Symmetries of b-symplectic manifolds

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Outline

Hamiltonian Group Actions

- An action by a group on a Poisson manifold is **Hamiltonian** if the infinitesimal generators of the action are Hamiltonian vector fields
- b-Hamiltonian vector fields can be transverse to the symplectic foliation

Definition

A ***b*-function** is a function on M of the form $c\log|f|+g$ where $c\in\mathbb{R}$, f is a defining function for Z, and g is a smooth function. We define $d(c\log|f|+g):=c\frac{df}{f}+dg$, where $\frac{df}{f}$ is a b-form.

Definition

b-Hamiltonian actions An action of $\mathbb{T}^r$ on a b-symplectic manifold (M^{2n},ω) is **b-Hamiltonian** if for all $X,Y\in\mathfrak{t}$:

- the one-form $\iota_{X^\#}\omega$ is exact, i.e., has a primitive $H_X\in {}^bC^\infty(M)$;
- $\omega(X^\#,Y^\#)=0$.

Cotangent lifts

Let G be a Lie group and let M be any smooth manifold. Given a group action $\rho : G \times M \longrightarrow M$, we define its cotangent lift as the action on T^*M given by $\rho^*_{g^{-1}}$.

$$\begin{tikzcd}
T^*M \arrow[r, "\rho^*_{g^{-1}}"] \arrow[d, "\pi"'] & T^*M \arrow[d, "\pi"] \\
M \arrow[r, "\rho_g"] & M
\end{tikzcd}$$

These actions are automatically Hamiltonian with respect to the canonical symplectic form on T^*M

canonical b-cotangent lift

- Take G a Lie group and $H \subset G$ a codimension one closed subgroup.
- Define the canonical b-symplectic form $\omega = \frac{dx}{x} \wedge dp + \sum dx_i \wedge dp_i$
- Define cotangent lifted action on ${}^bT^*G$ by pulling back b-forms.

$$\rho(h) : (h, \alpha_g) \mapsto (L_{h^{-1}})^* \alpha_g.$$

Theorem

Let ${}^bT^*G$ be endowed with the canonical b-Poisson structure. Then the Poisson reduction under the cotangent lifted action of H by left translations is

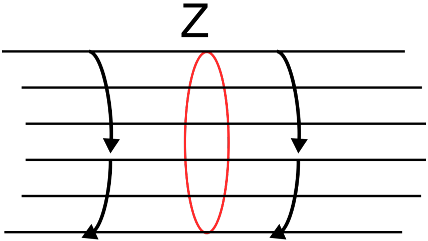
$$(({}^bT^*G)/H, \Pi_{red}) \cong (\mathfrak{h}^* \times {}^bT^*(G/H), \Pi_{L\text{-}P} + \Pi_{b\text{-}can})$$

where $\Pi_{L\text{-}P}$ is the Lie-Poisson structure on $\mathfrak{h}^*$ and $\Pi_{b\text{-}can}$ is the canonical b-symplectic structure on ${}^bT^*(G/H)$, where G/H is viewed as a b-manifold with critical hypersurface the point $[e]_\sim$.

another b-cotangent lift

an example

As vector fields on the cotangent bundle generated by the cotangent lifted action of a group on itself are horizontal, they are automatically elements of the b-tangent bundle ${}^{b}T(T^*\mathbb{S}^1)$ with critical set given by the zero section in $(T^*\mathbb{S}^1)$.



an invariant b-symplectic form

Given the cotangent lifted action of a torus action on itself on its cotangent bundle $\omega := \frac{c}{a_1} da_1 \wedge d\theta_1 + \sum_{i=2}^{n} da_i \wedge d\theta_i$ is invariant under the action

b-cotangent lifts

a non example

Proposition

There is no invariant b-symplectic structure on the cotangent bundle of a semi-simple Lie group.

Proof

Write $\omega = \frac{df}{f} \wedge \alpha + \beta$.
If α is closed, then, $d\alpha|_e([\eta, \nu]) = 0$ for all $\eta, \nu \in \mathfrak{g}$ and the Lie algebra is necessarily a semidirect product of $\ker(\alpha)$ and the linear subspace given by $\mu, \alpha(\mu) \neq 0$. So G is of the form $\mathbb{S}^1 \times H$ (if compact)

The best we can do is products $T^*(\mathbb{S}^1 \times H)$:(

Outline

The defining forms of the cosymplectic foliation

Definition

A form $\alpha \in \Omega^1$ is a **defining one-form** of the foliation if it is nowhere vanishing and $\iota_{\mathcal{L}}^* \alpha = 0$ for all leaves $\mathcal{L}$, $\iota_{\mathcal{L}}$ the canonical inclusion.

Definition

A form $\omega \in \Omega^2$ is a **defining two-form** of the foliation if $\iota_{\mathcal{L}}^* \omega$ is the symplectic form on each leaf of the foliation.

A cosympletic manifold is the critical hypersurface of a *b*-symplectic manifold iff we can choose these forms to be closed.

The Extension Problem

Theorem

Let ω_0 and ω_1 be two b-symplectic forms on (M, Z) such that $\omega_0|_Z = \omega_1|_Z$. Then ω_0 and ω_1 are locally b-symplectomophic around Z

Proof

$\omega_0 - \omega_1$ is an honest de Rham form. Use Moser's trick.

To see if two forms are *b*-symplectomorphic it is enough to check

- The induced Poisson structure on Z
- Their modular vector fields

Outline

Equivariant b-Moser theorem

Theorem

Let ω_0 and ω_1 be two b-symplectic forms on (M, Z) such that $\omega_0|_Z = \omega_1|_Z$ and ω_0 and ω_1 are invariant under the action of a group on M. Then ω_0 and ω_1 are locally b-symplectomorphic around Z and we can choose the b-symplectomorphism to be G invariant.

Proof

Moser's trick gives a vector field whose solution is a b-symplectic isotopy. Average this by G.

Corollary

Around Z, there is a tubular neighbourhood $(-\epsilon, \epsilon) \times Z$ where our b-symplectic form G-equivariantly looks like

$$\omega = \frac{dt}{t} \wedge p^*\alpha + p^*\beta,$$

Outline

1. Effective Hamiltonian toric actions
2. b-cotangent lifted actions
3. Normal Forms about Group Actions
   - The Structure of the Critical Hypersurface
   - Equivariant b-symplectic geometry
   - Product Structure on Z
   - A Normal Form Theorem

Product Structure on Z

Proposition

Let G act on a connected component of the critical hypersurface of a b-symplectic manifold. If an orbit of the group action is transverse (on Z) to the symplectic foliation at some point, then it is transverse to the symplectic foliation at every point.

Proof

The defining one form of the foliation must pull back to a defining one form. At p the restriction of α to $T^*\mathcal{O}_p$ is non zero. Consider any non-trivial action of ϕ on $\mathcal{O}_p$ then $\phi^*\alpha \neq 0$ implies the action is transverse to leaves.

Product Structure on Z

Proposition

If G is a Poisson action of a compact group with the action transverse (on Z) to the symplectic foliation then Z is a product of a symplectic leaf and $\mathbb{S}^1$.

Proof

We have a Poisson vector field transverse to the symplectic leaves with orbits isomorphic to $\mathbb{S}^1$.

Outline

A Normal Form Theorem

Proposition

Let G be a Lie group acting on b-symplectic manifold such that the action is transverse to the symplectic foliation. Then there exists a neighbourhood of the critical set such that the symplectic form is the G-equivariantly equivalent to the form

$$\omega := \frac{c}{a_1} da_1 \wedge d\theta_1 + \tilde{\omega}$$

$T^*\mathbb{S}^1 \times \mathcal{L}$ with the action of G on on $T^*\mathbb{S}^1 \times \mathcal{L}$ the cotangent lifted action times the action subgroup of G on a leaf.

A Normal Form Theorem

Proof

$T^*\mathbb{S}^1 \cong \mathfrak{g}^* \times \mathbb{S}^1$ is G-equivariantly diffeomorphic to $\mathbb{S}^1$ on $\mathfrak{g}^*$ with the trivial action of $\mathbb{S}^1$ on $\mathfrak{g}^*$. Construct the local diffeomorphism

$$\phi : \mathfrak{g}^* \times \mathbb{S}^1 \times U \to (-\epsilon, \epsilon) \times \mathcal{O}_z \times U$$

by sending $[g] \in \mathbb{S}^1/\mathbb{S}^1_z \cong \mathbb{S}^1$ to gz and where ϕ is the identity on U. Pull back the b-symplectic form. It has to look like

$$\omega := \frac{c}{a_1} da_1 \wedge d\theta_1 + \tilde{\omega}$$

References

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