{ (1-soliton).}$$

This potential is sometime called the periodic Lamé potential, or the cnoidal wave.



Known facts about Lieb-Thirring

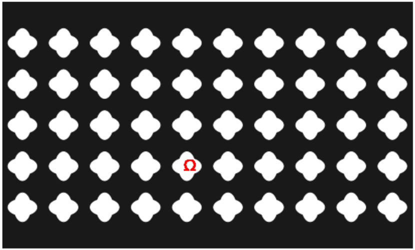
- $\gamma \mapsto L_{\gamma,d}/L^{\mathrm{sc}}_{\gamma,d}$ is decreasing (Aizenmann-Lieb, 1978), and ≥ 1.
  There is a unique point $\gamma_c(d) > 0$ so that $L_{\gamma,d} = L^{\mathrm{sc}}_{\gamma,d}$ iff $\gamma \geq \gamma_c(d)$.
- $\gamma = 3/2$ in dimension $d = 1$. $L_{\gamma,d} = L^{(N)}_{\gamma,d} = L^{\mathrm{sc}}_{\gamma,d} = \frac{3}{16}$. (Lieb-Thirring 1976).
- $\gamma \geq 3/2$ is semi-classical: $L_{\gamma,d} = L^{\mathrm{sc}}_{\gamma,d}$ for all $\gamma \geq \frac{3}{2}$. (Laptev-Weidl 2000) .
- $\gamma = 1/2$ in dimension $d = 1$. $L_{\frac{1}{2},1} = L^{(1)}_{\frac{1}{2},1}$. (Hundertmark-Lieb-Thomas, 1998).
- $\gamma < 1$ is not semi-classical. $L_{\gamma,d} > L^{\mathrm{sc}}_{\gamma,d}$ for all $\gamma < 1$. (Hellfer-Robert, 2010).



Appendix

A degenerate case

Consider $\Omega \subset \mathbb{R}^2$ a nice bounded set, and repeat it on a $\mathbb{Z}^2$ grid.
Consider $H = -\Delta$ on $L^2(\mathbb{R}^2)$, with Dirichlet boundary conditions «everywhere».



In the un-cut situation, the spectrum equals $\sigma\left(-\Delta|_\Omega\right)$, and each eigenvalue is of infinite multiplicities.

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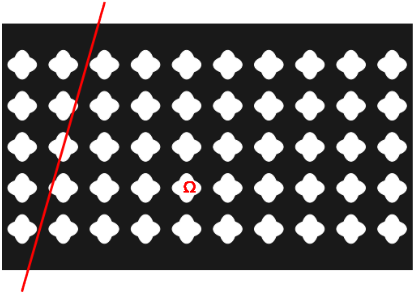

In the un-cut situation, the spectrum equals $\sigma\left(-\Delta|_\Omega\right)$, and each eigenvalue is of infinite multiplicities.
In the cut situation:

- If $\tan\theta \in \mathbb{Q}$, a finite number of new motifs appear, each one appears infinitely many times
  $\Longrightarrow$ finite number of new eigenvalues appear in each gap (all of infinite multiplicities)
- If $\tan\theta \notin \mathbb{Q}$, an infinite (countable) number of new motifs appear
  $\Longrightarrow$ pure-point spectrum everywhere.