ENTROPY METHODS, FUNCTIONAL INEQUALITIES AND APPLICATIONS

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ABSTRACT. Various functional inequalities are classically seen from a variational point of view in nonlinear analysis. They also have important consequences for evolution problems. For instance, entropy estimates are standard tools for relating rates of convergence towards asymptotic regimes in time-dependent equations with optimal constants of various functional inequalities. This point of view applies to linear diffusions and will be illustrated by some results on the Fokker-Planck equation based on the *carré du champ* method introduced by D. Bakry and M. Emery. In the recent years, the method has been extended from linear to nonlinear diffusions. This aspect will be illustrated by results on Gagliardo-Nirenberg-Sobolev inequalities on the sphere and on the Euclidean space. Even the evolution equations can be used as a tool for the study of detailed properties of optimal functions in inequalities and their refinements. There are also applications to other equations than pure diffusions: hypocoercivity in kinetic equations is one of them. In any case, the notion of entropy has deep roots in statistical mechanics, with applications in various areas of science ranging from mathematical physics to models in biology. A special emphasis will be put during the course on the corresponding models which offer many directions for new research developments.

PLAN OF THE COURSE

(1) history of entropy methods; heat flow, Nash and logarithmic Sobolev inequalities; the logarithmic Sobolev inequality and gaussian type interpolation inequalities by the carré du champ method; interpolation inequalities on the sphere; other classical inequalities; the large dimension limit
(2) functional inequalities on the Euclidean space; Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$ and fast diffusion flows; variational methods and entropy methods; the Sobolev inequality
(3) Some examples of applications in functional analysis and mathematical physics: 1) Legendre duality; 2) Keller duality; 3) Caffarelli-Kohn-Nirenberg inequalities; variational methods; symmetry and symmetry breaking
(4) an overview of stability results on $\mathbb{S}^d$ and on $\mathbb{R}^d$

NOTES AND REFERENCES OF THE COURSE

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/M2-Math2025/M2-Math2025.pdf

Date: January 20, 2025. *File:* M2-Math2025.tex.

SOME LINKS

Here is a short list of monographs and paper that are good entry points for *entropy methods* for PDEs:

• The book [2] by D. Bakry, I. Gentil and M. Ledoux presents in a mainly linear framework the results obtained by the *carré du champ* method and its extensions. It is a synthetic presentation of a very large number of works, which remains perfectly accessible with a basic training in analysis.

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/BGL2014.pdf
(login, password)

• In [5], A. Jüngel gave a concise presentation of entropy methods on a few well-chosen examples, which include the fast diffusion equation.

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/Juengel2016.pdf
(login, password)

This book is an excellent introduction to entropy methods applied to nonlinear parabolic equations.

• The book [6] by C. Villani on mass transport is an essential reference. Although very complete, it remains relatively easy to access. On the gradient flows, we can refer to [1] which is however much more technical and advanced.

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/Villani2009.pdf
(login, password)

• Although it is not (yet) a book, C. Schmeiser's lecture notes are available online at

https://homepage.univie.ac.at/christian.schmeiser/Entropy-course.pdf

and have the advantage of being accompanied by the full video

https://sciencesmaths-paris.fr/f/actualites-fr/entropy-methods

of the course he gave at the Institut Henri Poincaré in the fall of 2018.

• Parts of this course is taken from [4, Section 2]

https://www.ceremade.dauphine.fr/~dolbeaul/Preprints/Publi/1704.pdf
(login: Dolbeault ; password: Jean)

and from the Memoir [3] on the *Stability in Gagliardo-Nirenberg-Sobolev inequalities: flows, regularity and the entropy method*

https://hal.science/hal-02887010

• Dauphine-PSL - Master 2 : *Logarithmic Sobolev Inequalities Essentials*, by Djalil Chafai and Joseph Lehec

https://djalil.chafai.net/docs/M2/chafai-lehec-m2-lsie-lecture-notes.pdf

• For a course on *symmetry and nonlinear diffusion flows*, please refer to

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/UChile-2017/LectureNotes.pdf

• Stability results for Sobolev, logarithmic Sobolev, and related inequalities (Ghent, 2024):

https://www.ceremade.dauphine.fr/ dolbeaul/Preprints/Fichiers/Dolbeault-Ghent.pdf

• For general techniques in Analysis and functional inequalities, the book *Analysis* by Eliott H. Lieb and Michael Loss is highly recommended

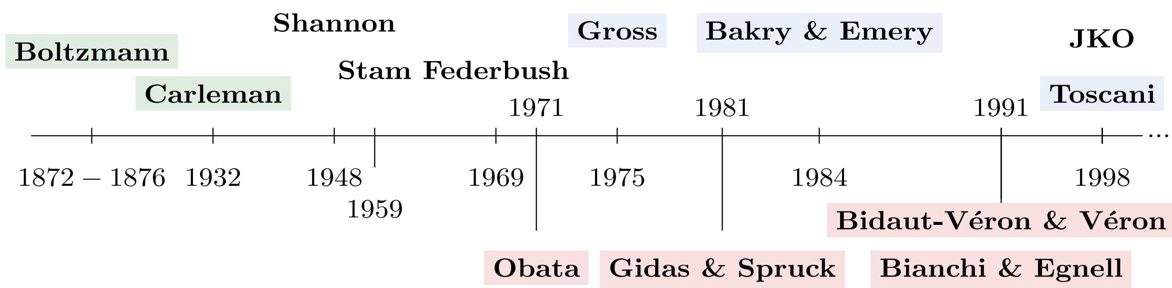
https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/Lieb-Loss2001.pdf

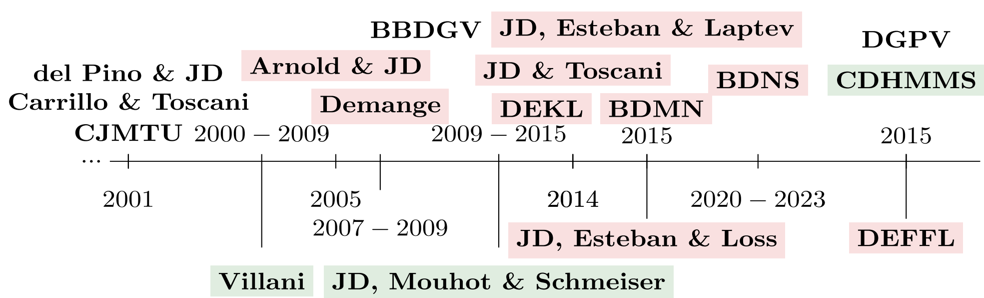
1. First course

1.1. **Introduction to entropy methods.**

1.1.1. *Notions of entropy.* A brief history of entropy methods. Some recent directions of research. A key idea: the link between functional inequalities and rates in evolution equations.

1.1.2. *An introduction to the literature.* What you can find in the main references of this course. Here is a *Time Line* with some important references (for this course).





kinetic equations and hypocoercivity

Semi-groups, linear diffusions and Markov processes

Nonlinear elliptic & parbolic PDEs, functional inequalities, symmetry, stability, *etc.*

1.2. **A list of functional inequalities.**

1.2.1. *Poincaré inequality.* The inequality for a bounded measure. Link with ground state estimates for the Schrödinger operator. Persson's lemma. Application to the measure $\exp(-|x|^\alpha)\,dx$ on $\mathbb{R}^d$ with $\alpha \ge 1$.

1.2.2. *Hardy inequality.* Proof by the expansion of the square method.

1.2.3. *Sobolev inequality.* A quick introduction to Sobolev, duality and Hardy-Littlewood-Sobolev inequalities, and Bianchi-Egnell stability results.

1.2.4. *Gagliardo-Nirenberg inequalities.* The interpolation of the L^p norm by the L^2 norm and the L^2 norm of the gradient, the case of dimension $d = 1$. The interpolation of the L^{2p} norm by the L^{p+1} norm and the L^2 norm of the gradient, gradient flow.

1.2.5. *Caffarelli-Kohn-Nirenberg inequalities.* A quick introduction symmetry versus symmetry breaking issues.

1.2.6. *Logarithmic Sobolev inequalities.* Logarithmic Sobolev inequalities as a limit case of the Gagliardo-Nirenberg inequalities. Standard Gaussian measure and various equivalent forms of the logarithmic Sobolev inequality: gaussian form, Euclidean form, scale invariant form.

1.2.7. *Nash inequality.* The original proof by Nash/Stein.

1.2.8. *Beckner inequalities with Gaussian weight.* Inequalities which interpolate between the Gaussian Poincaré inequality and the Gaussian logarithmic Sobolev inequality. The Gaussian logarithmic Sobolev inequality is the strongest of these inequalities. The proof by convex interpolation (Latala and Oleskiewicz).

1.3. **Heat and Fokker-Planck equation.**

1.3.1. *Heat equation.* Heat flow and Nash inequality: a classical interplay between decay rates and a simple evolution equation. Green function and the four steps method of J.L. Vázquez. Nash inequality and decay rate in L^2. The relative entropy method: self-similar rescaling. Fokker-Planck and Ornstein-Uhlenbeck equations. Carré du champ method in L^p with $1 \le p < 2$ and the limit case with $p = 2$. Convergence in L^1 for the solution of the Fokker-Planck equation and intermediate asymptotics for the solution of the heat equation. Side remark 1: Hermite functions. Side remark 2: Nelson's lemma and the hypercontractivity.

1.3.2. *The carré du champ method.* Proof of the Gaussian logarithmic Sobolev inequality by the Bakry-Emery approach. A generalization to φ-entropies and log-concave measures. Beckner type inequalities with construction of the constant and a first stability result. Properties of φ entropies based on [4]: Holley-Stroock, tensorization.

REFERENCES

[1] L. AMBROSIO, N. GIGLI, AND G. SAVARÉ, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2005.
[2] D. BAKRY, I. GENTIL, AND M. LEDOUX, *Analysis and geometry of Markov diffusion operators*, vol. 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Cham, 2014.
[3] M. BONFORTE, J. DOLBEAULT, B. NAZARET, AND N. SIMONOV, *Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method*, Preprint arXiv: 2007.03674 and hal-03160022, to appear in *Memoirs of the AMS*, (2022).
[4] J. DOLBEAULT AND X. LI, *Φ-Entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations*, Mathematical Models and Methods in Applied Sciences, 28 (2018), pp. 2637–2666.
[5] A. JÜNGEL, *Entropy Methods for Diffusive Partial Differential Equations*, SpringerBriefs in Mathematics, Springer International Publishing, 2016.
[6] C. VILLANI, *Optimal transport*, vol. 338 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2009. Old and new.

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