Global instability in the elliptic restricted three body problem

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The (planar) elliptic restricted three body problem (RPETB) describes the motion q of a massless particle (a comet) under the gravitational field of two massive bodies (the primaries, say the Sun and Jupiter) with mass ratio μ revolving around their center of mass on elliptic orbits with eccentricity ϵ_{J}.

Typical models:

- Sun–Jupiter–asteroid or comet: $\epsilon_{\mathrm{J}} = 0.048$
- Sun–Earth–Moon systems: $\epsilon_{\mathrm{J}} = 0.016$

We search for trajectories of motion which show a large variation of the angular momentum $G = q \times \dot{q}$.

So we search for global instability ("diffusion" is the term usually used) in the angular momentum of this problem.

Theorem (The Main Result)

There exist two constants $C > 0$, $c > 0$ *and* $\mu^* = \mu^*(C, c) > 0$ *such that for any* $0 < \epsilon_J < c/C$ *and* $0 < \mu < \mu^*$, *and for any two values of the angular momentum in the region* $C \leq G_1^* < G_2^* \leq c/\epsilon_J$, *there exists a trajectory of the RPETB such that* $G(0) < G_1^*$, $G(T) > G_2^*$ *for some* $T > 0$.

- If $\epsilon_J = 0$, the primaries revolve along circular orbits, and such diffusion is not possible, since the (planar) restricted circular three body problem (R3BP) is governed by an autonomous Hamiltonian with 2 degrees-of-freedom.
- This is not the case for the RPETB, which is a 2+1/2 degrees-of-freedom Hamiltonian system with time-periodic Hamiltonian.

Related results about oscillatory motions and diffusion for several Restricted Three Body Problems:

- Euler libration points: Llibre-Martínez-Simó85, Capinski-Zgliczynski11, D-Gidea-Roldán13-16
- Collisions: Bolotin06
- The (parabolic) infinity: Llibre-Simó80, Xia92-93, Moser01, Moeckel07, Martínez-Pinyol94, Gorodetski-Kaloshin11, Guàrdia-Martín-Seara12, Martínez-Simó14
- Mean motion resonances: Fejoz-Guàrdia-Kaloshin-Roldán14
- Aubry-Mather theory: Galante-Kaloshin13

The motion of the massless particle q (comet) is described by

$$\frac{d^2q}{dt^2} = (1-\mu)\frac{q_S - q}{|q_S - q|^3} + \mu\frac{q_J - q}{|q_J - q|^3}$$

where $1-\mu$ is the mass of the primary (Sun) at q_S and μ the mass of the primary (Jupiter) at q_J.
Introducing $p = dq/dt$, this is a 2+1/2 degree-of-freedom Hamiltonian system with time-periodic Hamiltonian

$$H_\mu(q,p,t;\epsilon_J) = \frac{p^2}{2} - U_\mu(q,t;\epsilon_J)$$

with self-potential

$$U_\mu(q,t;\epsilon_J) = \frac{1-\mu}{|q - q_S(t,\epsilon_J)|} + \frac{\mu}{|q - q_J(t,\epsilon_J)|}$$

Parameters: $0 < \mu, \epsilon_J < 1$ small.

When $\mu = 0$, there is no Jupiter in the equation of motion and the Sun is fixed at the origin: $q_S = 0$

The Sun q_S and the comet q form a two-body problem with the Hamiltonian $H_0(q, p, t; \epsilon_J) = H_0(q, p) = \dfrac{p^2}{2} - \dfrac{1}{|q|} = \dfrac{p^2}{2} - U_0(q)$.

The two–body problem is integrable, and there is no dependence on the eccentricity ϵ_J or the time t.

$$q_S = q_S(t, \epsilon_J) = \mu r(\cos f, \sin f)$$

$$q_J = q_J(t, \epsilon_J) = -(1-\mu) r(\cos f, \sin f)$$

with

$$r = r(t; \epsilon_J) = \frac{1-\epsilon_J^2}{1+\epsilon_J \cos f}, \qquad \frac{df}{dt} = \frac{(1+\epsilon_J \cos f)^2}{(1-\epsilon_J^2)^{3/2}},$$

where $f = f(t; \epsilon_J)$ is the true anomaly. If $q = \rho(\cos\alpha, \sin\alpha)$,

$$\begin{aligned} |q - q_S|^2 &= \rho^2 - 2\mu r\rho\cos(\alpha - f) + \mu^2 r^2, \\ |q - q_J|^2 &= \rho^2 + 2(1-\mu) r\rho\cos(\alpha - f) + (1-\mu)^2 r^2. \end{aligned}$$

Remark Also

$$r = r(t; \epsilon_J) = 1 - \epsilon_J \cos E, \qquad t = E - \epsilon_J \sin E,$$

where E is the eccentric anomaly.

Performing a standard polar-canonical change of variables $(q,p) \longmapsto (\rho, \alpha, P_\rho, P_\alpha)$

$$q = (\rho\cos\alpha, \rho\sin\alpha), \quad p = \left(P_\rho\cos\alpha - \frac{P_\alpha}{\rho}\sin\alpha, P_\rho\sin\alpha + \frac{P_\alpha}{\rho}\cos\alpha\right)$$

the Hamiltonian becomes

$$H_\mu^*(\rho, \alpha, P_\rho, P_\alpha, t; \epsilon_J) = \frac{P_\rho^2}{2} + \frac{P_\alpha^2}{2\rho^2} - U_\mu^*(\rho, \alpha, t; \epsilon_J)$$

with a self-potential U_μ^*

$$U_\mu^*(\rho, \alpha, t; \epsilon_J) = U_\mu(\rho\cos\alpha, \rho\sin\alpha, t; \epsilon_J) = \frac{1}{\rho} + O(\mu).$$

From now on we will write

$$G = P_\alpha, \qquad y = P_\rho,$$

so that Hamiltonian (8) becomes

$$H^*_\mu(\rho, \alpha, y, G, t; \epsilon_\mathrm{J}) = \frac{y^2}{2} + \frac{G^2}{2\rho^2} - U^*_\mu(\rho, \alpha, t; \epsilon_\mathrm{J}).$$

Remark

In the (planar) circular case $\epsilon_J = 0$ (RTBP), $r = 1$ and $f = t$, and $|q - q_S|, |q - q_J|$ depend on the time t and the angle α just through their difference $\alpha - t$. As a consequence, $U^*_\mu(\rho, \alpha, t; 0)$ as well as $H^*_\mu(\rho, \alpha, y, G, t; 0)$ depend also on t and α just through the same difference $\alpha - t$, the sinodic angle. This implies that the Jacobi constant $H^* + G$ is a first integral of the system.

Through McGehee non-canonical change of variables, for $x > 0$,

$$\rho = \frac{2}{x^2}$$

the infinity $\rho = \infty$ is sent to the origin $x = 0$ and the equations become

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2 x^6 - \frac{x^3}{4}\frac{\partial \mathcal{U}_\mu}{\partial x}$$

$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = \frac{\partial \mathcal{U}_\mu}{\partial \alpha},$$

where the self-potential $\mathcal{U}_\mu$ is given now by

$$\mathcal{U}_\mu(x,\alpha,t;\epsilon_{\mathrm{J}}) = U^*_\mu(2/x^2,\alpha,t;\epsilon_{\mathrm{J}}) = \frac{x^2}{2}\left(\frac{1-\mu}{\sigma_S} + \frac{\mu}{\sigma_J}\right)$$

with

$$|q - q_{\mathrm{S}}|^2 = \sigma_{\mathrm{S}}^2 = 1 - \mu r x^2 \cos(\alpha - f) + \frac{1}{4}\mu^2 r^2 x^4,$$

$$|q - q_{\mathrm{J}}|^2 = \sigma_{\mathrm{J}}^2 = 1 + (1-\mu) r x^2 \cos(\alpha - f) + \frac{1}{4}(1-\mu)^2 r^2 x^4.$$

Under McGehee change of variables $\rho = 2/x^2$ for $x > 0$,

$$d\rho \wedge dy + d\alpha \wedge dG \quad \text{is transformed to} \quad \omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG$$

which is a b^3-symplectic form, the new Hamiltonian reads as

$$\mathcal{H}_\mu(x, \alpha, y, G, t; \epsilon_J) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_\mu(x, \alpha, t; \epsilon_J),$$

and the the Hamiltonian equations become

$$\frac{dx}{dt} = -\frac{x^3}{4}\left(\frac{\partial \mathcal{H}_\mu}{\partial y}\right) \qquad \frac{dy}{dt} = -\frac{x^3}{4}\left(-\frac{\partial \mathcal{H}_\mu}{\partial x}\right)$$

$$\frac{d\alpha}{dt} = \frac{\partial \mathcal{H}_\mu}{\partial G} \qquad \frac{dG}{dt} = -\frac{\partial \mathcal{H}_\mu}{\partial \alpha}.$$

which can be written as $dz/dt = \{z, \mathcal{H}_\mu\}$ in terms of the Poisson bracket

$$\{f, g\} = -\frac{x^3}{4}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial f}{\partial G}\frac{\partial g}{\partial \alpha}.$$

- Some sources of b^m-symplectic structures can be found in [Scott13], [Kiesenhofer-Miranda-Scott15], [Guillemin-Miranda-Weitsman17-18].
- Other examples can be found in [Guardia-Martín-Seara16], [D-Kiesenhofer-Miranda17], [Braddell-D-Miranda-Oms-Planas17].
- New examples in [Baldomá-Fontich-Martín18].

For $\mu = 0$ and $G > 0$, Hamiltonian $\mathcal{H}_0$ becomes Duffing Hamiltonian:

$$\mathcal{H}_0(x, y, G) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_0(x) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \frac{x^2}{2}$$

$\mathcal{H}_0$ is autonomous and independent of ϵ_J and α. Its associated equations are

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2 x^6 - \frac{1}{4}x^4$$

$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = 0$$

The angular momentum G is a conserved quantity, $G > 0$ from now on. The phase space $(x, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+$ includes the set of equilibrium points

$$\mathcal{E}_\infty = \{z = (x = 0, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+\}.$$

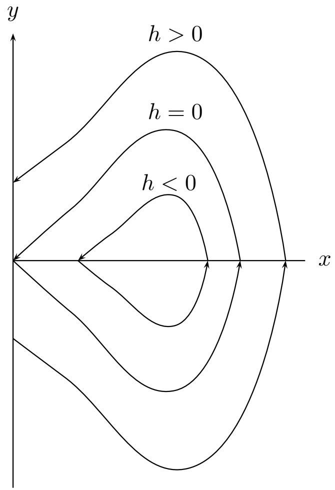

Figure: Level curves of $\mathcal{H}_0$ in the $(x \geq 0, y)$ plane, for fixed $G > 0$

For any fixed $\alpha \in \mathbb{T}$, $G \in \mathbb{R}$,

$$\Lambda_{\alpha,G} = \{(0, \alpha, 0, G)\}$$

is a parabolic equilibrium point, which is topologically equivalent to a saddle point, since it possesses stable and unstable 1D-invariant manifolds. The union of such points is the 2D-(symplectic) manifold of equilibrium points

$$\Lambda_\infty = \bigcup_{\alpha,G} \Lambda_{\alpha,G}.$$

which is the (parabolic) infinity manifold for the Kepler problem.
As we will deal with a time-periodic Hamiltonian, it is natural to work in the extended phase space

$$\tilde{z} = (z, s) = (x, \alpha, y, G, s) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}$$

just by writing s instead of t in the Hamiltonian and adding the equation

$$\frac{ds}{dt} = 1$$

The extended versions of the invariant sets $\Lambda_{\alpha,G}$, Λ_∞ for the Kepler problem are the 2π-periodic orbits with motion $ds/dt = 1$

$$\tilde{\Lambda}_{\alpha,G} = \{\tilde{z} = (0, \alpha, 0, G, s), s \in \mathbb{T}\},$$

and the 3D-invariant manifold (the "parabolic" infinity manifold)

$$\tilde{\Lambda}_\infty = \bigcup_{\alpha,G} \tilde{\Lambda}_{\alpha,G} = \{(0, \alpha, 0, G, s),\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}, \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T},$$

which is topologically equivalent to a normally hyperbolic invariant manifold (TNHIM).

Parameterizing the points in $\tilde{\Lambda}_\infty$ by

$$\tilde{\mathbf{x}}_0 = \tilde{\mathbf{x}}_0(\alpha, G, s) = (\mathbf{x}_0(\alpha, G), s) = (0, \alpha, 0, G, s) \in \tilde{\Lambda}_\infty \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$$

the inner dynamics on $\tilde{\Lambda}_\infty$ is trivial, since it is given by the dynamics on each periodic orbit $\tilde{\Lambda}_{\alpha,G}$:

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (0, \alpha, 0, G, s + t) = (\mathbf{x}_0(\alpha, G), s + t) = \tilde{\mathbf{x}}_0(\alpha, G, s + t),$$

where we denote by $\tilde{\phi}_{t,\mu}$ the flow of our system in the extended phase

The equilibrium points $\Lambda_{\alpha,G}$ have stable and unstable 1D-invariant manifolds which coincide:

$$\begin{aligned}\gamma_{\alpha,G} &= W^{\mathrm{u}}(\Lambda_{\alpha,G}) = W^{\mathrm{s}}(\Lambda_{\alpha,G}) \\ &= \left\{ z = (x, \hat{\alpha}, y, G),\ \mathcal{H}_0(x,y,G) = 0, \hat{\alpha} = \alpha - G \int_{\mathcal{H}_0=0} \frac{x}{y} dx \right\},\end{aligned}$$

whereas the 2D-manifold of equilibrium points Λ_∞ has stable and unstable 3D-invariant manifolds which coincide and are given by

$$\gamma = W^{\mathrm{u}}(\Lambda_\infty) = W^{\mathrm{s}}(\Lambda_\infty) = \{z = (x, \alpha, y, G),\ \mathcal{H}_0(x,y,G) = 0\}.$$

In the extended phase space, the surface

$$\tilde{\gamma}_{\alpha,G} = W^{\mathrm{u}}(\tilde{\Lambda}_{\alpha,G}) = W^{\mathrm{s}}(\tilde{\Lambda}_{\alpha,G})$$
$$= \left\{ \tilde{z} = (x, \hat{\alpha}, y, G, s), s \in \mathbb{T},\ \mathcal{H}_0(x, y, G) = 0, \hat{\alpha} = \alpha - G \int_{\mathcal{H}_0 = 0} \frac{x}{y} dx \right\}$$

is a 2D-homoclinic manifold to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$. The 4D-stable and unstable manifolds of the infinity manifold $\tilde{\Lambda}_\infty$ coincide along the 4D-homoclinic invariant manifold (the separatrix), which is just the union of the homoclinic surfaces $\tilde{\gamma}_{\alpha,G}$:

$$\tilde{\gamma} = W^{\mathrm{u}}(\tilde{\Lambda}_\infty) = W^{\mathrm{s}}(\tilde{\Lambda}_\infty) = \bigcup_{\alpha,G} \tilde{\gamma}_{\alpha,G}$$
$$= \{ \tilde{z} = (x, \alpha, y, G, s), (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T},\ \mathcal{H}_0(x, \alpha, y, G) = 0 \}$$

The homoclinic solutions to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$ are given by

$$x_h(t; G) = \frac{2}{G(1+\tau^2)^{1/2}} \qquad y_h(t; G) = \frac{2\tau}{G(1+\tau^2)}$$
$$\alpha_h(t; \alpha, G) = \alpha + \pi + 2\arctan\tau \qquad G_h(t; G) = G$$
$$s_h(t; s) = s + t \qquad ,$$

where α and G are the 2 free parameters and the relation between t and τ is

$$t = \frac{G^3}{2}\left(\tau + \frac{\tau^3}{3}\right) \quad \text{which is equivalent to} \quad \frac{dt}{d\tau} = \frac{2G}{x^2},$$

Due to the factor $-x^3/4$ in front of the equations, the convergence along the separatrix to the infinity manifold is power-like in τ and t:

$$x_h, y_h, \frac{\alpha - \alpha_h + \pi}{G} \sim \frac{2}{G\tau} \sim \frac{2}{\sqrt[3]{\pm 6t}}, \quad \tau, t \to \pm\infty.$$

Introducing the notation

$$\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s) = (x_h(\sigma; G), \alpha_h(\sigma; \alpha, G), y_h(\sigma; G), G, s)$$

we can parameterize any homoclinic surface $\tilde{\gamma}_{\alpha,G}$ as

$$\tilde{\gamma}_{\alpha,G} = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s),\ \sigma \in \mathbb{R}, s \in \mathbb{T}\}.$$

and the 4-dimensional separatrix $\tilde{\gamma} = W(\tilde{\Lambda}_\infty)$ as

$$\tilde{\gamma} = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s), \sigma \in \mathbb{R}, G \in \mathbb{R}_+, (\alpha, s) \in \mathbb{T}^2\}.$$

The motion on $\tilde{\gamma}$ and $\tilde{\Lambda}_\infty$ is given by

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) = \tilde{\mathbf{z}}_0(\sigma + t, \alpha, G, s + t) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t)$$
$$\tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (0, \alpha, 0, G, s + t) = (\mathbf{x}_0(\alpha, G), s + t) = \tilde{\mathbf{x}}_0(\alpha, G, s + t),$$

and the following asymptotic formula follows:

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) - \tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t) - (\mathbf{x}_0(\alpha, G), s + t) \xrightarrow[t \to \pm\infty]{} 0.$$

The scattering map $\widetilde{S}$ describes the homoclinic orbits to the infinity manifold $\tilde{\Lambda}_\infty$ (defined in (16)) to itself. Given $\tilde{\mathbf{x}}_-, \tilde{\mathbf{x}}_+ \in \tilde{\Lambda}_\infty$, we define

$$\widetilde{S}_\mu(\tilde{\mathbf{x}}_-) := \tilde{\mathbf{x}}_+$$

if there exists $\tilde{\mathbf{z}}^* \in W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty) \cap W^s_\mu(\tilde{\Lambda}_\infty)$ such that

$$\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}^*) - \tilde{\phi}_{t,\mu}(\tilde{\mathbf{x}}_\pm) \longrightarrow 0 \quad \text{for } t \to \pm\infty.$$

In the case $\mu = 0$ the previous asymptotic relation

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) - \tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t) - (\mathbf{x}_0(\alpha, G), s + t) \xrightarrow[t \to \pm\infty]{} 0.$$

implies $\widetilde{S}_0(\tilde{\mathbf{x}}_0) = \tilde{\mathbf{x}}_0$ so that that the scattering map $\widetilde{S}_0 : \tilde{\Lambda}_\infty \longrightarrow \tilde{\Lambda}_\infty$ is the identity.

For $\mu > 0$, the set $\mathcal{E}_\infty$ remains invariant as well as infinity manifold $\tilde{\Lambda}_\infty$, which is again a TNHIM, as well as all the periodic orbits $\tilde{\Lambda}_{\alpha,G}$.

The inner dynamics on $\tilde{\Lambda}_\infty$ is the same as in the case $\mu = 0$, so that the parametrization $\tilde{\mathbf{x}}_0$ as well as its trivial dynamics remain the same.

From [McGehee73, Guardia-Martín-Seara-Sabbagh17] we know that $W^s_\mu(\tilde{\Lambda}_\infty)$ and $W^u_\mu(\tilde{\Lambda}_\infty)$ exist for μ small enough and are 4-dimensional in the extended phase space.

The existence of scattering maps will depend on the transversal intersections between these two manifolds.

Introduce now the Melnikov potential $\mathcal{L} : \tilde{\Lambda}_\infty \longrightarrow \mathbb{R}$ [D-Gutiérrez00], [D-Llave-Seara06]

$$\mathcal{L}(\alpha, G, s; \epsilon_\mathrm{J}) = \int_{-\infty}^{\infty} \Delta\mathcal{U}_0(x_\mathrm{h}(t; G), \alpha_\mathrm{h}(t; \alpha, G), s + t; \epsilon_\mathrm{J})\, dt,$$

where $\Delta\mathcal{U}_0$ is defined by

$$\Delta\mathcal{U}_0(x, \alpha, s; \epsilon_\mathrm{J}) := \left.\frac{\partial \mathcal{U}\mu}{\partial \mu}\right|_{\mu=0} (x, \alpha, s; \epsilon_\mathrm{J}) = O(x^4) \quad \text{as} \quad x \to 0.$$

The asymptotics above follows from the asymptotic behavior of the solutions along the separatrix and of the self potential close to the parabolic infinity manifold, and guarantees that this integral is absolutely convergent.

Proposition (Transverse homoclinic points to the infinite manifold $\tilde\Lambda_\infty$)

Given $(\alpha, G, s) \in \mathbb{T} \times \mathbb{R}^+ \times \mathbb{T}$, *assume that the function*

$$\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R}$$

has a non-degenerate critical point $\sigma^* = \sigma^*(\alpha, G, s; \epsilon_J)$. *Then, there exists* $\mu^* = \mu^*(G, \epsilon_J)$, *such that for* $0 < \mu < \mu^*$, *close to the point* $\tilde{\mathbf{z}}_0^* = (\mathbf{z}_0(\sigma^*, \alpha, G), s) \in \tilde\gamma$ *there exists a locally unique point*

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*(\sigma^*, \alpha, G, s; \epsilon_J, \mu) \in W^s_\mu(\tilde\Lambda_\infty) \pitchfork W^u_\mu(\tilde\Lambda_\infty)$$

of the form $\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}_0^* + O(\mu)$, *and there exist unique points* $\tilde{\mathbf{x}}_\pm = (0, \alpha_\pm, 0, G_\pm, s) = (0, \alpha, 0, G, s) + O(\mu) \in \tilde\Lambda_\infty$ *such that*

$$\tilde\phi_{t,\mu}(\tilde{\mathbf{z}}^*) - \tilde\phi_{t,\mu}(\tilde{\mathbf{x}}_\pm) \longrightarrow 0 \quad \text{for } t \to \pm\infty.$$

Moreover, we have

$$G_+ - G_- = \mu \frac{\partial \mathcal{L}}{\partial \alpha}(\alpha, G, s - \sigma^*(\alpha, G, s; \epsilon_J)) + O(\mu^2).$$

Once we have found a critical point $\sigma^* = \sigma^*(\alpha, G, s; \epsilon_J)$ of

$$\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R}$$

on a domain of (α, G, s), we can define the reduced Poincaré function [D-Llave-Seara06]

$$\mathcal{L}^*(\alpha, G; \epsilon_J) := \mathcal{L}(\alpha, G, s - \sigma^*; \epsilon_J) = \mathcal{L}(\alpha, G, s^*; \epsilon_J)$$

with $s^* = s - \sigma^*$. Note that the reduced Poincaré function does not depend on the s chosen, since by the previous Proposition

$$\frac{\partial}{\partial s}\left(\mathcal{L}\left(\alpha, G, s - \sigma^*(\alpha, G, s; \epsilon_J); \epsilon_J\right)\right) \equiv 0.$$

Note also that if the function $\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R}$ has different non degenerate critical points there will exist different scattering maps.

The next Proposition gives an approximation of the scattering map in the general case $\mu > 0$.

Proposition (Expression of the scattering map)

The associated scattering map $(\alpha_+, G_+, s_+) = \widetilde{S}_\mu(\alpha, G, s)$ *for any non degenerate critical point* $\sigma^* = \sigma^*(\alpha, G, s; \epsilon_J)$ *of the function* $\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R}$ *is an exact symplectic map given by*

$$(\alpha, G, s) \longmapsto \left(\alpha - \mu \frac{\partial \mathcal{L}^*}{\partial G}(\alpha, G; \epsilon_J) + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*}{\partial \alpha}(\alpha, G; \epsilon_J) + O(\mu^2), s\right)$$

where $\mathcal{L}^*$ *is the Poincaré reduced function.*

Remark: the scattering map $\widetilde{S}_\mu$ follows closely the level curves of the Hamiltonians $\mathcal{L}^*$. More precisely, up to $O(\mu^2)$ terms, $\widetilde{S}_\mu$ is given by the time $-\mu$ map of the Hamiltonian flow of Hamiltonian $\mathcal{L}*$. The $O(\mu^2)$ remainder will be negligible as long as

$$|\mu| \ll \left|\frac{\partial \mathcal{L}^*}{\partial G}\right|, \left|\frac{\partial \mathcal{L}^*}{\partial \alpha}\right|.$$

$$\mathcal{L}(\alpha, G, s; \epsilon_J) = \int_{-\infty}^{\infty} \left[\frac{x_h^2}{\left[4 + x_h^4 r^2 + 4x_h^2 r \cos(\alpha_h - f)\right]^{1/2}} + \left(\frac{x_h^2}{2}\right)^2 r \cos(\alpha_h - f) - \frac{x_h^2}{2} \right] dt$$

where x_h and α_h, solutions on the separatrix, are evaluated at t, whereas r and f, concerning the primaries, are evaluated at $s + t$. Fourier expanding with respect to angular variables α, s, $\mathcal{L}$ is an even function α, s: $\mathcal{L}(-\alpha, G, -s; \epsilon_J) = \mathcal{L}(\alpha, G, s; \epsilon_J)$, and therefore $\mathcal{L}$ has a Fourier Cosine series with real coefficients $L_{q,k}$:

$$\mathcal{L} = L_{0,0} + 2\sum_{k \geq 1} L_{0,k} \cos k\alpha + 2\sum_{q \geq 1}\sum_{k \in \mathbb{Z}} L_{q,k} \cos(qs + k\alpha).$$

Using the method of steepest descent along adequate complex paths, and playing both with the eccentric and the true anomaly, it is possible to compute these Fourier coefficients.

Theorem (Computation of the Melnikov potential)

For $G \geq 32$, $\epsilon_J G \leq 1/8$, the Melnikov potential is given by

$$\mathcal{L}(\alpha, G, s; \epsilon_J) = \mathcal{L}_0(\alpha, G; \epsilon_J) + \mathcal{L}_1(\alpha, G, s; \epsilon_J) + \mathcal{L}_{\geq 2}(\alpha, G, s; \epsilon_J)$$

with

$$\begin{aligned}\mathcal{L}_0(\alpha, G; \epsilon_J) &= L_{0,0} + L_{0,1}\cos\alpha + \mathcal{E}_0(\alpha, G; \epsilon_J)\\ \mathcal{L}_1(\alpha, G, s; \epsilon_J) &= 2L_{1,-1}\cos(s-\alpha) + 2L_{1,-2}\cos(s-2\alpha)\\ &\quad + 2L_{1,-3}\cos(s-3\alpha) + \mathcal{E}_1(\alpha, G, s; \epsilon_J),\end{aligned}$$

where $L_{i,j} = L_{i,j}(G; \epsilon_J)$ with $L_{0,0} = \frac{\pi}{2G^3}(1 + E_{0,0})$ and

$$L_{0,1} = -\frac{15\pi\epsilon_J}{8G^5}(1 + E_{0,1}), \qquad 2\,L_{1,-1} = \sqrt{\frac{\pi}{8G}}e^{-G^3/3}(1 + E_{1,-1})$$

$$2\,L_{1,-2} = -3\sqrt{2\pi}\epsilon_J G^{3/2} e^{-G^3/3}(1 + E_{1,-2})$$

$$2\,L_{1,-3} = \frac{19}{8}\sqrt{2\pi}\epsilon_J^2 G^{5/2} e^{-G^3/3}(1 + E_{1,-3}).$$

Theorem (Continuation of the computation of the Melnikov potential)

The error functions satisfy

$$
\begin{aligned}
|E_{0,0}| &\leq 2^{12} G^{-4} + 2^2\, 49\, \epsilon_J^2 \\
|E_{0,1}| &\leq 2^{13} G^{-4} + \epsilon_J^2 \\
|E_{1-1}| &\leq 2^{21} G^{-1} + 2\, 49\, \epsilon_J^2 \\
|E_{1,-2}| &\leq 2^{17} G^{-1} + \frac{49}{3} \epsilon_J \\
|E_{1,-3}| &\leq 2^{17} G^{-1} + 15 \epsilon_J \\
|\mathcal{E}_0| &\leq 2^{14}\, \epsilon_J^2 G^{-7} \\
|\mathcal{E}_1| &\leq 2^{18} \epsilon_J e^{-G^3/3} \left[\epsilon_J^2 G^{7/2} + G^{-3/2} \right] \\
|\mathcal{L}_{\geq 2}| &\leq 2^{28} G^{3/2} e^{-2G^3/3}
\end{aligned}
$$

$s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$ is indeed a cosine-like function, that is, with a non-degenerate maximum (minimum) and no other critical points, so we can find easily its critical points.

Proposition

There exists $C > 32$ and $c < 1/8$ such that, for $G \geq C$ and $\epsilon_J G < c$, $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$ is a cosine-like function, and its two critical points are given by

$$s^*_+ = s^*_+(\alpha, G; \epsilon_J) = \alpha + \theta + \varphi^*, \qquad s^*_- = s^*_+ + \pi = \alpha + \theta + \pi + \varphi^*$$

where $\theta = \theta(\alpha, G; \epsilon_J)$ and $\varphi^* = O\left(G^{3/2} e^{-G^3/3}\right)$.

By the previous Theorem, for $G > C$ big enough and $G\epsilon_J < c$ small enough, the two critical points of $\mathcal{L}$ in the variable s are well approximated by the two critical points of the function $\mathcal{L}_0 + \mathcal{L}_1$ (in fact of $\mathcal{L}_1$ because $\mathcal{L}_0$ does not depend on s).

We can define two different reduced Poincaré functions

$$\begin{aligned}\mathcal{L}^*_\pm(\alpha, G; \epsilon_J) &= \mathcal{L}(\alpha, G, s^*_\pm; \epsilon_J)\\ &= \mathcal{L}_0(\alpha, G; \epsilon_J) \pm \mathcal{L}^*_1(\alpha, G; \epsilon_J) + \mathcal{E}_\pm(\alpha, G; \epsilon_J).\end{aligned}$$

and two different scattering maps $\widetilde{S}_\pm(\alpha, G, s) = (S_\pm(\alpha, G, s), s)$, where

$$S_\pm(\alpha, G, s) = \left(\alpha - \mu\frac{\partial \mathcal{L}^*_\pm}{\partial G}(\alpha, G; \epsilon_J) + O(\mu^2), G + \mu\frac{\partial \mathcal{L}^*_\pm}{\partial \alpha}(\alpha, G; \epsilon_J) + O(\mu^2)\right)$$

which follow closely the level curves of the Hamiltonians $\mathcal{L}^*_\pm$. More precisely, up to $O(\mu^2)$ terms, $S_\pm$ is given by the time $-\mu$ map of the Hamiltonian flow of Hamiltonian $\mathcal{L}^*_\pm$. The $O(\mu^2)$ remainder will be negligible as long as

$$|\mu| \ll \left|\frac{\partial \mathcal{L}^*_\pm}{\partial G}\right|, \left|\frac{\partial \mathcal{L}^*_\pm}{\partial \alpha}\right|,$$

which is true as long as $0 < \mu \ll \mu^* = e^{-(c/\epsilon_J)^3/3}$.

One has to check that the foliations of $\mathcal{L}^*_\pm =$ constant are different, since this will imply that the scattering maps $S_\pm$ are different. From

$$\{\mathcal{L}^*_+, \mathcal{L}^*_-\} = \{\mathcal{L}_0 + \mathcal{L}^*_1 + \cdots, \mathcal{L}_0 - \mathcal{L}^*_1 + \cdots\}$$
$$= -2\{\mathcal{L}_0, \mathcal{L}^*_1\} + \mathcal{E}_3$$

one computes

$$\{\mathcal{L}_0, \mathcal{L}^*_1\} = -\frac{15\pi\epsilon_J \mathcal{L}^*_1 d \sin\alpha}{8G^3B^2}.$$

The level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are transversal in the region $G \geq C > 32$ and $\epsilon_J G \leq c < 1/8$, except for the three curves $\alpha = 0$, $\alpha = \pi$ and $d = 0$, which are transversal to any of these level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$, see next slide.
Indeed, this is clear for the lines $\alpha = 0$ and $\alpha = \pi$, and the same happens for the curve $d = 0$ using its complete expression.

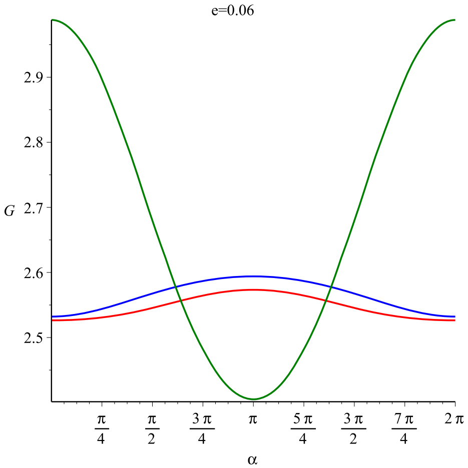

Figure: Illustration of the level Sets of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in Blue (Red) and $d = 0$ in Green

- Apart from these three curves $\alpha = 0, \pi$ and $d = 0$, at any point in the plane (α, G) the slopes $dG/d\alpha$ of the level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are different.
- We can choose which level curve increases more the value of G (see next slide).
- In the same way, we can find trajectories along which the angular momentum performs arbitrary excursions.
- Strictly speaking, this mechanism only produces pseudo-orbits, that is, heteroclinic connections between different periodic orbits in the infinity manifold which are commonly known as transition chains after Arnold.
- The existence of true orbits relies on shadowing methods [Moeckel02-07,Gidea-Llave06,Gidea-Llave-Seara14, Guardia-Martín-Seara-Sabbagh17].

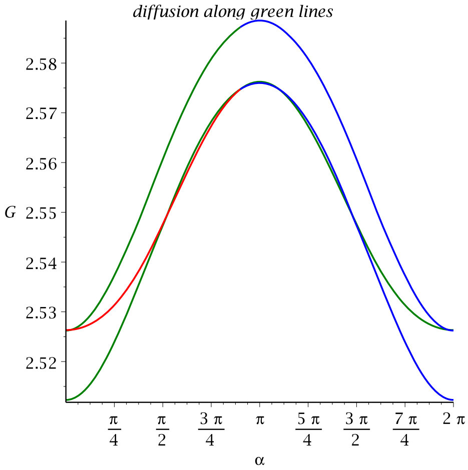

Figure: Zone of diffusion: Level curves of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in blue (red) and diffusion trajectories in green.

Theorem (Main Result again)

Let $G_1^* < G_2^*$ large enough and $\epsilon_J > 0$, $\mu > 0$ small enough. More precisely $C \leq G_1^* < G_2^* \leq c/\epsilon_J$ and $0 < \mu < \mu^* = \frac{c}{C} e^{-(8\epsilon_J)^{-3}/3}$, for $C < 32$ large enough and $c < 1/8$ small enough. Then, for any finite sequence of values $G_i \in (G_1^*, G_2^*)$, $i = 1, \ldots, n$, there exists a trajectory of the RPETB such that $G(T_i) = G_i$, $i = 1, \ldots, n$ for some $0 < T_i < T_{i+1}$. In particular, for any two values $G_1 < G_2 \in (G_1^*, G_2^*)$, there exists a trajectory such that $G(0) < G_1$, and $G(T) > G_2$ for some time $T > 0$.