Lambert's theorem on constant curvature spaces

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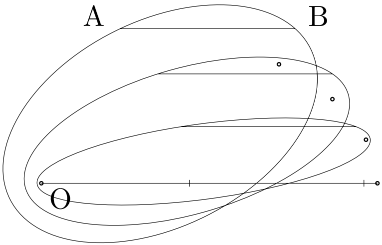
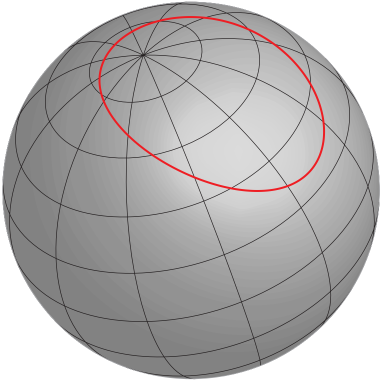
J.-H. Lambert (1728–1777) is one of the founders of non-Euclidean geometry, but he also discovered a strange and useful property of the Keplerian motion in a Euclidean space. The time required to reach a point B from a point A with a given energy, under the Newtonian attraction of a mass at a fixed point O, will not vary if we shift A and B in such a way that $\|AB\|$ and $\|OA\| + \|OB\|$ remain constant.



P. Serret (1827–1898) and W. Killing (1847–1923) introduced the Kepler problem in constant curvature spaces and listed its impressive analogies with the usual Kepler problem.



We complete this list by proving that the time required to reach a point B from a point A, under the attraction of a mass at a fixed point O of the curved space, with given energy, does not vary if we shift A and B in such a way that $d(\mathrm{A},\mathrm{B})$ and $d(\mathrm{O},\mathrm{A})+d(\mathrm{O},\mathrm{B})$ remain constant, where d is the geodesic distance.

We will also discuss the case of pseudo-Riemanian spaces with constant curvature. We will mainly use the well-known formulas of variational calculus that Hamilton introduced in 1834, and a simple property of the eccentricity vector. This work benefited from many discussions with Zhao Lei, from the University of Augsburg.

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Lambert's theorem is a strange statement. It is answering a question, but what was the question? For classical authors, the question was: how to compute less? Typically they wanted to determine the elements of the orbit of a new comet. Solving Kepler equation $l = u - e \sin u$ in each step of the computation makes it very long. With Lambert we can use the same Kepler equation for several Keplerian orbits.

But let us make the question more "Hamiltonian".

What was the question?

Given a natural dymamical system (with a configuration space and velocities), a departure point A and an arrival point B, there is a one-parameter family of trajectories going from A to B. This parameter may be locally the energy E: by the (Leibniz-) Maupertuis- Jacobi variational principle, namely, by considering stationary trajectories for the action $\int_{t_\mathrm{A}}^{t_\mathrm{B}} \|\dot{q}\|^2 dt$ for a given energy E, we find isolated paths.

Here q is the configuration, $\dot{q} = dq/dt$ is the velocity.

First question. Is there a pair of tangent vectors $(\delta\mathrm{A}, \delta\mathrm{B})$ depending on $(\mathrm{A}, \mathrm{B}, E)$ such that shifting (A, B) along the flow generated by $(\delta\mathrm{A}, \delta\mathrm{B})$, while keeping the same energy E, does not change the time Δt required to reach B from A?

answer: YES, of course

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Answer: yes, always, we just follow the levels of $\Delta t = t_\mathrm{B} - t_\mathrm{A}$.

Second question. Is there a pair of tangent vectors $(\delta\mathrm{A}, \delta\mathrm{B})$ depending on (A, B) such that shifting (A, B) along the flow generated by $(\delta\mathrm{A}, \delta\mathrm{B})$, while keeping the same energy E, does not change the time Δt required to reach B from A?

Answer: we have the symmetries of the system, but for some systems we have something else. We have Lambert's theorem for the Kepler problem.

Third question. Is there a pair of tangent vectors $(\delta\mathrm{A}, \delta\mathrm{B})$ depending on (A, B) such that shifting (A, B) along the flow generated by $(\delta\mathrm{A}, \delta\mathrm{B})$, while keeping the same energy E, does not change $w = \int_{t_\mathrm{A}}^{t_\mathrm{B}} \|\dot{q}\|^2 dt$?

Consistency of the third question

Third question. Is there a pair of tangent vectors $(\delta\mathrm{A}, \delta\mathrm{B})$ depending on (A, B) such that shifting (A, B) along the flow generated by $(\delta\mathrm{A}, \delta\mathrm{B})$, while keeping the same energy E, does not change $w = \int_{t_\mathrm{A}}^{t_\mathrm{B}} \|\dot{q}\|^2 dt$?

1) If we get $(\delta\mathrm{A}, \delta\mathrm{B})$ answering positively the third question, it does not change Δt either, due to a formula emphasized by Hamilton in 1834:

$$\Delta t = \left(\frac{\partial w}{\partial E}\right)_{\mathrm{A,B\ fixed}}$$

where E is the energy. Indeed w remains the same function of E when we follow the flow of $(\delta\mathrm{A}, \delta\mathrm{B})$, so Δt also remains the same function of E.

2) There is a nice and simple formula for δw also emphasized by Hamilton in 1834:

$$\delta w = \langle \delta\mathrm{B}, \dot{q}_\mathrm{B} \rangle - \langle \delta\mathrm{A}, \dot{q}_\mathrm{A} \rangle.$$

Examples