Martingale property and moments in signature stochastic volatility models

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Outline

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1. Path signature and SigVol models
2. Martingale property and moments
3. Result and proofs

Signature stochastic volatility models

General non-Markovian stochastic volatility model, of the form

$$dS_t = S_t \sigma_t dB_t,$$

$$\sigma_t = \sigma\left(t, (B_s)_{0 \leqslant s \leqslant t}, (W_s)_{0 \leqslant s \leqslant t}\right),$$

with B, W independent Brownian motions.

How to choose $\sigma(\cdot)$?

Natural choice :

$$\sigma = \langle \boldsymbol{\sigma}, \mathbb{X}_t \rangle$$

where $\mathbb{X}_t$ is the signature of (t, B, W) on $[0, t]$.

Path signature : definition

Let $\mathcal{A}_d = \{1, 2, \dots, d\}$, and

$$V_n := \{\mathrm{i}_1 \cdots \mathrm{i}_n : \mathrm{i}_k \in \mathcal{A}_d \text{ for } k = 1, \dots, n\}. \tag{1}$$

$$T((\mathbb{R}^d)) = \left\{ \ell \ = \sum_{n \geqslant 0} \sum_{\mathrm{w} \in V_n} \langle \ell, \mathrm{w} \rangle \, \mathrm{w} \right\}.$$

Let $X : [0, T] \to \mathbb{R}^d$ be a path (deterministic, or continuous semi-martingale).
Signature of X (Chen '57) : $\mathbb{X}_t \in T((\mathbb{R}^d))$ with

$$\langle \mathbb{X}_t, \mathrm{i}_1 \cdots \mathrm{i}_n \rangle = \int_{0<u_1<\cdots<u_n<t} dX^{i_1}_{u_1} \circ \cdots \circ dX^{i_n}_{u_n}$$

(Stratonovich integration in the case of semi-martingales).
e.g.

$$\langle \mathbb{X}_t, \mathrm{i} \rangle = \int_0^t \circ dX^i_u = X^i_t - X^t_0, \ \ \langle \mathbb{X}_t, \mathrm{ij} \rangle = \int_0^t X^i_u \circ dX^j_u, \dots$$

Path signature : properties (1)

- For $w = vi$,

$$\langle \mathbb{X}_t, w \rangle = \int_0^t \langle \mathbb{X}_s, v \rangle \circ dX_s^i .$$

- Shuffle product $⧢$ on words :

$$w ⧢ \emptyset = \emptyset ⧢ w = w,$$

$$vi ⧢ wj = (v ⧢ (wj))i + ((vi) ⧢ w)j$$

e.g.

$$1 ⧢ 2 = 12 + 21, \quad (12) ⧢ 3 = 123 + 132 + 312$$

Then for any u, w,

$$\langle \mathbb{X}_t, v \rangle \langle \mathbb{X}_t, w \rangle = \langle \mathbb{X}_t, v ⧢ w \rangle .$$

Consequence :
any polynomial function of the signature is linear (iterated integrals $\approx$ monomials on path-space)

Path signature : properties (2)

- $\mathbb{X}_t$ determines X on $[0,t]$ up to reparametrization and tree-like equivalence (Chen '58, Hambly-Lyons '10, Boedihardjo et al. '16).

  For $\hat{X}_t = (t, X_t)$, $\widehat{\mathbb{X}_t}$ determines X on $[0,t]$.

- **Universality properties** : for instance, let $\mathcal{K} \subset C^{1-var}([0,t], \mathbb{R}^d)$ be compact, then

$$\left\{ X \mapsto \left\langle \widehat{\mathbb{X}_t}, \ell \right\rangle, \ \ \ell \in T(\mathbb{R}^d) \right\}$$

  is dense in $C(\mathcal{K}, \mathbb{R})$ (for uniform convergence).
  (Here $T(\mathbb{R}^d) = \cup_N T^{(N)}(\mathbb{R}^d)$,
  $T^{(N)}(\mathbb{R}^d) = \{\ell = \sum_{n \leqslant N} \sum_{\mathrm{w} \in V_n} \ell_{\mathrm{w}} \mathrm{w}\}$).)
  Similar results also hold in other topologies than C^{1-var} (e.g. rough path topologies).
  (In particular, $X \mapsto Y$ where $dY = V(Y) \circ dX$ may be approximated by linear functionals of the signature).

Path signature : properties (3)

- **Tractability** due to various algebraic properties.
  In particular by linearity,

$$\mathbb{E}\left[\left\langle \widehat{\mathbb{X}_t}, \ell \right\rangle\right] = \left\langle \mathbb{E}\left[\widehat{\mathbb{X}_t}\right], \ell \right\rangle,$$

with $\mathbb{E}\left[\widehat{\mathbb{X}_t}\right]$ being explicit in certain cases.
For instance, if $\hat{X}_t = (t, B_t)$ (B scalar B.M.),

$$\mathbb{E}\left[\widehat{\mathbb{X}_t}\right] = \exp^{\otimes}\left(t\left(1 + \frac{1}{2}22\right)\right)$$
$$= \emptyset + t(1 + \frac{1}{2}22) + \frac{t^2}{2}\left(11 + \frac{1}{2}122 + \frac{1}{2}221 + \frac{1}{4}\ 2222\right) + \dots$$

(Fawcett's formula)

Signature models in finance 1/2

Volatility models of the form

$$\Sigma_t = \langle \sigma, \mathbb{X}_t \rangle$$

where $\mathbb{X}_t$ is the signature of an auxiliary process $\hat{X} = (t, X)$.

Note : $\sigma \in T^{(N)}(\mathbb{R}^d) \to (d^{N+1} - 1)/(d - 1)$ parameters

- **Perez Arribas, Salvi, Szpruch '20** $\hat{X} = (t, B)$ (time-augmented scalar BM). σ truncated at $N = 4$ (31 parameters). Numerical experiments (calibration, simulation, pricing)
- **Cuchiero, Gazzani, Svaluto-Ferro '22**. Further theoretical study of the models.
  $\hat{X} = (t, B, W)$ (multi-dimensional BM).
  σ truncated at $N = 2, 3, 4$, 10^2 to 10^5 parameters.

(In both these papers $dS_t = \Sigma_t dB_t$).

Signature models in finance 2/2

$$dS_t = S_t \Sigma_t dB_t,$$

$$\Sigma_t = \left\langle \sigma, \widehat{\mathbb{X}_t} \right\rangle$$

- **Cuchiero, Gazzani, Svaluto-Ferro '23**. Joint calibration of SPX/VIX options.
  Uses auxiliary processes $\hat{X} = (t, B, X)$, X Ornstein-Uhlenbeck processes (3-dimensional). Explicit formulae for VIX option prices in this model ("polynomial diffusions").
  σ truncated at $N = 3$, 81 parameters.

- **Abi Jaber, Gérard '24**. Pricing via Fourier, using affine structure and (infinite-dimensional) Ricatti equations (σ possibly infinite).
  $\mathbb{X}_t$ signature of (t, W), W a B.M. with $d\langle W, B\rangle = \rho dt$.
  Numerics for calibration : truncated at $N = 3$, 13 parameters.

Outline

Martingale property : discussion

Stochastic volatility model, price dynamics (no interest rate, under pricing measure).

$$dS_t = S_t \sigma_t dB_t.$$

S clearly always **local** martingale (and a supermartingale).
Q : is it a **true martingale** ? (Equivalent to $\mathbb{E}[S_t] = S_0$, for all $t \geqslant 0$).

Note : if S is a strict local martingale, then :

- The "model price" for holding the stock ($\mathbb{E}[S_t]$)until time t is $<$ market price ($= S_0$).
  Technically not an arbitrage opportunity in the classical (e.g. NFLVR) sense : shorting the stock is not an 'admissible strategy".
- Put-call parity does not hold.
- Has been suggested as a model for bubbles (Protter and co-authors).
- In any case : clearly a **pathological** model, should not be used for day-to-day activities ! (and it is important to rule this out when introducing a model).

Martingale property : literature

- Novikov criterion

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \sigma_s^2 ds\right)\right] < \infty$$

  can sometimes be used (Heston,...) but useless e.g. when σ has **superlinear** growth in a Gaussian process.
- Classical results. Sin '98, Jourdain '04, Lions Musiela '07, Bernard Cui McLeish '17,... for Markovian models. Typical result :

$$dS_t = S_t\sigma_t dB_t, \quad d\sigma_t = \alpha\sigma_t dW_t, \quad d\langle B, W\rangle_t = \rho dt,$$

  then

$$S \text{ true martingale} \quad \Leftrightarrow \quad \rho \leqslant 0. \tag{2}$$

- Few results in non-Markovian case. G. '19 rough Bergomi model,

$$\sigma_t = \exp\left(\int_0^t (t-s)^{H-1/2} dW_s\right), \quad d\langle B, W\rangle_t = \rho dt,$$

  same result (2) as above.

Existence of moments

For which $m > 1$, $T > 0$ does it hold that $\mathbb{E}[S_T^m] < \infty$?
($m = 1$ is clear by supermartingale property).
Importance :

- Monte Carlo error : CLT requires finite variance

$$\frac{1}{M}\sum_{i=1}^{M}(S_T^i - K)_+ - \mathbb{E}[(S_T - K)_+] \text{ of order } \frac{1}{\sqrt{M}} \text{ iff } \mathbb{E}[S_T^2] < \infty.$$

- Asymptotic formula : to go from LDP estimates

$$\mathbb{P}(S_t > K) \approx_{t\to 0} \exp\left(-\frac{I(K) + o(1)}{t^\alpha}\right)$$

to call price asymptotics

$$\mathbb{E}[(S_t - K)_+] \approx_{t\to 0} \exp\left(-\frac{I(K) + o(1)}{t^\alpha}\right)$$

(or finer asymptotics) requires some moments, e.g.$\exists m > 1$, $\mathbb{E}[S_t^m] < \infty$ for all $t > 0$ (Friz-G.-Pigato '19).

Moments : literature

- When volatility has Gaussian tails (e.g. Heston)

$$\forall m > 1, \ \ \exists T(m) > 0, \quad \forall t \leqslant T(m), \mathbb{E}[S_t^m] < \infty.$$

- 'superlinear' SDE models : Jourdain '04, Andersen Piterbarg '07, Lions Musiela '07.

$$dS_t = S_t \sigma_t dB_t, \ \ d\sigma_t = \alpha \sigma_t dW_t, \ \ d\langle B, W \rangle_t = \rho dt,$$

then, for $\rho \leqslant 0$, for any $t > 0$,

$$\mathbb{E}[S_t^m] < \infty \ \ \Leftrightarrow \ \ m \leqslant \frac{1}{1-\rho^2}$$

- Rough Bergomi model G. '19 Gulisashvili '19,

$$\sigma_t = \exp\left(\int_0^t (t-s)^{H-1/2} dW_s\right), \quad d\langle B, W \rangle_t = \rho dt,$$

Implication $\Rightarrow$ above holds.

Outline

Martingale property in truncated Sigvol model

Stock price dynamics

$$dS_t = S_t \sigma_t dB_t,$$

where

$$\sigma_t = \langle \boldsymbol{\sigma}, \mathbb{X}_t \rangle$$

and $\mathbb{X}_t$ is the signature of $(t, B_t, W_t^i; i = 1, \ldots, d)$, where B, W^i are independent Brownian motions, and

$$\boldsymbol{\sigma} = \sum_{0 \leqslant n \leqslant N} \sum_{\mathrm{w} \in V_n} \sigma_{\mathrm{w}} \mathrm{w},$$

where words are over the alphabet $\{1, 2, 3, \ldots, \mathrm{d} + 2\}$.

Results : martingale property

(Recall that 2 corresponds to integrals in dB, where B drives S).

Theorem

Assume that $N \geqslant 2$ and $\boldsymbol{\sigma}$ is such that $\sigma_{2^{\otimes N}} \neq 0$.
Then

$$S \text{ is a true martingale} \iff N \text{ is odd and } \sigma_{2^{\otimes N}} < 0.$$

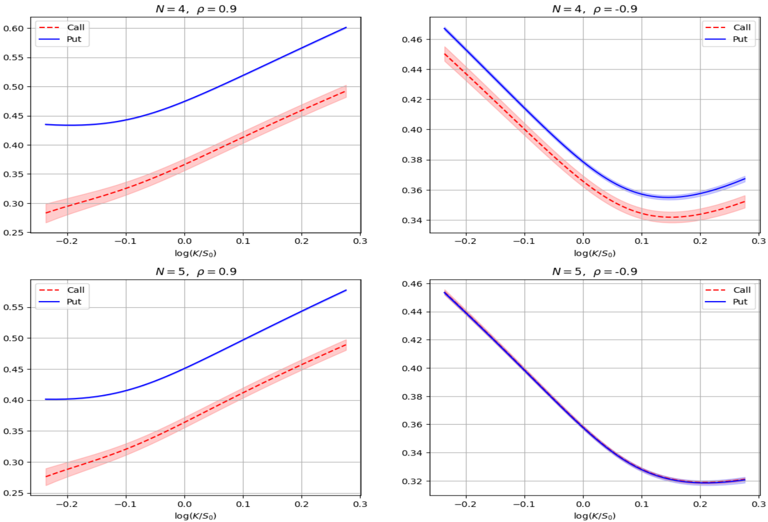
Note : as a special case, in the Abi Jaber-Gérard model

$$\sigma_t = \langle \boldsymbol{\sigma}, Sig(t, \rho B + \bar{\rho} W) \rangle ,$$

then if $\sigma_{2^{\otimes N}} \neq 0$,

$$S \text{ is a true martingale} \iff N \text{ is odd and } \rho \sigma_{2^{\otimes N}} < 0.$$

Numerical illustration



Results : moments

Consider the Abi Jaber-Gérard model

$$dS_t = S_t\sigma_t(\rho dB_t + \sqrt{1-\rho^2}dW_t),$$

$$\sigma_t = \langle \boldsymbol{\sigma}, Sig(t,B)_t \rangle,$$

with $\boldsymbol{\sigma} \in T_N(\mathbb{R}^d)$, $N \geqslant 3$ is odd and

$$\sigma_{2^{\otimes N}} > 0, \quad \rho \leqslant 0.$$

Theorem

For any $T > 0$, it holds that

$$m < \frac{1}{1-\rho^2} \quad \Rightarrow \quad \mathbb{E}[S_T^m] < \infty,$$

$$m > \frac{1}{1-\rho^2} \quad \Rightarrow \quad \mathbb{E}[S_T^m] = \infty.$$

Idea of proofs : martingale (1/2)

- Classical idea (Sin '98, see Ruf '15 for a general statement): for $\sigma_t = \sigma(t, B)$, it holds that

$$\mathbb{E}[S_T] = S_0 \quad \Leftrightarrow \quad \text{almost surely, } \int_0^T \sigma(t, X)^2 dt < \infty,$$

where X is solution to the fixed point equation

$$dX_t = dB_t + \sigma(t, X) dt.$$

(Follows from Girsanov, with weight $S_{T \wedge \tau_n}/S_0$)

- In our context : explosion / no-explosion of solution to

$$dX_t = dB_t + \langle \boldsymbol{\sigma}, Sig(t, X, W) \rangle \, dt.$$

(SDE with signature drift and additive noise)

Idea of proofs : martingale (2/2)

- Assumption on $\boldsymbol{\sigma}$:

$$dX_t = -cX_t^N dt + (l.o.t.) + dB_t,$$

and explosion can be related to the sign of the leading coefficient.

- More precisely, a crucial lemma shows that the other terms are of lower order in expectation, e.g.

$$\int_0^t X_s dW_s \lesssim_{L^p} \sup_{s \leqslant t} |X_s|$$

$$\begin{aligned} &\int_{0 \leqslant u_1 \leqslant u_2 \leqslant u_3 \leqslant t} dX_{u_1} dW_{u_2} dX_{u_3} \\ &= \left(\int_0^t X_u dW_u\right)\left(\int_0^t dX_u\right) - \int_0^t X_u^2 dW_u \\ &\lesssim_{L^p} \sup_{s \leqslant t} |X_s|^2. \end{aligned}$$

(and similarly for arbitrary words by induction on word length and some shuffle identities)

Idea of proofs : moments (1/2)

$$dS_t = S_t\sigma_t(\rho dB_t + \sqrt{1-\rho^2}dW_t),$$
$$\sigma_t = \langle \boldsymbol{\sigma}, Sig(t,B)_t \rangle,$$

- After conditioning,

$$\mathbb{E}[S_T^m]/S_0^m = \mathbb{E}\left[\exp\left(\int_0^T C_{\rho,m}\sigma(t,B)^2 dt + \rho m \sigma(t,B)dB_t\right)\right],$$

where $C_{\rho,m} = \frac{(1-\rho^2)m^2-m}{2}$ (assumption on $\rho, m \leftrightarrow$ sign of C_m).

- As in G. '19, apply Boué-Dupuis formula

$$\log \mathbb{E}\exp(F(B)) = \sup_v \mathbb{E}\left[F\left(B + \int_0^\cdot v\right) - \frac{1}{2}\int v^2\right]$$

to obtain that finiteness of the moments is equivalent to that of

$$\mathcal{V} := \sup_{(v_t)\text{ adapted}} \mathbb{E}\left[\int_0^T \left(C_{\rho,m}\sigma\left(t, B + \int_0^\cdot v\right)^2 + \rho m \sigma\left(t, B + \int_0^\cdot v\right)v_t - \frac{1}{2}v_t^2\right)dt\right]$$

Idea of proofs : moments (2/2)

- The case where $C_{m,\rho} > 0$: use feedback controls and explosion as in martingale part to obtain $\mathcal{V} = +\infty$
- If $C_{m,\rho} < 0$: letting $V = \int_0^\cdot v$, we can rewrite

$$\sigma(t, B+V) = \langle \widetilde{\boldsymbol{\sigma}}, Sig(t, B, V)_t \rangle = \sigma_{2^{\otimes N}} V_t^N + (l.o.t.)$$

and as a result

$$\mathbb{E}\left[\int_0^T C_{\rho,m}\sigma(t, B+V)^2 dt\right] = \mathbb{E}\left[-c_1 \int_0^T V_t^{2N} dt + (l.o.t.)\right] \leqslant C_1,$$

and

$$\mathbb{E}\left[\int_0^T \rho m \sigma(t, B+V) dV_t\right] = \mathbb{E}[-c_2 V_T^{N+1} + (l.o.t.)] \leqslant C_2,$$

so that $\mathcal{V} < \infty$.

Conclusion

We prove sharp conditions on martingale / moments in SigVol models. Correlation (and leading coefficient) matter. So does (parity of the) **truncation order**.

Still not completely general result :

- More general driving processes (e.g. OU as in Cuchiero et al. ?)
- Moments for more general models ? (not clear there would be a simple condition)

Methodology : here, we can treat the non-Markovian case as perturbation of the Markovian one. Could we obtain robust **genuinely non-Markovian** methods ? (could maybe help for **moments in rough Bergomi** ?)