Some dynamical aspect of Minimum time affine control systems

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A closing conference for the Chair d'excellence of Eva Miranda

Joint work with Jean-Baptiste Caillau, Robert Roussarie

Introduction

Restricted circular 3 body problem

$$\ddot{q} + \nabla V_\mu(q) - 2i\dot{q} = u, \|u\| \leq 1 \tag{1}$$

in the rotating frame (RC3BP), u being the control and $V_\mu(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}$.

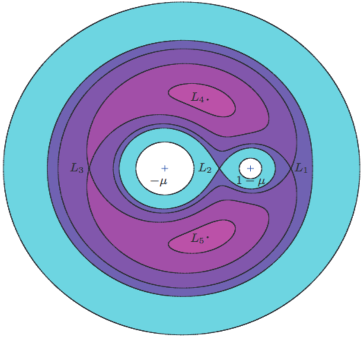

Figure: Hill's region and Lagrange points for the RC3BP, figure from [1].

Control affine system

→ *Optimization problem* :

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \; u_1^2 + u_2^2 \leq 1 \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min . \end{cases} \tag{2}$$

F_i are smooth, $i = 0, 1, 2$, $x_0, x_f \in M$ a 4 dimensional manifold (can be generalized to $2n$ with n controls).

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Remark

(1) can be written that way with $x = (q, v)$.

Notation : $F_{ij} = [F_i, F_j]$, $H_{ij} = \{H_i, H_j\}$, $i, j = 0, 1, 2$.

Hypothesis

Assumption :

$$(\mathcal{A}) : \text{rank}(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) = 4, \ \text{for all } x \in M.$$

Check for the RC3BP.

→ Link with controllability when F_0 is *recurrent* ($\mu = 0$ or certain energy levels of the RC3BP.)

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Proposition

Any system of the form $\ddot{q} + g(q, \dot{q}) = u$ *verifies* $(\mathcal{A})$.

We will use later the following hypothesis : $(\mathcal{B}) : [F_1, F_2] = 0$.

Hamiltonian formulation

Consider an optimal control problem

$$\begin{cases} \dot{x} = f(x,u) \\ x(0) = x_0, \;\; x(t_f) = x_f \\ \int_0^{t_f} \varphi(x(t),u(t))dt \to \min \end{cases}$$

$f : M \times U \to TM$ a family of smooth vector fields, $U \subset \mathbb{R}^m$.

Définition (Pseudo-Hamiltonian)

$\forall (x,p) \in T_x^*M$, $H(x,p,u) = \langle p, f(x,u)\rangle - \varphi(x,u)$.

Here $H(x,p,u) = H_0(x,p) + u_1 H_1(x,p) + u_2 H_2(x,p)$, with $H_i(x,p) = \langle p, F_i(x)\rangle$.

Necessary condition : the PMP

Théorème (P.M.P.)

(x, u) *minimum time trajectory then there exists a Lipschitz curve* $p(t) \in T_{x(t)}M^* \setminus \{0\}$ *s.t.*

- (x, p) *is solution of :*

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, p, u) \\ \dot{p} = -\frac{\partial H}{\partial x}(x, p, u). \end{cases} \tag{3}$$

- $H(x(t), p(t), u(t)) = \max_{\tilde{u} \in U} H(x(t), p(t), \tilde{u})$.
- $H(x(t), p(t), u(t)) \geq 0$.

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- $H(x(t), p(t), u(t)) \geq 0$.

Pros : **Autonomous** Hamiltonian system.
Cons : Dimension doubled, only necessary condition → existence of optimal control, **singularities**.
Solutions of (3) maximizing the Hamiltonian are called extremals. Their projection on M are extremal trajectories.

Singularities

Pseudo-Hamiltonian : $H(x, p, u) = H_0(x, p) + u_1 H_1(x, p) + u_2 H_2(x, p)$
Maximized Hamiltonian : $H^*(x, p) = H_0(x, p) + \sqrt{H_1(x, p)^2 + H_2(x, p)^2}$
$u = \frac{1}{\sqrt{H_1^2 + H_2^2}}(H_1, H_2)$: discontinuities of the control u are called **switchs**.

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Définition (Singular locus.)

A switch is a discontinuity of the reference control.
The singular locus, or switching surface is defined by
$\Sigma = \{z = (x,p) \in T^*M,\ H_1(x,p) = H_2(x,p) = 0\}$.

Définition

$\Sigma = \Sigma_0 \cup \Sigma_- \cup \Sigma_+$ with :
$\Sigma_- = \{H_{12}(z)^2 < H_{02}(z)^2 + H_{01}(z)^2\}$, $\Sigma_+ = \{H_{12}(z)^2 > H_{02}(z)^2 + H_{01}(z)^2\}$,
$\Sigma_0 = \{H_{12}(z)^2 = H_{02}(z)^2 + H_{01}(z)^2\}$.

Structure of the extremal flow

Stratification

Théorème (J.-B. Caillau, M. O.)

There exists unique solution for system (1) in a neighborhood $O_{\bar z}$ *of* $\bar z$*, and there is at most one switch on* $O_{\bar z}$*.*

- *If* $\bar z \in \Sigma_-$ *: The local extremal flow* $z : (t, z_0) \in [0, t_f] \times O_{\bar z} \mapsto z(t, z_0) \in M$ *is piecewise smooth, and smooth on each strata :*

  $$O_{\bar z} = S_0 \sqcup S_1 \sqcup \Sigma$$

  - where S_1 *is the codimension one submanifold of initial conditions leading to the switching surface,*
  - $S_0 = O_{\bar z} \setminus (S_1 \cup \Sigma)$*.*
- *If* $\bar z \in \Sigma_+$*, no extremal intersects the singular locus, and therefore, the flow is smooth on* $O_{\bar z}$*.*

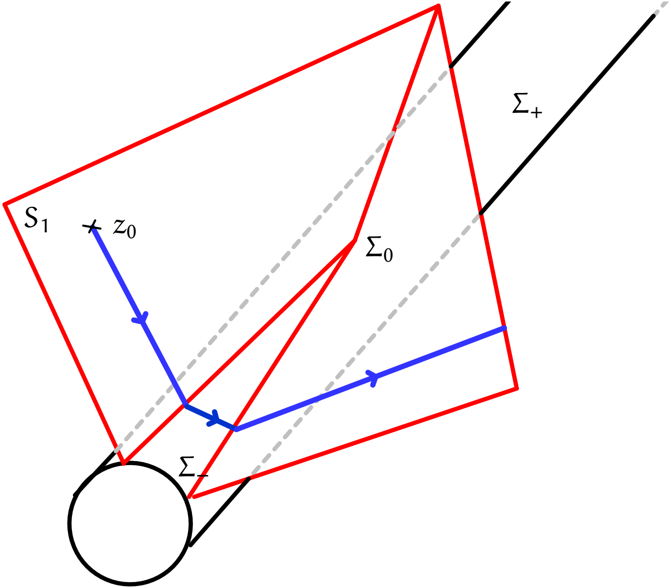

Figure 1 - Stratification of the flow into regular submanifolds.

Regular-singular transition

Théorème (J.-B. Caillau, M. O., R. Roussarie)

The singular-regular transition is continuous, with singularities in "$z \ln z$".

$(x, p) \mapsto (x, H_1, H_2, H_{01}, H_{02})$ then, in polar coordinates : $(H_1, H_2) = (\rho \cos s, \rho \sin s)$, with a time rescaled :

$$Y : \begin{cases} \rho' = \rho \cos s \\ s' = g(\rho, s, \xi) - \sin s = G(\rho, s, \xi) \\ \xi' = \rho h(\rho, s, \xi). \end{cases} \tag{4}$$

(i) g, h are smooth functions on an open subset of $\mathbb{R} \times \mathbb{R} \times D$, $D \subset \mathbb{R}^k$ compact, h has values in $\mathbb{R}^k$; Semi-hyperbolic equilibria when $\rho = 0$, $G = 0$.
(ii) g is smooth in $(\rho \cos \psi, \rho \sin \psi)$ and $|g| < 1$ on O.

Normal form theorem

Proposition (C^∞-normal form, Caillau, O., Roussarie)

Let $u = \rho s$, *then there exist* A, B, C *smooth functions on a neighborhood of* $D \times 0_u$ *such that* Y *is* C^∞ *equivalent to*

$$Y^\infty : \begin{cases} \rho' = -\rho(1 + uA(u,\xi)) \\ s' = s(1 + uB(u,\xi)) \\ \xi' = uC(u,\xi) \end{cases} \tag{5}$$

The global stable manifold has become $S_- = \{s = 0\}$.
For $\rho_0, s_f \geq 0$ consider the two sections $\Sigma_0 \subset \{\rho = \rho_0\}$, parameterized by (s,ξ) and $\Sigma_f \subset \{s = s_f\}$ parameterized by (ρ,ξ).

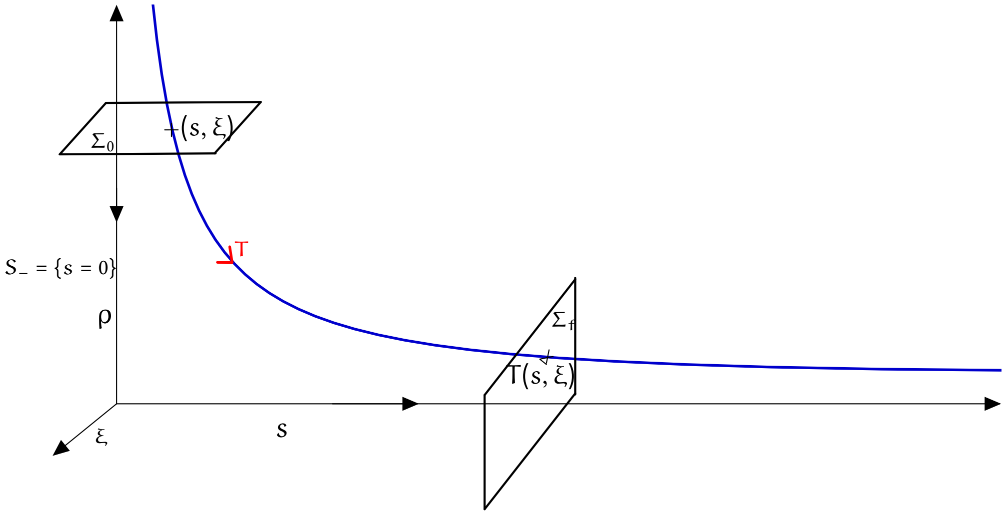

Figure 2 - Poincaré map between the two sections.

Théorème (J.-B. Caillau, M. O., R. Roussarie)

Let $T : \Sigma_0 \to \Sigma_f$ *be the Poincaré mapping between the two sections,* $T(s_0, \xi_0) = (\rho(s_0, \xi_0), \xi(s_0, \xi_0))$. *Then,* T *is a smooth function in* $(s_0 \ln s_0, s_0, \xi_0)$ *as there exist smooth functions* R *and* X *defined on a neighborhood of* $\{0\} \times \{0\} \times D$ *such that*

$$T(s_0, \xi_0) = (R(s_0 \ln s_0, s_0, \xi_0), X(s_0 \ln s_0, s_0, \xi_0)).$$

Proof of the Lemma. Step 1: Make the Jacobian diagonal. Y is equivalent to :

$$X : \begin{cases} \rho' = -\rho(1 + O(\rho)) \\ s' = s + O((\rho + |s|)^2) \\ \xi' = \rho O(\rho + s) \end{cases} \tag{6}$$

Proof of the Lemma.

Step 2 : Generalization of the Poincaré-Dulac theorem.
X a vector field, we say g is **resonant** with X if $[X, g] = 0$

Lemme

Let $X(x, \xi)$ *be a smooth vector field in* $\mathbb{R}^n \times \mathbb{R}^k$, $X(0, \xi) = 0$. *Note* X_1 *its linear part. Then, if* X_1 *does not depend on* ξ, *it can be formally develop along its resonant monomials up to a flat term.*

The proof of the initial theorem can be adapted since the bracket $[X_1, .]$ doest not see ξ : we reason by induction on the space of homogeneous monomials.

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The proof of the initial theorem can be adapted since the bracket $[X_1, .]$ doest not see ξ : we reason by induction on the space of homogeneous monomials.
Here

$$X_1 = -\rho \frac{\partial}{\partial \rho} + s \frac{\partial}{\partial s}.$$

Resonant monomials are

$$a(\xi)\rho u^k \frac{\partial}{\partial \rho},\ b(\xi) s u^k \frac{\partial}{\partial s},\ c(\xi) u^k \frac{\partial}{\partial \xi},\ k \in \mathbb{N}.$$

So X is **formally** conjugate to

$$W : \begin{cases} \rho' = -\rho(1 + \sum_{k\geq 1} a_k(\xi)u^k) \\ s' = s(1 + \sum_{k\geq 1} b_k(\xi)u^k) \\ \xi' = \rho \sum_{k\geq 1} c_k(\xi)u^k \end{cases}$$

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Step 4: Kill the flat perturbation. Path method : equivalent to solve

$$[X_t, Z_t] = R_\infty. \tag{7}$$

with X_t a path of field joining X and X^∞, with unknown Z_t.
Using normal hyperbolicity, (7) has a solution (Roussarie, 1975).

Sketch of the proof

A consequence of the normal form theorem.

$$(Y^\infty) \text{ equivalent to } \begin{cases} s' = s \\ \rho' = -\rho(1 + uA(u,\xi)) \\ \xi' = uC(u,\xi) \end{cases} \tag{8}$$

Transition time : $t(s_0) = \ln(s_f/s_0)$,
$(u = \rho s)$

$$Z : \begin{cases} s_0' = 0, \\ u' = -u^2 A(u,\xi), \\ \xi' = uC(u,\xi), \end{cases} \tag{9}$$

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Sketch of the proof

→ Rescale the time : $\tilde{Z} = \frac{1}{s_0} Z$:

$$\begin{cases} s_0' = 0, \\ u' = -(u^2/s_0)A(u,\xi), \\ \xi' = (u/s_0)C(u,\xi). \end{cases} \tag{10}$$

Its flow $\tilde{\varphi}$ is well defined and the Poincaré mapping is obtained by evaluating it in time $s_0 \ln(s_f/s_0)$:

$$T(s_0,\xi_0) = \tilde{\varphi}(s_0 \ln(s_f/s_0), s_0, \rho_0 s_0, \xi_0).$$

Issue : $\tilde{Z}$ is not smooth.
Blow up on $\{u = s = 0\}$: $f(u,s,\xi) = (\eta, s, \xi)$ with $\eta = u/s$
f^{-1} sends a rectangle $-\eta_0 \leq \eta \leq \eta_0$, $-s_0 \leq s \leq s_0$ on a cone $-\eta_0 s \leq u \leq \eta_0 s$. Lemma ⇒ the flow of $\tilde{Z}$ is contained in that cone.

Sketch of the proof

The blown up vector field writes:

$$\hat{Z} : \begin{cases} s_0' = 0, \\ \eta' = -\eta^2 A(\eta s_0, \xi), \\ \xi' = \eta\, C(\eta s_0, \xi). \end{cases} \tag{11}$$

and is smooth.
Denote $\hat{\varphi} = (\hat{\eta}, \hat{\xi})$ its flow, we only need to evaluate it on a small band $s_0 \in [-s_1, s_1]$, $\eta_0 \in [-M, M]$, on which it is smooth.

$$T(s_0, \xi_0) = (\hat{\eta}(s_0 \ln(s_f/s_0), s_0, \rho_0, \xi_0), \hat{\xi}(s_0 \ln(s_f/s_0), s_0, \rho_0, \xi_0),$$

which has the desired regularity.

Sufficient conditions for optimality

Smooth case

Let

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0, \ \ x(t_f) = x_f \\ \int_0^{t_f} \varphi(x(t), u(t))dt \to \min \end{cases}$$

be an optimal control problem. Recall, that its pseudo-Hamiltonian is $H(x, p, u) = \langle p, f(x)\rangle - \varphi(x, u)$, assume that its maximized Hamiltonian H^* is **smooth**.

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Let $z = (x, p)$ be an extremal and assume :
(($\mathcal{B}_0$): The reference extremal is normal (meaning $p^0 \neq 0$).)
($\mathcal{B}_1$): $\frac{\partial x}{\partial p_0}(t, x_0, p_0)$ is invertible for $t \in]0, t_f[$.

Théorème

Under those hypothesis, the reference trajectory $x = \Pi(z)$ *is a local minimizer along all trajectories with same endpoints.*

Idea

Compare using the **Poincaré-Cartan** invariant along an extremal $z(t)$:

$$\int_z p\,dx - H^* dt = \int_0^{t_f} \langle p(t), f(x(t), u(t))\rangle - H^*(z(t))dt = \int_0^{t_f} \varphi(x(t), u(t))dt.$$

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To compare with every C^0 curves on M, one has to lift them to T^*M.
→ Make the canonical projection $\Pi : T^*M \to M$ invertible: build a Lagrangian submanifold $\mathcal{L}_0$ transverse to $T^*_{x_0}M$ on which Π is invertible (tangent space transversal to $\ker d\Pi$) and propagate it by the extremal flow.

$$\mathcal{L} = \{(t, z), \exists z_0 \in \mathcal{L}_0, z = \exp(t\vec{H})(z_0)\}$$

Π is still invertible on $\mathcal{L}$.

Idea

$\mathcal{L}_0$ can be chosen so that $\alpha = p dx - H^* dt_{|\mathcal{L}}$ is exact on $\mathcal{L}$.
Let $(\tilde{x}, \tilde{u})$ be any admissible trajectory with same endpoints, denote $\tilde{z} = (\tilde{x}, \tilde{p})$ its lift it to T^*M (through Π).

$$\int_0^{t_f} \varphi(\tilde{x}(t), \tilde{u}(t))dt = \int_0^{t_f} \tilde{p}(t).\dot{\tilde{x}}(t) - H(\tilde{z}(t), \tilde{u}(t))dt \geq \int_0^{t_f} \tilde{p}(t).\dot{\tilde{x}}(t) - H^*(\tilde{z}(t))dt \tag{12}$$

but

$$\int_0^{t_f} \tilde{p}(t).\dot{\tilde{x}}(t) - H^*(\tilde{z}(t))dt = \int_{\tilde{z}} \alpha = \int_z \alpha = \int_0^{t_f} p(t).\dot{x}(t) - H^*(z(t))dt = \int_0^{t_f} \varphi(x(t), u(t))dt.$$

Application to Kepler and the RC3BP

In this case : $(\mathcal{B})$: $[F_1, F_2] = 0 \Rightarrow H_{12} = 0$.
Previous results apply directly to the controlled RC3BP, and

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Proposition

In the controlled Kepler problem and RC3BP switching are instantaneous rotations of angle π of the control u: if t is a switching time, $u(t_-) = -u(t_+)$.

We call such switchings π-singularities.

π-singularities

We can globally bound the number of π-singularities on a time interval $[0, t_f]$.

Définition (Distance to collisions)

We define $\delta = \inf_{[0,t_f]} |q(t)|$,
$\delta_1 = \inf_{[0,t_f]} |q(t) + \mu|$,
$\delta_2 = \inf_{[0,t_f]} |q(t) - (1 - \mu)|$.
Finally note $\delta_{12}(\mu) = \frac{\delta_1 \delta_2}{((1-\mu)\delta_2^3 + \mu\delta_1^3)^{1/3}}$.

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Proposition

- Keplerian case : Time interval of length $\pi\delta^{3/2}$ between two π-singularities. On a time interval $[0, t_f]$ the number of such singularities is at most $N_0 = [\frac{t_f}{\pi\delta^{3/2}}]$.

- Controlled RC3BP : Time interval of length $\pi\delta_{12}(\mu)^{3/2}$ between two π-singularities. On a time interval $[0, t_f]$ there is at most $N_\mu = [\frac{t_f}{\pi\delta_{12}(\mu)^{3/2}}]$ π-singularities.

→ Sturm type estimations.

Conclusion and open problems

- Well known structure of the extremal flow → Good criteria for optimality in our case (lack of regularity).
- More general way to treat sufficient conditions for optimal control problems using degenerate symplectic geometry?
- Global answer to the sufficient condition questions by Fillipov's theorem: construct a compact containing the extremals.

Thank you for
your attention !

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