Université Paris-Dauphine
Master Masef
Année 2023-2024

Written exam : Evaluation des actifs financiers et arbitrage
December 11, 2023 — 2h.

Exercise 1 : No-shortselling constraint in a one-period binomial model

We consider a classical one-period binomial market. The underlying probability space is $\Omega = \{\omega_u, \omega_d\}$, with $\mathbb{P}(\omega_u) = \mathbb{P}(\omega_d) = 1/2$. The market consists in two tradable assets : a risk-free asset with price values $B_0 = 1$ at time 0 and $B_1 = 1 + r$ at time 1, for some fixed $r > 0$, and a risky asset with price $S_0 = 1$, and $S_1(\omega_u) = 1 + u$, $S_1(\omega_d) = 1 + d$ with $-1 < d < u$.

In this market, a trading strategy (for a self-financing portfolio) is simply a number $\phi \in \mathbb{R}$, representing the number of units of the risky asset held in the portfolio between times 0 and 1. We assume that **no shortselling in the risky asset is allowed** in this market, which means that the only admissible strategies are $\phi \in \mathbb{R}_+$.

We then denote by $X_1^{x,\phi}$ the value at time 1 of a portfolio starting with wealth x at time 0 and trading with strategy ϕ.

As usual, if X_i, $i = 0, 1$ is a stochastic process we will call $\tilde{X}$ its discounting, i.e. $\tilde{X}_0 = X_0$ and $\tilde{X}_1 = X_1/(1 + r)$.

The No Arbitrage condition in this context is

$$\forall \phi \geq 0, \quad \left(X_1^{0,\phi}(\omega_i) \geq 0 \text{ for } i = u, d\right) \Rightarrow \left(X_1^{0,\phi}(\omega_u) = X_1^{0,\phi}(\omega_d) = 0\right). \qquad (NA^+)$$

(1) For any $x \in \mathbb{R}$ and $\phi \geq 0$, give a formula for $X_1^{x,\phi}$. Show that (NA^+) implies $d < r$.

(2) We let

$$\mathcal{M}_{super}(\tilde{S}) = \left\{\mathbb{Q} \sim \mathbb{P} \text{ such that } \mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_1\right] \leq S_0\right\}.$$

(2a) Compute all the possible $\mathbb{Q}$ in $\mathcal{M}_{super}(\tilde{S})$ (parametrized by $q = \mathbb{Q}(S_1 = 1 + u)$).

(2b) Show that $(NA^+) \Leftrightarrow \mathcal{M}_{super}(\tilde{S}) \neq \emptyset \Leftrightarrow d < r$.

We now assume that $d < r$ and consider an option with payoff $G = g(S_1)$, for a given function $g : \{1 + u, 1 + d\} \to \mathbb{R}$.

(3) The superreplication price in this market is defined by

$$p^+(G) = \inf\left\{p \in \mathbb{R},\ \exists \phi \geq 0,\ \ X_1^{p,\phi} \geq G\right\}.$$

(3a) We define $\theta^+(G) = \sup_{\mathbb{Q} \in \mathcal{M}_{super}(\tilde{S})} \mathbb{E}^{\mathbb{Q}}\left[\tilde{G}\right]$. Show that

$$(1 + r)\theta^+(G) = \begin{cases} g(1 + d) & \text{if } g(1 + d) > g(1 + u), \\ g(1 + u) & \text{if } g(1 + d) \leq g(1 + u) \text{ and } r > u, \\ \frac{(r-d)g(1+u)+(u-r)g(1+d)}{u-d} & \text{otherwise.} \end{cases}$$

(3b) Show that $p^+(G) \geq \theta^+(G)$. Give an admissible strategy ϕ such that $X_1^{\theta^+(G),\phi} \geq G$. Conclude that $p^+(G) = \theta^+(G)$.

(3c) Under which condition is the claim G exactly replicable in this market ? In that case, is the price necessary for replication the same as the super-replication price ?

(4) (4a) Give the definition of a viable price for G in this market (assuming that both buyer and seller of the option are faced with the no-shortselling constraint).

(4b) Show that the set of viable prices is given by an interval I with upper bound $p^+(G)$ and lower bound $-p^+(-G)$. Further show that

$$I = \left\{ \mathbb{E}^{\mathbb{Q}}\left[\tilde{G}\right], \quad \mathbb{Q} \in \mathcal{M}_{super}(\tilde{S}) \right\}$$

and discuss whether $p^+(G)$ and $-p^+(-G)$ are viable prices. Is there any relation between whether G is exactly replicable and the form of I ?

(5) (5a) Recall (without proof) the results seen in class for questions 2-3-4 in the case of the unconstrained model ($\phi \leq 0$ allowed). Compare with the above results.

(5b) Assume $d < r < u$. Show that one can find a function $\hat{g}$ such that the replication price $p^+(G)$ in the constrained market is equal to the price *in the unconstrained market* of an option with payoff $\hat{g}(S_1)$.

Exercise 2 : Asian option pricing

Let us consider a continuous time market on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a single risky asset $S = (S_t)_{0 \le t \le T}$, and a riskfree asset with interest rate $r = 0$. The goal of this exercise is to discuss pricing and hedging of asian options whose payoff is given by

$$G := g(A_T), \quad \text{with } A_T := \int_0^T S_t dt \text{ and some function } g : \mathbb{R}_+ \to \mathbb{R}.$$

Part I : Black-Scholes model

In this section, we assume that S follows the dynamic

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and $\sigma > 0$ are constants, W is a standard Brownian motion under $\mathbb{P}$.

(1) Recall how to construct a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that

$$dS_t = \sigma S_t d\bar{W}_t,$$

where $\bar{W}$ is a Brownian motion under $\mathbb{Q}$.

(2) Let $u : [0, T] \times \mathbb{R}_+^2 \to \mathbb{R}$ be a smooth function, and define $A_t := \int_0^t S_r dr$ so that $dA_t = S_t dt$. Apply the Itô formula on $u(t, S_t, A_t)$ to obtain the dynamic of

$$du(t, S_t, A_t)$$

(in terms of dt and $d\bar{W}_t$).

(3) Let us define

$$p(t, s, a) := \mathbb{E}^{\mathbb{Q}}\big[g(A_T)\big|S_t = s, A_t = a\big].$$

Assume that $p \in C^{1,2}([0, T) \times \mathbb{R}_+ \times \mathbb{R}_+) \cap C([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$, prove that p satisfies the following equation:

$$\partial_t u + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} u + s\partial_a u = 0, \qquad u(T, s, a) = g(a).$$

(4) Given a smooth solution u to the above PDE, provide a replicating portfolio for the payoff G, and justify it.

Part II: Robust hedging.

In this section, we only assume that S is a continuous semi-martingale under $\mathbb{P}$, and we want to obtain hedges which do not rely on further assumptions on the model.

(5) We first consider the case when $g(A_T) = A_T$. Using integration by parts, show that

$$A_T = TS_0 + \int_0^T (T - u) dS_u$$

and deduce a replicating strategy for that particular payoff.

(6) We now consider the case when $g(A_T) = A_T^2$ and we will use (without proof) that

$$A_T^2 = 2\int_0^T S_t^2(T-t)dt + 2\int_0^T A_t(T-t)dS_t.$$

(i) Recall the definition of a semi-static hedging strategy.

(ii) Show that for all $x \geq 0$,

$$x^2 = 2\int_0^\infty (x-k)_+ \, dk.$$

(iii) Assuming that call options of all possible maturities $t \in [0,T]$ and strikes $k > 0$ may be bought or sold at time 0 (at price $C(t,k)$), deduce a semi-static replicating strategy for the payoff A_T^2 (give the price as well as the strategy).

(7) We now consider the payoff of an Asian call option

$$g(A_T) = (A_T - K)_+$$

with $K > 0$ fixed.

(i) Show that for each $0 < L < K$, it holds that

$$\forall A \in \mathbb{R}_+, \quad \frac{(A-L)^2}{4(K-L)} \geq (A-K)_+.$$

(ii) Deduce from the previous questions semi-static super-hedging strategies for this payoff.