Kinetic Model for Myxobacteria with Directional Diffusion

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2. Kinetic Modelling of Myxobacteria Dynamics
3. Existence and Asymptotic Behavior
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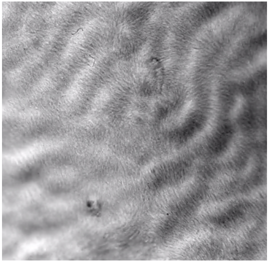
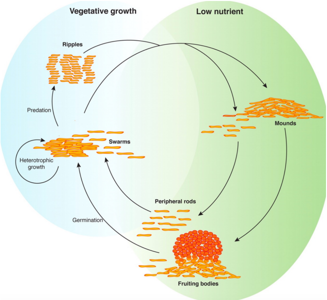
Myxobacteria





[1] E.M.F. Mauriello, T. Mignot, Z. Yang, and D. R. Zusman,
Gliding Motility Revisited: How Do the Myxobacteria Move without Flagella?, Microbiol Mol Biol Rev. 2010 Jun; 74(2): 229–249

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Kinetic Modelling

Bacterium: (x, φ), center of mass $x \in \mathbb{R}^2$, velocity direction $\varphi \in \mathbb{T}^1$
Free flight: $\mu \geq 0$ diffusion constant

$$\mathrm{d}x = \omega(\varphi)\mathrm{d}t, \;\; \mathrm{d}\varphi = \sqrt{2\mu}\mathrm{d}B_t$$

Bacteria Collisions:

Collisions between two bacteria $(x, \varphi), (x_*, \varphi_*) \in \Gamma$ can lead to *alignment*, if the bacteria meet with an angle less than $\pi/2$, or to *reversal*, if they meet with an angle greater than $\pi/2$:

- **Alignment:** $(x, \varphi), (x_*, \varphi_*) \longrightarrow \left(\frac{x+x_*}{2}, \frac{\varphi+\varphi_*}{2}\right), \left(\frac{x+x_*}{2}, \frac{\varphi+\varphi_*}{2}\right)$

- **Reversal:** $(x, \varphi), (x_*, \varphi_*) \longrightarrow (x, \varphi + \pi), (x_*, \varphi_* + \pi)$

Kinetic Equation for Myxobacteria with Directional Diffusion

$$
\begin{aligned}
\partial_t f + \omega(\varphi)\cdot\nabla_x f &= \mu\partial_\varphi^2 f + Q_{AL}(f,f) + Q_{REV}(f,f) \\
&= \mu\partial_\varphi^2 f + 2\int_{\mathbb{T}^1_{AL}} b(\tilde\varphi,\varphi_*)\Big(f(\tilde\varphi)f(\varphi_*) - f(\tilde\varphi_*)f(\varphi)\Big)\,\mathrm{d}\varphi_* \\
&+ \int_{\mathbb{T}^1_{REV}} b(\varphi,\varphi_*)\Big(f(\varphi+\pi)f(\varphi_*+\pi) - f(\varphi)f(\varphi_*)\Big)\,\mathrm{d}\varphi_*,
\end{aligned}
$$

where $\tilde\varphi = 2\varphi - \varphi_*$.
Initial conditions: $f(x,\varphi,0) = f_I(x,\varphi)$

Kinetic Equation for Myxobacteria with Directional Diffusion

$$\partial_t f + \omega(\varphi)\cdot\nabla_x f = \mu\partial_\varphi^2 f + Q_{AL}(f,f) + Q_{REV}(f,f)$$

$$= \mu\partial_\varphi^2 f + 2\int_{\mathbb{T}^1_{AL}} b(\tilde{\varphi},\varphi_*)\Big(f(\tilde{\varphi})f(\varphi_*) - f(\tilde{\varphi}_*)f(\varphi)\Big)\,\mathrm{d}\varphi_*$$

$$+ \int_{\mathbb{T}^1_{REV}} b(\varphi,\varphi_*)\Big(f(\varphi+\pi)f(\varphi_*+\pi) - f(\varphi)f(\varphi_*)\Big)\,\mathrm{d}\varphi_*,$$

where $\tilde{\varphi} = 2\varphi - \varphi_*$.
Initial conditions: $f(x,\varphi,0) = f_I(x,\varphi)$

Collision operator: $\mathbf{Q}(\mathbf{f},\mathbf{f}) := Q_{AL}(f,f) + Q_{REV}(f,f)$.
Collision kernel:

$$b(\varphi,\varphi_*) = \begin{cases} |\sin(\varphi-\varphi_*)| & \text{(rod shaped bacteria)},\\ 1 & \text{(Maxwellian myxos)}.\end{cases}$$

Literature

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Diffusion vs. No-Diffusion

Kinetic Equation for Myxobacteria

$$\partial_t f + \omega(\varphi) \cdot \nabla_x f = \cancel{\mu \partial_\varphi^2 f} + Q_{AL}(f,f) + Q_{REV}(f,f)$$

No diffusion $\mu = 0$:

- *Existence* of a unique global solution $f \in C\left([0,\infty); \mathbf{L}^1_+(\mathbb{T}^1)\right)$ for the spatially homogeneous equation.
- *Equilibria:*
  - Uniform equilibrium $f_0 := \frac{M}{2\pi}$ unstable (total mass $M := \int f \, d\varphi \, dx$)
  - Measure equilibrium $f_\infty(\varphi) := M_+ \delta_{\varphi_\infty}(\varphi) + M_- \delta_{\varphi_\infty - \pi}(\varphi)$ stable (equilibrium angle φ_∞)

[1] S. Hittmeir, L. Kanzler, A. Manhart, C. Schmeiser,
Kinetic Modelling of Colonies of Myxobacteria, Kinteic & Related Models **14** (2021), pp. 1-24

Diffusion vs. No-Diffusion

Kinetic Equation for Myxobacteria with Diffusion in Velocity

$$\partial_t f + \omega(\varphi) \cdot \nabla_x f = \mu \partial_\varphi^2 f + Q_{AL}(f, f) + Q_{REV}(f, f) \qquad (1)$$

Diffusion $\mu > 0$:

- *Existence* ?
- *Equilibria:*
  - Stability of the uniform equilibrium $f_0 := \frac{M}{2\pi}$?
  - Existence and stability of non-trivial equilibria ?

In dependence of the diffusivity $\mu > 0$!

Decay to the uniform equilibrium

Theorem

Let $f_I \in H^2_{x,\varphi}(\mathbb{T}^2 \times \mathbb{T}^1)$, $f_I \geq 0$, and let μ/M be "large enough". Let furthermore $\|f_I - f_0\|_{H^2_{x,\varphi}(\mathbb{T}^2\times\mathbb{T}^1)}$ be "small enough". Then equation (1) subject to the initial condition $f(t=0) = f_I$ has a unique global mild solution $f \in C([0,\infty), H^2_{x,\varphi}(\mathbb{T}^2 \times \mathbb{T}^1))$, satisfying

$$\|f(t) - f_0\|_{H^2_{x,\varphi}(\mathbb{T}^2\times\mathbb{T}^1)} \leq Ce^{-\lambda t}\|f_I - f_0\|_{H^2_{x,\varphi}(\mathbb{T}^2\times\mathbb{T}^1)}\,, \qquad C, \lambda > 0\,.$$

Decay to the uniform equilibrium

Theorem

Let $f_I \in H^2_{x,\varphi}(\mathbb{T}^2 \times \mathbb{T}^1)$, $f_I \geq 0$, *and let* μ/M *be "large enough". Let furthermore* $\|f_I - f_0\|_{H^2_{x,\varphi}(\mathbb{T}^2\times\mathbb{T}^1)}$ *be "small enough". Then equation* (1) *subject to the initial condition* $f(t=0) = f_I$ *has a unique global mild solution* $f \in C([0,\infty), H^2_{x,\varphi}(\mathbb{T}^2 \times \mathbb{T}^1))$, *satisfying*

$$\|f(t) - f_0\|_{H^2_{x,\varphi}(\mathbb{T}^2\times\mathbb{T}^1)} \leq Ce^{-\lambda t}\|f_I - f_0\|_{H^2_{x,\varphi}(\mathbb{T}^2\times\mathbb{T}^1)}, \qquad C, \lambda > 0.$$

Proof (essentials):

- Spectral stability of f_0 by L^2-*hypocoercivity*
- Nonlinear stability of f_0 by control of quadratic nonlinearities in H^2, existence by Picard-argument

Spectral stability by hypocoercivity

Linearization of (1) around f_0:

$$\partial_t f + Tf = (L + Q_L) f, \tag{2}$$

with

- the dissipative operator $L := \mu \partial_\varphi^2$
- the conservative transport operator $T := \omega(\varphi) \cdot \nabla_x$
- linearized collision operator $Q_L f := Q(f_0, f) + Q(f, f_0)$ (perturbation).

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[1] J. Dolbeault, C. Mouhot, C. Schmeiser,
Hypocoercivity for linear kinetic equations conserving mass, Trans. AMS 367 (2015), pp. 3807-3828.

Spectral stability by hypocoercivity in L^2

- $L + Q_L - T$ generates the strongly continuous semigroup $e^{(L+Q_L-T)t}$ on $\mathcal{H}$, with

$$\mathcal{H} := \left\{ f \in L^2(\mathbb{T}^2 \times \mathbb{T}^1) : \int_{\mathbb{T}^2 \times \mathbb{T}^1} f \,\mathrm{d}\varphi \mathrm{d}x = 0 \right\} .$$

- Orthogonal projection to the nullspace $\mathcal{N}(L)$ of L is given by the average with respect to the angle:

$$\Pi f := \frac{1}{2\pi} \int_{\mathbb{T}^1} f \,\mathrm{d}\varphi .$$

- Modified entropy

$$H[f] := \frac{1}{2} \|f\|^2 + \varepsilon \langle Af, f \rangle ,$$

fulfilling $\frac{1-\varepsilon}{2} \|f\|^2 \le H[f] \le \frac{1+\varepsilon}{2} \|f\|^2$ with an appropriately chosen small parameter $0 < \varepsilon < 1$, with the operator

$$A = (1 + (T\Pi)^* T\Pi)^{-1} (T\Pi)^* . \qquad (3)$$

Spectral stability by hypocoercivity in L^2

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -D[f] := \langle Lf, f\rangle + \langle Q_L f, f\rangle - \varepsilon\langle AT\Pi f, f\rangle \\ - \varepsilon\langle AT(1-\Pi)f, f\rangle + \varepsilon\langle ALf, f\rangle + \varepsilon\langle TAf, f\rangle + \varepsilon\langle AQ_L f, f\rangle$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -D[f] := \langle Lf, f\rangle + \langle Q_L f, f\rangle - \varepsilon\langle AT\Pi f, f\rangle - \varepsilon\langle AT(1-\Pi)f, f\rangle + \varepsilon\langle ALf, f\rangle + \varepsilon\langle TAf, f\rangle + \varepsilon\langle AQ_L f, f\rangle$$

- Microscopic- and macroscopic coercivity

$$-\langle Lf, f\rangle + \varepsilon\langle AT\Pi f, f\rangle \geq \mu\|(1-\Pi)f\|^2 + \varepsilon\frac{4\pi^2}{1+4\pi^2}\|\Pi f\|^2$$

- $\|Af\| \leq \frac{1}{2}\|(1-\Pi)f\|$, $\|TAf\| \leq \|(1-\Pi)f\|$ due to $\Pi T\Pi = 0$
- AT bounded due to elliptic regularity

Spectral stability by hypocoercivity in L^2

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -D[f] := \langle Lf, f\rangle + \langle Q_L f, f\rangle - \varepsilon\langle AT\Pi f, f\rangle \\ - \varepsilon\langle AT(1-\Pi)f, f\rangle + \varepsilon\langle ALf, f\rangle + \varepsilon\langle TAf, f\rangle + \varepsilon\langle AQ_L f, f\rangle$$

- Boundedness of the collision kernel $0 \le b \le 1$
- Conservation of total mass M

Spectral stability by hypocoercivity in L^2

$$
\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} = -D[f] &:= \langle Lf, f\rangle + \langle Q_L f, f\rangle - \varepsilon\langle AT\Pi f, f\rangle \\
&\quad - \varepsilon\langle AT(1-\Pi)f, f\rangle + \varepsilon\langle ALf, f\rangle + \varepsilon\langle TAf, f\rangle + \varepsilon\langle AQ_L f, f\rangle \\
&\le -\left(\mu - \frac{13}{2}M - \varepsilon\right)\|(1-\Pi)f\|^2 - \varepsilon\frac{8\pi^3}{1+8\pi^3}\|\Pi f\|^2 \\
&\quad + \varepsilon\left(\sqrt{2} + \frac{\mu}{2} + 3M\left(\sqrt{\frac{2}{\pi}} + \frac{1}{2}\right)\right)\|\Pi f\|\,\|(1-\Pi)f\|\,.
\end{aligned}
$$

For μ/M "big enough" and for ε small enough, we have

$$
\frac{d}{dt}H[f] \le -2\lambda H[f]\,,
$$

hence *spectral stability* of f_0 in $\mathcal{H}$.

Spectral stability by hypocoercivity in H^2

Spectral stability of f_0 in $H^2 \cap \mathcal{H} \subset L^\infty$ via recursive arguments.

- Decay for x-derivatives analogous.
- Decay of derivatives involving φ via recursive arguments: E.g.: $g := \partial_\varphi f$ solves the equation

$$\partial_t g + (T - L)g = -\omega(\varphi)^\perp \cdot \nabla_x f + Q_L g$$

Lemma (Spectral stability in H^2)

For μ/M large enough there exist positive constants λ and C, such that for any initial datum $f_I \in H^2(\mathbb{T}^2 \times \mathbb{T}^1) \cap \mathcal{H}$, we have

$$\left\| e^{t(L+Q_L-T)} f_I \right\|_{H^2(\mathbb{T}^2\times\mathbb{T}^1)} \le Ce^{-\lambda t} \left\| f_I \right\|_{H^2(\mathbb{T}^2\times\mathbb{T}^1)}, \qquad t \ge 0.$$

Nonlinear stability of the uniform equilibrium

The perturbation $h := f - f_0 \in H^2(\mathbb{T}^2 \times \mathbb{T}^1) \cap \mathcal{H}$, satisfies

$$\partial_t h + Th = Lh + Q_L h + Q(h,h),$$

with Q being *local* $\bar{Q}$*-Lipschitz continuous* on $H^2(\mathbb{T}^2 \times \mathbb{T}^1) \cap \mathcal{H}$.

Nonlinear stability of the uniform equilibrium

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- Local existence and uniqueness of a *mild solution*.

Nonlinear stability of the uniform equilibrium

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$$\partial_t h + Th = Lh + Q_L h + Q(h,h),$$

with Q being *local $\bar{Q}$-Lipschitz continuous* on $H^2(\mathbb{T}^2 \times \mathbb{T}^1) \cap \mathcal{H}$.

- Local existence and uniqueness of a *mild solution*.
- Estimating the mild formulation gives

$$\begin{aligned}\|h(t)\|_{H^2(\mathbb{T}^2\times\mathbb{T}^1)} &\le Ce^{-\lambda t}\|f_I - f_0\|_{H^2(\mathbb{T}^2\times\mathbb{T}^1)} \\ &\quad + C\bar{Q}\int_0^t e^{\lambda(s-t)}\|h(s)\|^2_{H^2(\mathbb{T}^2\times\mathbb{T}^1)}\mathrm{d}s\,.\end{aligned}$$

for $\|f_I - f_0\|_{H^2(\mathbb{T}^2\times\mathbb{T}^1)} \le \frac{\lambda}{4C^2\bar{Q}}$, Picard iteration preserves the inequality

$$\|h(t)\|_{H^2(\mathbb{T}^2\times\mathbb{T}^1)} \le 2Ce^{-\lambda t}\|f_I - f_0\|_{H^2(\mathbb{T}^2\times\mathbb{T}^1)}\,.$$

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Bifurcation form the Uniform Equilibrium

Linearization of (1) around f_0:

$$\partial_t f + Tf = (L + Q_L)f,$$

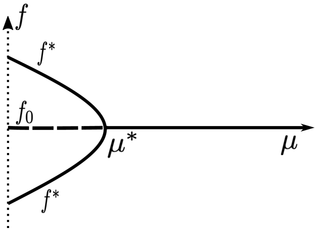
Fourier series expansion

$f(\varphi, t) = \sum_{n=1}^{\infty} a_n(t)\cos(n\varphi) + \sum_{n=1}^{\infty} b_n(t)\sin(n\varphi)$ leads to

$$\dot{a}_n = \lambda_n a_n, \qquad \dot{b}_n = \lambda_n b_n,$$

with

- $\lambda_n(\mu/M) < 0$ for $n \neq 2$
- $\lambda_2(\mu/M) \begin{cases} > 0 & \text{for } \mu/M < c_* \\ < 0 & \text{for } \mu/M > c_* \end{cases}$
- Occurrence of a *supercritical pitchfork bifurcation*



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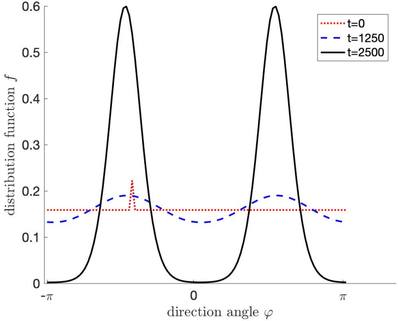

Figure: $\mu/M < c_*$

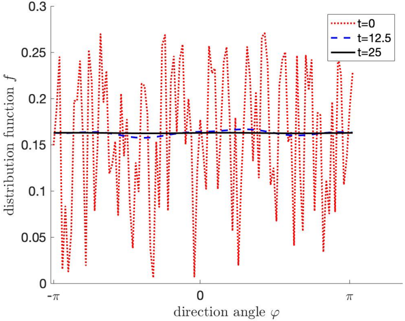

Figure: $\mu/M > c_*$

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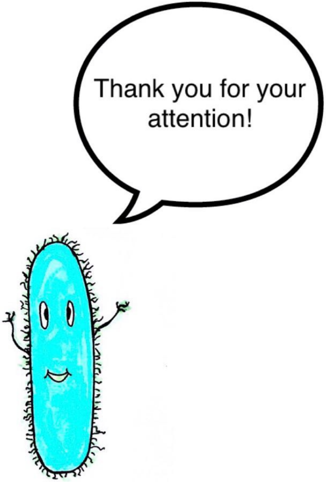
Thank you for your attention!