On Periodic Solutions of N-Vortex Problem

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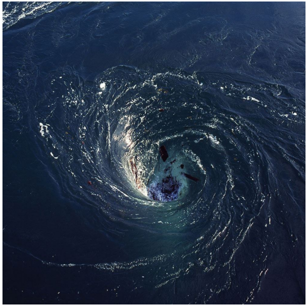
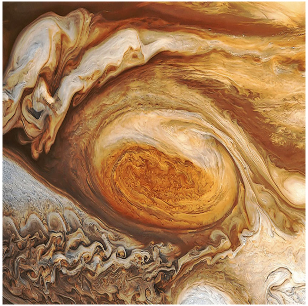
1. Vortex Problem as a Hamiltonian System

Vortices as we might see in real life:



(a) A vortex in Atlantic Ocean



(b) The Jupitor Red Spot

(c) When One flushes the Toilet

Figure 1: Some Examples of Vortices

Vortex System

- The study of vortices goes back to Helmholtz since 1858

Figure 2: Hermann Von Helmholtz, 1821-1894

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Figure 2: Hermann Von Helmholtz, 1821-1894

- Its Hamiltonian structure is first formulated by Kirchhoff in 1876

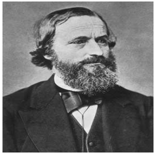

Figure 3: Gustav Robert Kirchhoff, 1824-1887

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$$\begin{cases} \Gamma_i \dot{x}_i(t) = \frac{\partial H}{\partial y_i} \\ \Gamma_i \dot{y}_i(t) = -\frac{\partial H}{\partial x_i} \end{cases} \tag{1}$$

with

$$H = -\frac{1}{4\pi} \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j \log |z_i - z_j|^2$$

Hamiltonian Structure of Vortex Dynamics

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$$H = -\frac{1}{4\pi} \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j \log |z_i - z_j|^2$$

- The energy surface is neither compact, nor convex

Define the Poisson Bracket

$$\{f, g\} = \sum_{1 \le i \le N} \frac{1}{\Gamma_i}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right)$$

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- The system is invariant under translation

$\Rightarrow X = \sum_{1 \leq i \leq N} \Gamma_i x_i = CST, Y = \sum_{1 \leq i \leq N} \Gamma_i y_i = CST$

- The system is invariant under rotation

$\Rightarrow I = \sum_{1 \leq i \leq N} \Gamma_i |z_i|^2 = CST$

Integrability

- There are three independent first integrals in involution: $H, I, P^2 + Q^2$

$$\{H, P^2 + Q^2\} = 0, \quad \{H, I\} = 0, \quad \{I, P^2 + Q^2\} = 0$$

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- The 3-vortex problem is integrable.
- The N-vortex problem is in general not integrable when $N > 3$
(S. Ziglin 1980; J. Koiller and S. P. Carvalho 1989).

Variational Approach for Hamiltonian System

- Analogy in Celestial Mechanics
(A. Chenciner and R. Montgomery, 2000)

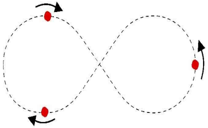

Figure 4: The eight-figure curve

Motivation

- Analogy in Celestial Mechanics
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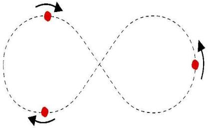

Figure 4: The eight-figure curve

- Give an example that linking techniques apply to Superquatratic Hamiltonian with physical background

- Let $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ denote the set of 2N-tuples of 2π periodic functions which are square integrable. The Fourrier expansion hence exists, i.e., for $z \in L^2(\mathbb{S}^1, \mathbb{R}^{2N})$,

$$\mathbf{z} = \sum_{k \in \mathbb{Z}^{2n}} a_k e^{ikt}$$

Define the norm

$$||\mathbf{z}||_{W^{2,p}} = (\sum_{k \in \mathbb{Z}} (1 + |k|^{2p})|a_k|^2)^{\frac{1}{2}}$$

It has been noticed that a proper functional space for Hamiltonian system is the space $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n})$, where

$$H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n}) = \{\mathbf{z}(t) \in H^{\frac{1}{2}}(\mathbb{S}^1; \mathbb{R}^{2n}) | \, \mathbf{z}(0) = \mathbf{z}(T)\}$$

- The space $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n})$ admits the following decomposition $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n}) = E^+ \oplus E^- \oplus E^0$:

$$\begin{aligned}
E^+ &= \mathit{span}\{(\sin\frac{2\pi jt}{T})e_k - (\cos\frac{2\pi j}{T}t)e_{k+n}, (\cos\frac{2\pi j}{T}t)e_k + (\sin\frac{2\pi j}{T}t)e_{k+n}\} \\
E^- &= \mathit{span}\{(\sin\frac{2\pi j}{T}t)e_k + (\cos\frac{2\pi j}{T}t)e_{k+n}, (\cos\frac{2\pi j}{T}t)e_k - (\sin\frac{2\pi j}{T}t)e_{k+n}\} \\
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$$E^- = \mathit{span}\{(\sin \frac{2\pi j}{T}t)e_k + (\cos \frac{2\pi j}{T}t)e_{k+n}, (\cos \frac{2\pi j}{T}t)e_k - (\sin \frac{2\pi j}{T}t)e_{k+n}\}$$

$$E^0 = \mathit{span}\{e_1, e_2, ..., e_{2n}\}$$

- Given the vortex Hamiltonian function, we define the following functionals for variational argument,

$$\forall z(t) \in H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n}), \quad \mathcal{A}(z) = \int_0^T y\Gamma dx dt$$

$$\mathcal{H}(\mathbf{z}) = \int_0^T H(x, y) dt$$

$$\mathcal{I}_H(\mathbf{z}) = \mathcal{A}(z) - \mathcal{H}(z) = \int_0^T y\Gamma dx - H(x, y) dt$$

Variational Formulation

- Given $\mathbf{z} = \mathbf{z}^+ + \mathbf{z}^- + \mathbf{z}^0$, and $\Gamma_i > 0, \forall 1 \leq i \leq n$

$$\mathcal{A}(z^+) > 0, \mathcal{A}(z^-) < 0, \mathcal{A}(z^0) = 0$$

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- One can define an equivalent norm $||.||_E \simeq ||.||_{H_T^{\frac{1}{2}}(\mathbb{S}^1,\mathbb{R}^{2n})}$, where

$$||.||_E^2 := \mathcal{A}(\mathbf{z}^+) - \mathcal{A}(\mathbf{z}^-) + |\,\mathbf{z}^0\,|^2$$

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- The subspaces E^+, E^-, E^0 are mutually orthogonal not only in $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n})$ but also in $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$

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$$H_1 = \prod_{i,j=1,i<j}^{N} |z^i - z^j|^2$$

$$H_2 = \prod_{i,j=1,i<j}^{N} |z^i - z^j|^2 + f(I(z))$$

where $f(\lambda) = \mu\lambda^k$, for an integer $k > 0$ fixed large enough whose value is to be precised later on.while

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- $H_0 \to H_1$ replaces collision singularity by fixed point
- $H_1 \to H_2$ ensures the compactness: the validity of Palais-Smale condition

Strategy

$$H_0 = \sum_{i,j=1,i<j}^{N} \log |z^i - z^j|^2, H_1 = \prod_{i,j=1,i<j}^{N} |z^i - z^j|^2, H_2 = \prod_{i,j=1,i<j}^{N} |z^i - z^j|^2 + f(I(z))$$

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- 1. We show that $\mathcal{J}_{H_2}$ possesses a critical point $\mathbf{z}_{H_2}$ in $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n})$ by the construction of topological linking,Standard argument then shows that this critical point is indeed a classical solution $\mathbf{z}_{H_2}$ of the Hamltonian H_2

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- 4. Now by taking logarithm of H_1 (which is a legal operation when $H_1 \neq 0$), $\mathbf{z}_{H_1}$ will become, after a reparametrization of time, a relative periodic solution $\mathbf{z}_{H_0}$ for H_0

Existence of Critical Point for $\mathcal{J}_{H_2}$

Subquadratic Hamiltonian

Theorem (Rabinowitz-Benci, 1979)

Suppose that the Hamiltonian H *is of class* $\mathcal{C}^1(\mathbb{S}^1, \mathbb{R}^{2n})$ *and satisfies that*

1. $H(z) > 0$
2. $H(z) = o(||z||^2)$ *when* $||z|| \to 0$
3. $\exists r > 0$ *and* $\mu > 2$ *s.t.* $0 < \mu H(z) \leq \nabla H(z), z >$ *when* $||z|| > r$

Then for any $T > 0$ *the Hamiltonian system has a non-constant T-periodic solution.*

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- This critical point is characterized by minmax method through topological linking.
- Apply this to H_2, we can find a non-constant periodic solution $\mathbf{z}_{H_2}$.
- It can be proved that the corresponding critical value c satisfies that

$$c \leq (1 + \epsilon_k)^2 \pi$$

for a small ϵ_k depending on k

From H_2 to H_1

$$H_1 = \prod_{i,j=1,i<j}^{N} |z^i - z^j|^2$$

$$H_2 = \prod_{i,j=1,i<j}^{N} |z^i - z^j|^2 + f(I(z))$$

- Note that in our setting

$$X_{H+f(I)} = \mathbb{J}\nabla(H + f(I)) = X_H + X_{f(I)}$$

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- $\{f(I), H\} = 0$, as a result $[X_H, X_{f(I)}] = 0$
- $\mathbf{z}_{H_2}$ induces a relative T-periodic solution $\mathbf{z}_{H_1}$ for H_1

From H_1 to H_0

- If $\mathbf{z}_{H_1}$ does not have any collision, then it is, up to a reparametrization of time, a relative T_0-periodic solution $\mathbf{z}_{H_0}$ of the Hamltonian H_0.

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- Adding a rotationing frame work does not change the mutual distances.
- If $\mathbf{z}_{H_2}$ does not have any collision, then it is, up to a reparametrization of time, a relative T_0-periodic solution $\mathbf{z}_{H_0}$ of the Hamltonian H_0.

Exclusion of Collision for z_{H_2}

What if a Collision Happened

- Suppose that there is a collision. It implies that $\nabla H_1 = 0$, and $\mathbf{z}_{H_2}$ becomes a centered uniform rotation.

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- The critical value satisfies that

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\begin{aligned}
c = I_{H_2}(\mathbf{z}_{H_2}) &= \int_0^T y\,dx - H_2(\mathbf{z}_{H_2})dt \\
&= \frac{T\omega}{2} I(\mathbf{z}_{H_2}) - Tf(I(\mathbf{z}_{H_2})) \\
&= \frac{T\omega}{2} I(\mathbf{z}_{H_2}) - T\frac{1}{k} f'(I(\mathbf{z}_{H_2}))I(\mathbf{z}_{H_2}) \\
&= \frac{T\omega}{2} I(\mathbf{z}_{H_2}) - \frac{1}{k}\frac{T\omega}{2} I(\mathbf{z}_{H_2}) \\
&= \frac{T\omega}{2} I(\mathbf{z}_{H_2})(1 - \frac{1}{k}) \\
&= m\pi I(\mathbf{z}_{H_2})(1 - \frac{1}{k})
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What if a Collision Happened

- The collision cannot happen when $I(\mathbf{z}_{H_2})$ too small. If $I(\mathbf{z}_{H_2}) \leq 1$, then :

$$
\begin{aligned}
|\omega| = 2\frac{df}{dI}(I(\mathbf{z}_{H_2})) &= \mu k I^{k-1} \\
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- We conclude that $I(\mathbf{z}_{H_2}) > 1$.

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- The collision cannot happen when $I(\mathbf{z}_{H_2})$ too small. If $I(\mathbf{z}_{H_2}) \leq 1$, then :

$$|\omega| = 2\frac{df}{dI}(I(\mathbf{z}_{H_2})) = \mu k I^{k-1}$$
$$\leq \frac{\alpha}{T} < \frac{2\pi}{T}$$

- We conclude that $I(\mathbf{z}_{H_2}) > 1$.

$$c = m\pi I(\mathbf{z}_{H_2})(1 - \frac{1}{k}) > m\pi(1 - \frac{1}{k})$$

- Recall that

$$c \leq (1 + \epsilon_k)^2 \pi$$

Lemma

Suppose that the solution $\mathbf{z}_{H_2}$ that we have found ***does*** *have a collision, then this solution must verify that T is its minimal period.*

Symmetry and Palai's Principle

Theorem (Palais, 1979)

(Palai's Principle of symmetric criticality)Let $\boldsymbol{G}$ *be a group of isometries of a Riemannian manifold* $\boldsymbol{M}$ *and let* $f : \boldsymbol{M} \to \boldsymbol{R}$ *be a* $\mathcal{C}^1$ *function invariant under* $\boldsymbol{G}$*. Then the set* Γ *of stationary points of* $\boldsymbol{M}$ *under the action of* $\boldsymbol{G}$ *is a totally geodesic smooth submanifold of* $\boldsymbol{M}$*, and if* $p \in \Gamma$ *is a critical point of* $f|\Gamma$ *then* p *is in fact a critical point of* f

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- Let G be a finite subgroup of $O(2) \times \Sigma_N \times O(2)$. Let Λ be T-periodic loops in the configuration space of our vortex system (Note that for the vortex problem, the configuration space coincides with the phase space). Let $g = (\tau, \sigma, \rho) \in G$ acts on $z(t) = (z_1(t), z_2(t), ..., z_n(t)) \in \Lambda$ be such that:

$$gz_i(t) = \rho y_{\sigma^{-1}(j)}(\tau^{-1}(t))$$

In the special case, let $\rho = I$, $\sigma^{-1}(j) = j - 1$, with the convention that $z_n = z_{0-1}$ $\tau^{-1}(t) = t - \frac{T}{n}$, then the group thus generated is called the **group of choregraphy**

Choregraphy

Consider the simple choregraphy of N vortices

$$z_i\left(t + \frac{T}{N}\right) = z_{i-1}(t), \quad i = 1, 2, \ldots, N$$

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Consider the simple choregraphy of N vortices

$$z_i(t + \frac{T}{N}) = z_{i-1}(t), \quad i = 1, 2, ..., N$$

This gives us a solution $\mathbf{z}_{H_2}$ that is a simple choregraphy - Suppose to the contrary that $\mathbf{z}_{H_2}$ has a collision. Then it becomes a uniform rotation with $T^* = T$. Moreover, Without loss of generality we could assume the collision involes $z^1_{H_2}$, i.e.,

$$z^i_{H_2}(t) = z^1_{H_2}(t), \forall 1 \leq i \leq N$$

Now by the definition of choregraphy again, we see that $\forall t \in [0, T]$

$$z^{2i-1}_{H_2}(t + \frac{T(i-1)}{N}) = z^i_{H_2}(t) = z^1_{H_2}(t) = z^i_{H_2}(t + \frac{T(i-1)}{N})$$

It turns out that

$$z^{2i-1}_{H_2}(t) = z^i_{H_2}(t) = z^1_{H_2}(t)$$

- It is clear how we can define an equivalent class for vortices collided in this way. The index of vortices in one equivalent class will be a subgroup of the cyclic group S^N, thus each equivalent will at least have two elements. Dividing $\mathcal{S}^1$ parameterized by $[0, T]$ into two equal parts $[0, \frac{T}{2}]$ and $[\frac{T}{2}, T)$. Now by Pigeonhole principle there must be at least two elements falling into the same part, i.e., the time gap is less or equal to $\frac{T}{2}$. In other words, any collision will imply that

$$T^* \leq \frac{T}{2}$$

In other words, the collision will lead to $m \geq 2$

Choregraphy

- We have thus proved the following theorem:

Theorem

$\forall N \in \mathbb{N}^*$, *the identical N-vortex system has a relative periodic choregraphy.*

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$\forall N \in \mathbb{N}^*$, *the identical N-vortex system has a relative periodic choregraphy.*

- Is the solution the Thomson's N-polygon?...

Thank you!

References

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