Geometric Quantization, Semi-classical limits, and Formal quantization : work in common with Paul-Emile Paradan

SUMMARY :

1) G torus acting on M compact even dimensional manifold :
L line bundle, $\Phi : M \to \mathfrak{g}^*$ moment map.
Semi-classical behavior of geometric quantization.

2) G compact connected, M non necessarily compact but Φ proper

3) The semi-classical behavior determines the quantization :
Application to formal quantization.

Moment map and Marsden-Weinstein reduction

(M, Ω) compact symplectic manifold
Liouville measure : $\Omega^{\dim M/2}$
G torus acting on M in a Hamiltonian way
$\mathfrak{g}$: Lie algebra of G, X_M vector field on M produced by $X \in \mathfrak{g}$.
$\Phi : M \to \mathfrak{g}^*$: moment map :

FUNDAMENTAL RELATION

$X \in \mathfrak{g}$:

$$d\langle \Phi, X \rangle = \iota(X_M)\Omega.$$

$\iota(X_M)$ contraction of differential forms by X_M.

Marsden Weinstein reduction

The symplectic (orbifold) manifold $M_{red}(a) = \Phi^{-1}(a)/G$.

Here $a \in \mathfrak{g}^*$, regular value of Φ.

Volume of the fiber : the Duistermaat-Heckman measure

$a \in \mathfrak{g}^*$ regular value of $\Phi : M \to \mathfrak{g}^*$:

The Duistermaat-Heckman measure

$DH(a) = vol(\Phi^{-1}(a)/G)$

A priori, $DH(a)$ defined only for regular values.
BUT

$DH(a)$ extends to a continuous and piecewise polynomial function on the convex polytope $\Phi(M) \subset \mathfrak{g}^*$.

More generally

(M, Ω, Φ), M even dimensional oriented. Ω not necessarily non degenerate. $\Omega^{(\dim M)/2}$ a signed measure on M.

We say : Φ is the moment map if

$$d\langle \Phi, X\rangle = \iota(X_M)\Omega.$$

Assume that Φ admits regular values. Ω descend to a two-form on $\Phi^{-1}(a)/G$ for a regular. We obtain a corresponding signed "Liouville measure" on $\Phi^{-1}(a)/G$.

The Duistermaat-Heckman measure

$DH(a) = vol(\Phi^{-1}(a)/G)$.
$DH(a)$ extends to a piecewise polynomial function on $\mathfrak{g}^*$ supported on a union of convex polytopes. We call $DH(a)da$ the Duistermaat-Heckman measure

An example : Hamiltonian action of G on M

In the next frame,

$$M = O[1,0,-1] \times O[1,0,-1]$$

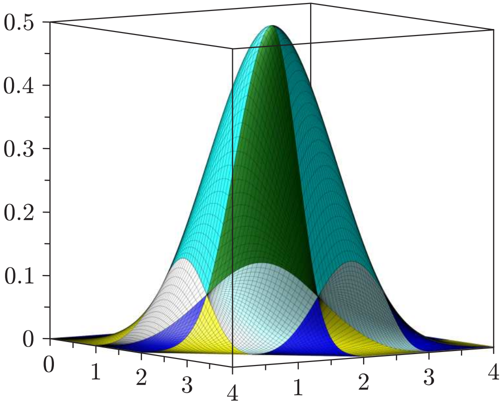
with $O([1,0,-1])$= Hermitian matrices (x_{ij}) with eigenvalues $(1,0,-1)$.

$$G = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

On M, with coordinates (x_{ij}, y_{ij}), $\Phi = (\phi_1, \phi_2) : M \to \mathbf{R}^2$

$$\phi_1 = x_{11} + y_{11}, \qquad \phi_2 = x_{22} + y_{22}.$$

Then the reduced manifolds $\Phi^{-1}(a)/G$ are of dimension $8 = 6 + 6 - 4$. The volume is given by local polynomials of degree 4 :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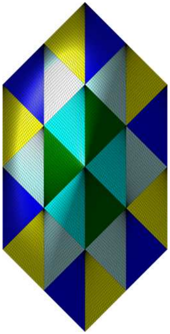
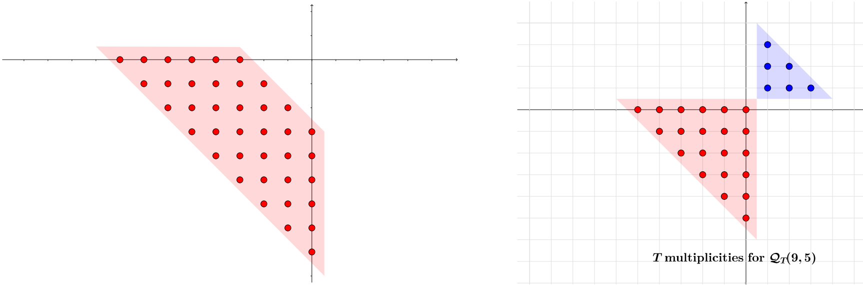


Varying the moment map : M toric manifold corresponding to the Delzant polytope on the left.



On the left : moment map and Duistermaat-Heckman measure associated to a symplectic form.
On the right : moment map associated to a degenerate form.
The red is $+1$, and the blue is -1.

Kostant line bundle

(M, Ω, Φ) as before.
Let L be a G-equivariant line bundle with connection ∇

Definition : L is a Kostant line bundle for (M, Ω, Φ)

if :

$$\nabla^2 = -i\Omega \quad \text{and} \quad L(X) - \nabla_X = i\langle \Phi, X \rangle.$$

That is Ω is the curvature of the line bundle, and do not need to be non degenerate. The moment map is determined by the connection.

Quantization with Dirac operators

Let (M, Ω, Φ) and L a Kostant line bundle for this data.
To simplify, assume M with a G-equivariant spin structure.
We consider the Dirac operator D_L

$$D_L := C^\infty(M, S^+ \otimes L) \to C^\infty(M, S^- \otimes L)$$

by $D_L = \sum_i \nabla^{S\otimes L}(e_i) \otimes \mathit{Cliff}(e_i)$ where e_i is a local orthonormal frame, $S = S^+ \oplus S^-$ the spin bundle, and $\mathit{Cliff}(e_i)$ the Clifford action of e_i on the spin module.

$$Q^G(M, L) = \mathit{Index}(D_L) = \mathit{Ker}(D_L) - \mathit{Coker}(D_L).$$

We call $Q^G(M, L)$ the quantization of (M, L). This is a virtual finite dimensional representation of G.
If $G = \{1\}$, $Q(M, L) = Q^G(M, L) \in \mathbb{Z}$ is the index of the elliptic operator D_L.

Quantization and multiplicities

$\widehat{G}$:= a lattice $\Lambda \in \mathfrak{g}^*$.
If $\lambda \in \Lambda$, we denote by π_λ the corresponding character of G.
Example $G = S^1 = \{e^{i\theta}\}$. Then $\widehat{G} = \mathbb{Z}$: $\pi_n(e^{i\theta}) = e^{in\theta}$ is the corresponding character.

$$Tr_{Q^G(M,L)}(t) = \sum_\lambda m(\lambda)\pi_\lambda(t).$$

We write this as

$$Q^G(M,L) = \sum_{\lambda\in\Lambda} m(\lambda)\pi_\lambda$$

Quantization commutes with reduction

$$Q^G(M, L) = \sum_{\lambda \in \Lambda} m(\lambda)\pi_\lambda$$

$\lambda \in \Lambda$ a regular value of Φ.

$$M_{red}(\lambda) = \Phi^{-1}(\lambda)/G$$

$L \otimes [\mathbb{C}_\lambda]$ descends to a line bundle L_λ on $M_{red}(\lambda)$.

Theorem (Cannas da Silva-Karshon-Tolman (2000))

λ regular value of Φ :

$$m(\lambda) = Q(M_{red}(\lambda), L_\lambda).$$

This is Guillemin-Sternberg conjecture proved by Meinrenken-Sjamaar (1999) for G compact and any λ in the symplectic case. For Dirac quantization by Paradan-Vergne (2015)

Multiplicity function under dilation

$$Q^G(M, L^k) = \sum_{\lambda \in \Lambda} m(\lambda, k)\, \pi_\lambda, \quad k \geq 1.$$

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Theorem :

$(\lambda, k) \mapsto m(\lambda, k)$ is a piecewise quasi-polynomial function on $\Lambda \times \mathbb{N}$. The domains of quasi-polynomiality are convex polyhedral cones.

Meinrenken-Sjamaar (1999) in the symplectic case and any compact connected Lie group G. For Dirac quantization, Paradan-Vergne (2015)

Behavior for large k

Let *test* be a smooth function on $\mathfrak{g}^*$. It is natural to consider

$$\langle V_k, test\rangle = \sum_{\lambda\in\Lambda} m(\lambda,k)\, test(\lambda/k).$$

Theorem (Paradan-Vergne (2017))

The family of distributions V_k admits an asymptotic expansion

$$V_k \equiv k^{(\dim M)/2} \sum_{n\geq 0} \frac{1}{k^n}\theta_n$$

with θ_0 the Duistermaat-Heckman measure.

AIM : GIVE AN EXPLICIT FORMULA FOR THE FULL ASYMPTOTIC EXPANSION

Example : Variation on the Euler-MacLaurin formula

$M = P_1(\mathbb{C})$ with $L = \mathcal{O}^a$:

$$\mathrm{Tr}_{Q^G(M,L^k)}(t) = \frac{t^{ak} - t^{-ak}}{t - t^{-1}} = \sum_{j \in [ka,-ka] \cap (ka+(2\mathbb{Z}+1))} t^j.$$

So

$$(V_k, test) = \sum_{\xi \in [a,-a] \cap a + \frac{2\mathbb{Z}+1}{k}} test(t)$$

$$\equiv \frac{k}{2}\int_{-a}^{a} test(t)dt + \sum_{n=1}^{\infty} (\frac{2}{k})^{n-1} \frac{B_n(1/2)}{n!} (test^{(n-1)}(a) - test^{(n-1)}(-a)).$$

Here $B_n(t)$ is the n-th Bernoulli polynomial.

Equivalent formulation with $\hat{A}$ operator

Use

$$\frac{x}{e^{x/2} - e^{-x/2}} = \sum_{n=0}^{\infty} B_n(1/2)\frac{x^n}{n!}.$$

Then we obtain :

$$V_k \equiv \frac{k}{2}\frac{i\partial/k}{\sin(i\partial/k)}\mathbf{1}_{[-a,a]}.$$

The main results

The semi-classical expansion can be computed explicitly in term of the graded equivariant class $\hat{A}(M)$ of the manifold M.

The semi-classical expansion determines $Q^G(M, L)$.

De Rham model for Equivariant cohomology

Equivariant form : equivariant polynomial map

$$\eta : \mathfrak{g} \longrightarrow \mathcal{A}(M)$$

Differential :

$$(D\eta)(X) = d(\eta(X)) - \iota(X_M)\eta(X).$$

Here $\mathcal{A}(M)$ = differential forms, d De Rham differential.
Equivariant curvature : $\Omega(X) := \Omega - \langle \Phi, X \rangle$.
Fundamental relation $\equiv \Omega(X)$ is a closed equivariant two-form.

Twisted Duistermaat-Heckman distributions

If $X \mapsto \eta(X)$ is an equivariant form, we define

$$\langle DH(M,\Omega,\eta), test\rangle = \int\int_{M\times\mathfrak{g}} e^{-i\Omega(X)}\eta(X)\hat{t}est(X)dX.$$

Here *test* is a test function on $\mathfrak{g}^*$, and $\hat{t}est$ its Fourier transform.
$DH(M,\Omega,\eta)$ depends only of the cohomology class of η.
When M non-compact, $DH(M,\Omega,\eta)$ is still defined if Φ is proper.

The equivariant class $\hat{A}(M)$

Let $p = \dim M$. If x is a $p \times p$ matrix, let

$$J(x) := \det\left(\frac{e^{x/2} - e^{-x/2}}{x}\right).$$

and

$$\frac{1}{J^{1/2}(x)} = \sum_{n \geq 0} p_n(x)$$

with $p_n(x)$ homogeneous invariant polynomial of degree n.

Chern-Weil morphism : $p_n(x) \rightsquigarrow$ equivariant form $\hat{A}_n(M)$

The series

$$\hat{A}(M)(X) = \sum_{n=0}^{\infty} \hat{A}_n(M)(X)$$

converges for $X \in \mathfrak{g}$ small enough.

Berline-Vergne equivariant Riemann-Roch formula

For $X \in \mathfrak{g}$ small,

$$Tr_{KerD_L}(expX) - Tr_{CokerD_L}(expX) =$$

$$\frac{1}{(2i\pi)^{dimM/2}} \int_M e^{i\Omega(X)} \hat{A}(M)(X).$$

Formula for the asymptotic expansion

Theorem :

The measure

$$< V_k, test >= \sum_{\lambda} m(\lambda, k) test(\lambda/k)$$

admits the asymptotic expansion

$$k^{\dim M/2} \sum_{n=0}^{\infty} k^{-n} < DH(M, \Omega, p_n), test >$$

as a series of distributions supported on $\Phi(M)$.

When M is toric : Asymptotic expansion : asymptotic of a Riemann sum (Guillemin-Sternberg ; Berline-Vergne) : given by various derivatives of the faces.

Why this is true ?

By Fourier transform, this means that

$$Tr_{KerD_{L^k}}(exp(X/k)) - Tr_{CokerD_{L^k}}(exp(X/k)) =$$

$$\frac{1}{(2i\pi)^{dimM/2}} \int_M e^{ik\Omega(X/k)} \hat{A}(M)(X/k)$$

provided we replace the right hand side by its Laurent series in $1/k$, and the left hand side by its semi-classical approximation. In other words, B-V formula holds for the semi-classical expansion, if we replace the $\hat{A}$ class by its series in the graded ring of equivariant cohomology.
The proof indeed follows from B-V formula.

M non compact with proper moment map

Assume the fibers of the moment map compact. Then various definitions (Weitsman,Paradan-Vergne,Braverman, Ma-Zhang) of $Q^G(M,L)$ as an infinite dimensional representation of G can be given. They all coincide.

• Defined as the index of a G-transversally elliptic operator D_L^{push} by Paradan-Vergne (using the Kirwan vector field)

$$Q^G(M,L) = Ker(D_L^{push}) - CoKer(D_L^{push}).$$

• Defined as formal quantization by J Weitsman

$$Q^G(M,L) = \sum_{\lambda \in \Lambda} Q(M_{red,\lambda}, L_\lambda)\pi_\lambda$$

Here $Q(M_{red,\lambda}, L_\lambda)$ can be defined since $M_{red,\lambda} = \Phi^{-1}(\lambda)/G$ is compact (defined by "continuity" of the index if λ is not a regular value using a nearbye regular value.)

Example 1

$M = T^*S^1$, with coordinates $(e^{i\theta}, t)$,

$$\Omega = d\theta \wedge dt, \qquad \Phi(e^{i\theta}, t) = t.$$

$L = [\mathbb{C}]$ with connection $d - itd\theta$ is a Kostant line bundle.

$$Q^G(M, L) = L^2(S^1) = \sum_{n \in \mathbb{Z}} e^{in\theta}.$$

Example 2

$M = \mathbb{C}$, $a \in \mathbb{Z}$, $G = S^1$, $L = M \times \mathbb{C}$

with action $e^{i\theta}(z, v) = (e^{2i\theta} z, e^{ia\theta} v)$.

$$\nabla = d - \frac{i}{2} Im(z d\bar{z}).$$

Moment map $\phi_G(z) = a + |z|^2$ proper. Then

$$Q^G(M, L) = e^{ika\theta} \sum_{j \geq 0} e^{i(2j+1)\theta}.$$

The same theorem holds with same proof

Use Berline-Vergne-Paradan formula for index of transversally elliptic operators.
(M, Ω, Φ) ; L Kostant line bundle : Φ proper moment map.

$$Q^G(M, L^k) = \oplus m(\lambda, k)\Pi_\lambda.$$

Theorem :

The measure

$$< V_k, test >= \sum_\lambda m(\lambda, k) test(\lambda/k)$$

admits the asymptotic expansion

$$k^{\dim M/2} \sum_{n=0}^{\infty} k^{-n} < DH(M, \Omega, p_n), test >$$

G compact connected

$$(M, L, \Phi), \qquad \Phi : M \to \mathfrak{g}^* \text{ proper moment map.}$$

L Kostant line bundle : then we can define (as index of a transversally elliptic operator)

$$Q^G(M, L^k) = \sum_{\lambda \in \hat{G}} m(\lambda, k)\pi_\lambda.$$

We now describe $\hat{G}$ and the classical analog of the right hand side.
$\hat{G}$ can be identified to a discrete subset of coadjoint orbits in $\mathfrak{g}^*$.
T maximal torus, choice of positive roots, $\mathfrak{t}^*_{\geq 0}$ positive Weyl chamber.

Coadjoint orbits

Function $j_{\mathfrak{g}}(X) = \det_{\mathfrak{g}}(\frac{e^{X/2} - e^{-X/2}}{X})$, $X \in \mathfrak{g}$.
Coadjoint orbits $G\xi \in \mathfrak{g}^*$ have a canonical symplectic form and Liouville measure $d\beta_\xi$.

$$\hat{G} \subset \mathfrak{g}^*/G$$

by

Kirillov formula

$$Tr(\pi_\lambda(\exp X)) j_{\mathfrak{g}}^{1/2}(X) = \int_{O_\lambda} e^{i<\xi,X>} d\beta_\xi$$

The corresponding set of all the O_λ is the set of regular admissible orbits in the sense of Duflo.
Identification

$$\hat{G} \simeq \{\textit{dominant regular admissible weights}\} \subset \mathfrak{t}^*_{\geq 0}$$

Assume G simply connected.

Theorem :

$(\lambda, k) \mapsto m(\lambda, k)$ extends to a piecewise quasi-polynomial W-antiinvariant function on $\Lambda \times \mathbb{N}$. The domains of quasi-polynomiality are convex polyhedral cones.

Asymptotics

Restrict here (for simplicity) to the case of abelian generic stabilizer. Write :

$$Q^G(M, L^k) = \sum_\lambda m(G, \lambda, k)\pi_\lambda.$$

The function $j_{\mathfrak{g}}(X) = \det_{\mathfrak{g}} \frac{e^{adX/2} - e^{-adX/2}}{adX}$ gives rise to an infinite series of constant coefficients differential operators on $\mathfrak{g}^*$.

Theorem : The measure

$$\sum_{\lambda} m(G, \lambda, k) Measure(O_{\lambda/k})$$

admits an asymptotic expansion

$$k^{\dim M/2} j_{\mathfrak{g}}^{-1/2}(\partial/k) \cdot \left(\sum_{n=0}^{\infty} k^{-n} DH(M, \Omega, p_n \right)$$

as a series of distributions supported on $\Phi(M)$.

The semi-classical behavior determines $Q^G(M, L)$

If M is compact, knowing the behavior of $Q^G(M, L^k)$ for k large determines $Q^G(M, L)$ for $k = 1$.
For M non compact, we need to consider also the asymptotic development of

$$Tr_{Q^G(M,L^k)}(s \exp X/k)$$

where $s.X = X$. Similar formulae can be proven for the asymptotics in terms of equivariant differential forms.

Application

Annoying feature of the definition of $Q^G(M, L)$ for M non compact : Not clear on any of the definitions of $Q^G(M, L)$ that if H is a subgroup of G, $Q^G(M, L)$ restricts to $Q^H(M, L)$! ! !
Proof that this is true by Paradan using cutting and compactifications.
Here we can give a natural proof.

Restrictions of the formal quantization to a subgroup

If H is a compact subgroup of G such that $\Phi : M \to \mathfrak{h}^*$ is proper, we can define $Q^G(M, L)$ and $Q^H(M, L)$.
It is obvious that the asymptotic development of $Q^H(M, L)$ is the push forward of the asymptotic development of $Q^G(M, L)$ by the map $\mathfrak{g}^* \to \mathfrak{h}^*$.
So comparing the asymptotic developments we obtain a natural proof.