Rigorous Derivation of Macroscopic PDEs from Microscopic stochastic particle systems

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Quantitative Hydrodynamical limits from stochastic interacting particle systems

• Goal: present an abstract method (quantitative) to prove a hydrodynamic limit for

- Zero-Range process (ZRP)
- Simple Exclusion process (SEP)
- Kawasaki dynamics with Ginzburg-Landau type potentials

[Joint work with D. Marahrens and C. Mouhot]

• $\mathbb{T}^d = [0,1)^d$, $N \in \mathbb{N}$ the inverse of the distance among sites.
We are interested in the limit as $N \to \infty$.
Notation: $\eta(x)$ = number of particles on $\left[\frac{x}{N}, \frac{x+1}{N}\right)$

$$\eta = \{\eta(0), \dots, \eta(N-1)\} \in X_N := \mathbb{N}^{\mathbb{T}_N^d}, \quad \mathbb{T}_N^d = \{0, 1, \dots, N-1\} = \mathbb{Z}/N\mathbb{Z}.$$

- macroscopic scale $\mathbb{T}^d$ $(u, v, w, \dots)$
- microscopic scale $\mathbb{T}_N^d$ $(x, y, z, \dots)$ correspond to points of the form $\frac{x}{N}, \frac{y}{N}, \dots \in \mathbb{T}^d$.

Let $\eta \in X_N$ and functions $\{c(x,y,\cdot) : x,y \in \mathbb{T}_N^d\} \Rightarrow$ Markov process with generator:

$$(\mathcal{L}f)(\eta) := \sum_{x \sim y \in \mathbb{T}_N^d} c(x,y,\eta)(f(\eta^{x,y}) - f(\eta))$$

where

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

Generator satisfies then

$$\frac{d}{dt}\langle \mu_t^N, f \rangle = \langle \mu_t^N, \mathcal{L}f \rangle, \quad f \in C_b(X_N).$$

Assume that $\forall \alpha > 0, \exists!$ equilibrium measure ν_α with density α s.t.

$$\int \mathcal{L} f d\nu_\alpha = 0, \quad \int \eta(0) d\nu_\alpha(\eta) = \alpha, \quad \int \tau_x f(\eta) d\nu_\alpha = \int f(\eta) d\nu_\alpha(\eta).$$

Definition (mesaure with slowly varying parameter)

$\forall f_0$ smooth function, $\nu^N_{f_0(\cdot)}$: the product measure on X_N s.t.

$$\nu^N_{f_0(\cdot)}\big(\{\eta : \eta(x) = k\}\big) = \nu^N_{f_0(x/N)}\big(\{\eta : \eta(0) = k\}\big)$$

and under $\nu^N_{f_0(\cdot)}$ the variables $\{\eta(x) : x \in \mathbb{T}^d_N\}$ are independent.

• Empirical measure $\alpha_\eta^N(du) := N^{-d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{x/N}(du) \in \mathcal{M}^+(\mathbb{T}^d)$.

• Does α_η^N, where η has law μ_t^N, approach a deterministic profile f_t satisfying a macroscopic equation? i.e. for $\phi \in C(\mathbb{T}^d)$, $\forall \delta > 0$:

$$\mathbb{P}_{\mu_t^N}\left(\left| \langle \alpha_\eta^N, \phi \rangle - \langle f_t, \phi \rangle \right| > \delta \right) \overset{N \to \infty}{\to} 0, \quad \text{where } \partial_t f_t = L f_t.$$

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About L:

- asymmetric process: rescale time t by N and space x by N (hyperbolic scaling) $\Rightarrow$

$$\partial_t f_t = \gamma \cdot \nabla\sigma(f_t), \gamma := \sum_z z p(z)$$

- mean-zero process: rescale time t by N^2 and space x by N (diffusive scaling) $\Rightarrow$

$$\partial_t f_t = \Delta_c \sigma(f_t),\ \Delta_c = \sum_{1\le i,j\le d} c_{ij}\partial_{u_i}\partial_{u_j}.$$

Simple Exclusion Process (SEP)

- We allow *at most* 1 particle per site: state space $X_N = \{0,1\}^{\mathbb{T}_N^d}$.
- Generator:

$$(\mathcal{L}f)(\eta) = \sum_{x\sim y} \eta(x)(1-\eta(y))p(x-y)\big(f(\eta^{x,y}) - f(\eta)\big).$$

- Equilibrium measure: For $\alpha \in (0,1)$, ν_α^N the Bernoulli product measure

$$\nu_\alpha^N(\eta) = \prod_x \alpha^{\eta(x)}(1-\alpha)^{1-\eta(x)}$$

- particle densities of Sym. SEP $\Rightarrow$ as $N \to \infty$, $\partial_t f_t = \Delta_c f_t$.
- particle densities of Asym. SEP $\Rightarrow$ as $N \to \infty$, $\partial_t f_t = \gamma \cdot \nabla f_t(1-f_t)$.

Zero-Range Process (ZRP)

- No restrictions on the total number of particles per site: state space $X_N = \mathbb{N}^{\mathbb{T}_N^d}$.
- Rate function $g : \mathbb{N} \to \mathbb{R}_+, g(0) = 0, g(n) > 0$ for all $n \in \mathbb{N}^*$, and
- $g^* := \sup_k |g(k+1) - g(k)| < \infty$.
- Generator: $(\mathcal{L}f)(\eta) = \sum_{x\sim y} g(\eta(x))p(x-y)\big(f(\eta^{x,y}) - f(\eta)\big)$.
- Invariant measure: $\forall \alpha > 0$,

$$\nu_\alpha^N(\eta) = \prod_{x\in\mathbb{T}_N^d} \frac{\sigma(\alpha)^{\eta(x)}}{g(\eta(x))!Z(\sigma(\alpha))}, \quad Z(\varphi) = \sum_{k\geq 0} \frac{\varphi^k}{g(k)!}.$$

Zero-Range Process (ZRP), Invariant measure structure

- Nonlinearity functional σ prescribed s.t. $\langle \nu_\alpha^N, \eta(0) \rangle = \alpha$. [As the inverse of $R : [0,\infty) \to \mathbb{R}, R(\varphi) = \mathbb{E}_{\bar{\nu}_\varphi^N}(\eta(0))$, where $\bar{\nu}_\varphi^N(\{\eta(x) = k\}) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!}$.]
- ν_α^N translation invar p.m. with $\langle \nu_\alpha^N, \eta(x) \rangle = \alpha$, $\langle \nu_\alpha^N, g(\eta(x)) \rangle = \sigma(\alpha)$.

Assumptions for the hydrodynamic limit:

(iii) $g(n) - g(m) \geq \delta$ for $\delta > 0$ & $n - m \geq n_0$ for $n_0 > 0$.

(iv) $g(n+1) \geq g(n)$.

- particle densities of Sym. ZRP $\Rightarrow$ as $N \to \infty$, $\partial_t f_t = \Delta_c \sigma(f_t)$.
- particle densities of Asym. ZRP $\Rightarrow$ as $N \to \infty$, $\partial_t f_t = \gamma \cdot \nabla \sigma(f_t)$.

Ginzburg-Landau with Kawasaki dynamics

- To each lattice site $x \in \mathbb{T}_N$ we associate a variable $\eta(x) \in \mathbb{R}$. State space $X_N = \mathbb{R}^{\mathbb{T}_N}$.
- Hamiltonian: $H(\eta) = \sum_{x \in \mathbb{T}_N} V(\eta(x))$, where V is one-body potential: $V(u) = V_0(u) + V_1(u)$ with

$$V_0''(u) \geq \lambda > 0 \quad \text{and } \|V_1\|_{L^\infty(\mathbb{T})}, \|V_1'\|_{L^\infty(\mathbb{T})} \leq C.$$

- Kawasaki dynamics: the SDE

$$d\eta_t(x) = \frac{N^2}{2}\Delta^N V'(\eta(x))dt + N(dW_t(x) - dW_t(x+1)).$$

Ginzburg-Landau with Kawasaki dynamics

Generator:

$$\mathcal{L} := \frac{N^2}{2} \sum_{x\sim y\in\mathbb{T}_N} \left(\frac{\partial}{\partial\eta(x)} - \frac{\partial}{\partial\eta(y)}\right)^2 - \\ - \frac{N^2}{2} \sum_{x\sim y\in\mathbb{T}_N} \left(\frac{\partial V}{\partial\eta(x)} - \frac{\partial V}{\partial\eta(y)}\right)\left(\frac{\partial}{\partial\eta(x)} - \frac{\partial}{\partial\eta(y)}\right).$$

symmetric w.r.t. the invariant product measure:

$$\nu^N(\eta) = e^{-\sum_{x\in\mathbb{T}_N} V(\eta(x_i))}.$$

- Let $M(\lambda) = \int e^{\lambda u - V(u)}$. Define

$$p(\lambda) := \log M(\lambda),\ \ h(y) := \sup_{\lambda\in\mathbb{R}}(\lambda y - p(\lambda))$$

- h, p so that $h'(y) = \lambda$ iff $y = p'(\lambda)$.
- particle densities of Sym. Ginzburg-Landau process $\Rightarrow \partial_t f_t = \Delta h'(f_t)$.

Some state of the art

• [J. Fritz'89]: Hydrodynamic limit for the Ginzburg-Landau model
• [Guo-Papanicolaou-Varadhan'88]: 'Entropy method for the hydrodynamic limit'- general method

For ZRP: Assume on the initial data $\mu_0^N \in L^\infty(\mathbb{T}^d)$

$$H\left(\mu_0^N \middle| \nu_\rho^N\right) = \int_{X_N} \ln\left(\frac{d\mu_0^N}{d\nu_\rho^N}\right) \mathrm{d}\mu_0^N \lesssim N^d, \ \left\langle \mu_0^N, \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta(x)^2 \right\rangle \lesssim 1$$

Then there is propagation in time of the deterministic limit:

$$\mathbb{P}_{\mu_0^N}\left(\left\{\left|\langle \alpha_{\eta_0}^N, \varphi\rangle - \langle f_0, \varphi\rangle\right| > \varepsilon\right\}\right) \overset{N\to\infty}{\to} 0$$

implies for later times $t > 0$

$$\mathbb{P}_{\mu_t^N}\left(\left\{\left|\langle \alpha_{\eta_t}^N, \varphi\rangle - \langle f_t, \varphi\rangle\right| > \varepsilon\right\}\right) \overset{N\to\infty}{\to} 0.$$

Some state of the art

- [Yau1991] If furthermore $f \in C^3(\mathbb{T}^d)$ (smooth solution at the limit) and

$$\frac{1}{N^d} H\left(f_0^N | \nu_{f_0}^N\right) \overset{N\to\infty}{\to} 0$$

for the local equilibrium product measure

$$\nu_f^N := \prod_{x\in\mathbb{T}_N^d} \frac{\sigma(f(x/N))^{\eta(x)}}{Z(\sigma(f(x/N)))g(1)\cdots g(\eta(x))}$$

then at later times $t > 0$,

$$\frac{1}{N^d} H\left(f_t^N | \nu_{f_t}^N\right) \overset{N\to\infty}{\to} 0$$

Some state of the art

- [Grunewald-Otto-Villani-Westdinckenberg ‘09] for Ginzburg-Landau model - 1st step towards quantitative results
- [Dizdar-Menz-Otto-Wu ‘18] Quantitative hydrodynamic limit for GL model
- [E.Kosygina ‘05],[M.Fathi ‘12] - Convergence of the entropy
- Hyperbolic scaling: [F. Rezakhanlou ‘91] - Hydrodynamic limit *for all times* towards a conservation law

Questions and motivations

- Quantitative rate of convergence? not in GPV, almost in GOVW, Yau's relative entropy methods can be made explicit, but rate $O(e^{\lambda t})$ with $\lambda > 0$ large, and for smooth solutions.
- However both the many-particle and limit systems are dissipative, hence ergodicity and relaxation should win over stochastic fluctuations at the level of the laws.

$$\begin{array}{ccc} \mu_t^N \in \mathcal{P}(X_N) & \xrightarrow{N\to\infty} & f_t \in L^\infty(\mathbb{T}^d) \\ \Big\downarrow {\scriptstyle t\to\infty} & & \Big\downarrow {\scriptstyle t\to\infty} \\ \nu_\rho^N \in \mathcal{P}(X_N) & \xrightarrow{N\to\infty} & f_\infty \end{array}$$

Abstract Theorem - Assumptions

(H1) Microscopic Stability. We define a coupling among 2 processes, generator $\widetilde{\mathcal{L}}$, with

$$\widetilde{\mathcal{L}}\left(N^{-d}\sum_{x\in\mathbb{T}_N^d}|\eta(x)-\zeta(x)|\right)\leq 0$$

(H2) Macroscopic Stability. Let $(H,\|\cdot\|_H)$ be the space of solutions to the limit PDE. $\exists T>0$:

$$\|\nabla^s f_t\|_H\leq K,\forall\ t\in[0,T],\ \forall\ s \text{ multi-index } |s|\leq 4.$$

When $T=\infty$, $\|\nabla^s(f_t-f_\infty)\|_H\lesssim R(t)\in L^1_t$.

Abstract Theorem - Assumptions

(H3) Consistency Estimate. For $k > 0, \rho \geq 0$, there is $\mathcal{E}^N \to 0$ s.t.

$$\iint_{X_N^2} N^{-d} \sum_x |\eta(x) - \zeta(x)|^k (\partial_t - \mathcal{L}^*) \psi_t^N(\zeta) d\nu_\rho^N(\eta) d\nu_\rho^N(\zeta) \lesssim \mathcal{E}^N \max_{s\in\{1,\dots,4\}} \|D^s(f_t - f_\infty)\|_H.$$

Theorem (Marahrens-M.-Mouhot, 2021)

*Let $d = 1$, $F \in \mathrm{Lip}(\mathbb{R})$, $\phi \in C(\mathbb{T}^d)$. Under **(H1)-(H2)-(H3)** and if initially, $\exists\ \mathcal{R}^N \to 0$:*

$$\int_{X_N} \left| N^{-d} \sum_x \eta(x) \phi\left(\frac{x}{N}\right) - \int f_0(u)\phi(u) du \right| d\mu_0^N(\eta) \leq C_0 \mathcal{R}^N,$$

$$\int_{X_N^2} \sum_x |\eta(x) - \zeta(x)|^k G_0^N(d\eta, d\zeta) \leq C_0 \mathcal{R}^N, \Rightarrow$$

Abstract Theorem (Marahrens-M.-Mouhot, 2021)

$\exists\ 0 < C_1, C_2 < \infty$ independent of N, t and

$$r(t) = \begin{cases} \in L^1((0,\infty)) & \text{if } T = \infty, \\ tK & \text{if } T < \infty. \end{cases}$$

such that for all $t \geq 0$

$$\left| \int_{X_N} F\left(N^{-d} \sum_{x \in \mathbb{T}_N^d} \eta(x)\phi\left(\frac{x}{N}\right)\right) - F\left(\int_{\mathbb{T}^d} f_t(u)\phi(u)du\right) d\mu_t^N(\eta) \right| \leq$$

$$\leq C_1 r(t)\mathcal{E}^N + \mathcal{R}^N + C_2 N^{-\frac{d}{d+2}}.$$

proof of Theorem

$F \in \mathrm{Lip}(\mathbb{R})$ so we need to bound

$$\left| \int_{X_N} \frac{1}{N^d} \sum_x \eta(x) \phi\left(\frac{x}{N}\right) d\mu_t^N(\eta) - \int_{\mathbb{T}^d} f_t(u)\phi(u)du \right|$$

- Using the coupling density G_t^N we have

$$\iint_{X_N^2} \left| \frac{1}{N^d} \sum_x (\eta(x) - \zeta(x))\phi\left(\frac{x}{N}\right) \right| G_t^N(\eta,\zeta) d\nu_\lambda^N(\eta) d\nu_\lambda^N(\zeta)$$
$$+ \int_{X_N} \left| \frac{1}{N^d} \sum_x \zeta(x)\phi\left(\frac{x}{N}\right) - \int_{\mathbb{T}^d} f_t(u)\phi(u)du \right| d\nu_{f_t(\cdot)}^N(\zeta)$$
$$\leq C_1 \iint_{X_N^2} \frac{1}{N^d} \sum_x |\eta(x) - \zeta(x)| G_t^N(\eta,\zeta) d\nu_\lambda^N(\eta) d\nu_\lambda^N(\zeta) + C_2 N^{-d/(d+2)}$$

proof cont.

For the first term:

$$\frac{d}{dt}\iint_{X_N^2} N^{-d}\sum_x |\eta(x)-\zeta(x)|\,G_t^N(\eta,\zeta)d\nu_\lambda^N(\eta)d\nu_\lambda^N(\zeta)$$

$$\overset{(H3)}{\leq} \iint_{X_N^2}\frac{1}{N^d}\sum_x |\eta(x)-\zeta(x)|\widetilde{\mathcal{L}}_N^* G_t^N(\eta,\zeta)d\nu_\lambda^N(\eta)d\nu_\lambda^N(\zeta) + \max_k \|D^k(f_t-f_\infty)\|_H \mathcal{E}^N$$

$$\overset{(H1)}{\leq} \max_k \|D^k(f_t-f_\infty)\|_H \mathcal{E}^N.$$

proof cont.

Integrate in time:

$$\iint_{X_N^2} \frac{1}{N^d} \sum_x |\eta(x) - \zeta(x)| G_t^N(\eta,\zeta) d\nu_\lambda^N(\eta) d\nu_\lambda^N(\zeta)$$
$$\leq \iint_{X_N^2} \frac{1}{N^d} \sum_x |\eta(x) - \zeta(x)| G_0^N(\eta,\zeta) d\nu_\lambda^N(\eta) d\nu_\lambda^N(\zeta)$$
$$+ \mathcal{E}^N \int_0^t \max_k \|D^k(f_s - f_\infty)\|_H ds.$$

The second term, due to **(H2)** equals to $\mathcal{E}^N Kt$ if $T < \infty$, while in the case of $T = \infty$, the second term equals $\mathcal{E}^N \int_0^t R(s)ds$ (integrable in time).

How to meet (H1)-(H2)-(H3) for ZRP

- Microscopic Stability. Consider the Wasserstein coupling

$$\begin{aligned}\widetilde{\mathcal{L}}f(\eta,\zeta) &= N^2\sum_{x,y} p(y-x)g(\eta(x))\wedge g(\zeta(x))(f(\eta^{xy},\zeta^{xy})-f(\eta,\zeta))\\ &+ N^2\sum_{x,y} p(y-x)\Big(g(\eta(x))-g(\eta(x))\wedge g(\zeta(x))\Big)(f(\eta^{xy},\zeta)-f(\eta,\zeta))\\ &+ N^2\sum_{x,y} p(y-x)\Big(g(\zeta(x))-g(\eta(x))\wedge g(\zeta(x))\Big)(f(\eta,\zeta^{xy})-f(\eta,\zeta)).\end{aligned}$$

explicit calculations give $\widetilde{\mathcal{L}}(|\eta(x)-\zeta(x)|)\leq 0$.

How to meet (H1)-(H2)-(H3) for ZRP

- Macroscopic Stability. Known for the diffusion equation, $\sigma'(0) > 0$, $\sigma \nearrow$.
- Consistency Estimate.

$$\iint_{X_N^2} N^{-d} \sum_x |\eta(x) - \zeta(x)|^k (\partial_t - \mathcal{L}^*) \psi_t^N(\zeta) d\nu_\rho^N(\eta) d\nu_\rho^N(\zeta) \lesssim$$

$$c e^{-ct} N^{-d/(d+2)}.$$

Calculations on the 'artificial ' process ψ_t^N.

- Replace $g(\zeta(x))$ with $\overline{g(\zeta(x))}^\ell$ for intermediate scale $0 < \ell < N$.
- And $\overline{g(\zeta(x))}^\ell$ with $\sigma(\overline{\zeta(x)}^\ell)$ (Local Law of Large Numbers).

Thank you for listening!