Extremal functions in some interpolation inequalities: Symmetry, symmetry breaking and estimates of the best constants

Caffarelli-Kohn-Nirenberg interpolation inequalities: regions of symmetry and symmetry breaking

```
Off[General::"spell1"]
```

Symmetry: Schwarz' symmetrization

$$\text{Theta}[p\_, d\_] := d\,\frac{p-2}{2\,p}$$

$$\text{S}[d\_] := \frac{2\,\pi^{\frac{d}{2}}}{\text{Gamma}[\frac{d}{2}]}$$

$$\text{K}[\theta\_, p\_] := \left(\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right)^{\frac{p-2}{2\,p}} \left(\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right)^{\theta} \left(\frac{4}{p+2}\right)^{\frac{6-p}{2\,p}} \left(\frac{\text{Gamma}[\frac{2}{p-2}+\frac{1}{2}]}{\sqrt{\pi}\,\text{Gamma}[\frac{2}{p-2}]}\right)^{\frac{p-2}{p}}$$

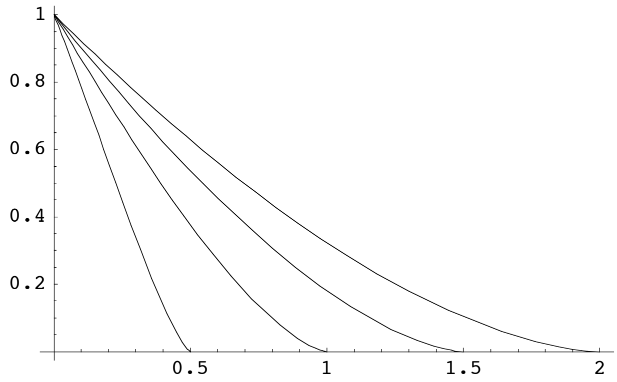
$$\text{Cstar}[\theta\_, p\_, d\_] := \text{S}[d]^{\frac{2}{p}-1}\,\text{K}[\theta, p]$$

$$\text{ac}[d\_] := \frac{d-2}{2}$$

$$\text{as}[d\_, p\_] := \text{ac}[d] - \text{ac}[d]\left(\frac{\text{Theta}[p, d]\,\text{Cstar}[\text{Theta}[p, d], p, d]^{\frac{1}{\text{Theta}[p,d]}}}{\text{Cstar}[1, \frac{2\,d}{d-2}, d]}\right)^{\frac{d}{2\,(d-1)}}$$

$$\text{s}[d\_] := \text{ParametricPlot}\left[\{\text{as}[d, p], \text{Theta}[p, d]\}, \left\{p, 2, \frac{2\,d}{d-2}\right\}, \text{DisplayFunction} \to \text{Identity}\right]$$

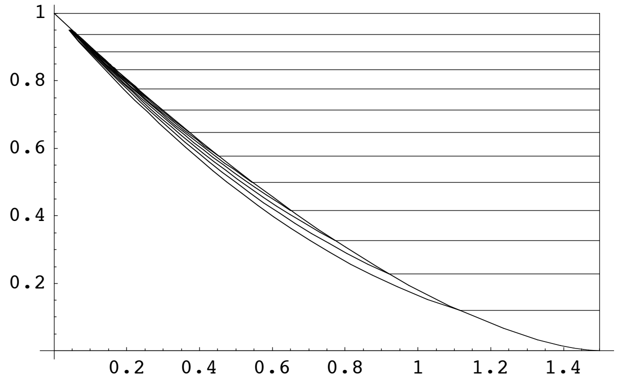
```
AS = Show[Table[s[d], {d, 3, 6}], DisplayFunction → $DisplayFunction];
```



Comparison with previous results: dimension d=5

```
R[a_, θ_, p_, d_] :=
 Evaluate[(t + (a - ac[d])^2)^θ - (Cstar[1, 2d/(d-2), d] ac[d]^(-2 (d-1)/d))^Theta[p,d] / Cstar[θ, p, d] (t + ac[d]^2)^Theta[p,d]
     (ac[d] - a)^(2θ - 2/d Theta[p,d]) /. t -> (θ ac[d]^2 - (ac[d] - a)^2)/(1 - θ)]
Iter[θ_, p_, d_, a_, h_, ϵ_, Nmax_, n_] := Module[{M = Evaluate[R[a, θ, p, d]]},
  If[n > Nmax || Abs[M] < ϵ, {M, a, h, n}, If[M > ϵ,
    Iter[θ, p, d, a + h, h, ϵ, Nmax, n + 1], Iter[θ, p, d, a - h / 2, h / 2, ϵ, Nmax, n + 1]]]]
F[θ_, p_, d_, h_, ϵ_, Nmax_] := Iter[θ, p, d, ac[d] - h, -h, ϵ, Nmax, 0]
L[p_, d_, h_, Nmax_] := Module[{M = F[Theta[p, d], p, d, h, eps, Nmax][[2]]},
  ListPlot[{{M, Theta[p, d]}, {ac[d], Theta[p, d]}},
   PlotJoined → True, DisplayFunction → Identity]]
G[d_, p_, DF_, Marge_, h_, Nmax_] := ParametricPlot[{F[s, p, d, h, eps, Nmax][[2]], s},
  {s, Theta[p, d], 1 - Marge}, DisplayFunction → DF ]
Off[Greater::"nord"]
Off[ParametricPlot::"pptr"]
Off[Graphics::"gptn"]
dim = 5;
eps = 10^-8;
Orgn = {0, 0};
LL = ListPlot[{{Orgn[[2]], 0}, {ac[dim], 0}, {ac[dim], 1}, {Orgn[[2]], 1}},
   PlotJoined → True, DisplayFunction → Identity];
Schwarz[d_] := ParametricPlot[{as[d, p], Theta[p, d]},
  {p, 2, 2d/(d - 2)}, DisplayFunction → Identity]
```

```
Show[Schwarz[dim], Table[G[dim, p, Identity, 0.05, 0.01, 200], {p, 2.1, 3.2, 0.1}],
  Table[L[p, dim, 0.01, 200], {p, 2.1, 3.2, 0.1}], LL, PlotRange → All,
  DisplayFunction → $DisplayFunction, AxesOrigin → {Orgn[[1]], Orgn[[2]]}];
```



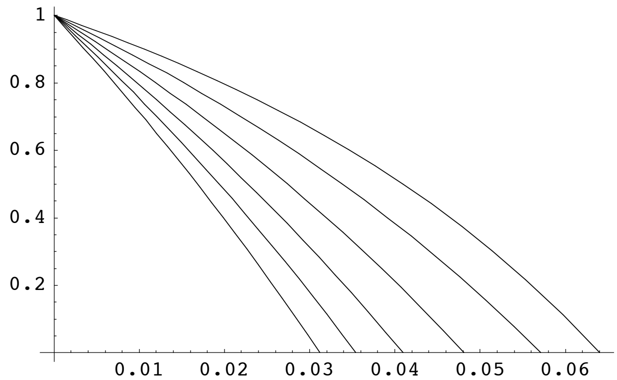
Existence: a priori estimates

```
aae[d_, p_] := ac[d] - ac[d]
```

$$\sqrt{\text{Min}\left[\left(\frac{\text{Cstar}[\text{Theta}[p,d],p,d]^{\frac{1}{\text{Theta}[p,d]}}}{\text{Cstar}[1,\frac{2d}{d-2},d]}\right)^{\frac{d}{d-1}},\left(\frac{\text{Cstar}[1,\frac{2d}{d-2},d]}{\text{Cstar}[\text{Theta}[p,d],p,d]^{\frac{1}{\text{Theta}[p,d]}}}\right)^{d}\right]}$$

```
ae[d_] :=
  ParametricPlot[{aae[d, p], Theta[p, d]}, {p, 2, 2 d/(d - 2)}, DisplayFunction → Identity]

AAE = Show[Table[ae[d], {d, 3, 8}], DisplayFunction → $DisplayFunction];
```



The endpoint p=2 for d=5

```
FullSimplify[
  PowerExpand[(Integrate[r^(d+1) ⅇ^(-r^2), {r, 0, ∞}, Assumptions -> Re[d] > 0]^Theta[p,d]
      Integrate[r^(d-1) ⅇ^(-r^2), {r, 0, ∞}, Assumptions -> Re[d] > 0]^(1-Theta[p,d])) /
    Integrate[r^(d-1) ⅇ^(-p r^2/2), {r, 0, ∞}, Assumptions -> Re[d] > 0 && Re[p] > 0]^(2/p)]]
```

$$2^{-\frac{-2+d+p}{p}}\, p^{d/p}\, \text{Gamma}\left[1+\frac{d}{2}\right]^{\frac{d(-2+p)}{2p}}\, \text{Gamma}\left[\frac{d}{2}\right]^{-\frac{(-2+d)(-2+p)}{2p}}$$

$$\texttt{H[p\_, d\_] := } 2^{-\frac{-2+d+p}{p}}\, p^{d/p}\, \texttt{Gamma}\left[1+\frac{d}{2}\right]^{\frac{d(-2+p)}{2p}}\, \texttt{Gamma}\left[\frac{d}{2}\right]^{-\frac{(-2+d)(-2+p)}{2p}}$$

$$\texttt{LimCGN = Limit}\left[\frac{\texttt{H[p, d]} - 1}{p-2},\ p \to 2\right]\ /.\ d \to 5;$$

$$1.5 - \sqrt{\texttt{Limit}\left[\left((1+\texttt{LimCGN}\,(p-2))\ \texttt{K[Theta[p, 5], p]}\right)^{\frac{d}{(d-1)\,\texttt{Theta[p, d]}}}\ /.\ d \to 5,\ p \to 2\right]};$$

$$\texttt{PlimExist = \{\{N[\%], 0\}\}}$$

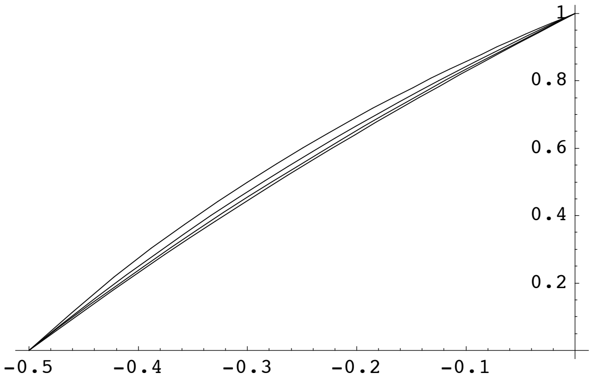
```
{{-0.32461, 0}}
```

Symmetry breaking: Felli & Schneider method

$$\texttt{afs[d\_, p\_] := ac[d]} - \frac{2\sqrt{d-1}}{p+2}\sqrt{\frac{2\,p\,\texttt{Theta[p, d]}}{p-2} - 1}$$

$$\texttt{fs[d\_] :=}$$
$$\texttt{ParametricPlot}\left[\texttt{\{afs[d, p], Theta[p, d]\}},\ \left\{p,\ 2,\ \frac{2d}{d-2}\right\},\ \texttt{DisplayFunction} \to \texttt{Identity}\right]$$

$$\texttt{AFS = Show[Table[fs[d], \{d, 3, 6\}], DisplayFunction} \to \texttt{\$DisplayFunction];}$$



Symmetry breaking: comparison with Gagliardo-Nirenberg

Sobolev's constant

$$\texttt{FullSimplify}\left[\texttt{Cstar}\left[1,\ \frac{2d}{d-2},\ d\right]\ \texttt{ac[d]}^{-2\,\frac{d-1}{d}}\right]$$

$$\texttt{Table}\left[\texttt{N}\left[\frac{1}{\%}\right],\ \{d, 3, 5\}\right]$$

$$\frac{2^{2-\frac{2}{d}}\,(-2+d)^{-\frac{1+d}{d}}\,(-1+d)^{3/d}\left(\frac{1}{2-3d+d^2}\right)^{\frac{1}{d}}\pi^{-1/d}\left(\frac{\texttt{Gamma}\left[\frac{1}{2}(-1+d)\right]}{\texttt{Gamma}\left[-1+\frac{d}{2}\right]}\right)^{2/d}\left(\frac{\pi^{d/2}}{\texttt{Gamma}\left[\frac{d}{2}\right]}\right)^{-2/d}}{d}$$

```
{5.4779, 10.2604, 14.8119}
```

```
FullSimplify[d (d - 2)/4 S[d + 1]^(2/d)]
Table[N[%], {d, 3, 5}]
```

$$4^{-1+\frac{1}{d}}\ (-2+d)\ d\ \left(\frac{\pi^{\frac{1+d}{2}}}{\mathrm{Gamma}\left[\frac{1+d}{2}\right]}\right)^{2/d}$$

{5.4779, 10.2604, 14.8119}

```
Sobolev[d_] := 1/(π d (d - 2)) (Gamma[d]/Gamma[d/2])^(2/d)
Sobolev[d]
Table[N[1/%], {d, 3, 5}]
```

$$\frac{\left(\frac{\mathrm{Gamma}[d]}{\mathrm{Gamma}\left[\frac{d}{2}\right]}\right)^{2/d}}{(-2+d)\ d\ \pi}$$

{5.4779, 10.2604, 14.8119}

The optimal function for Sobolev's inequality

```
Off[Integrate::"idiv"]
```

```
uu[r_, d_] := (1 + r^2)^(-(d-2)/2)
```

```
At[d_] := Integrate[r^(d-1) D[uu[r, d], r]^2, {r, 0, ∞}]
```

```
Table[At[d], {d, 3, 5}]
```

$$\left\{\frac{3\pi}{16},\ \frac{2}{3},\ \frac{45\pi}{256}\right\}$$

```
Bt[d_] := Integrate[r^(d-1) uu[r, d]^2, {r, 0, ∞}]
```

```
Table[Bt[d], {d, 3, 5}]
```

$$\left\{\int_0^\infty \frac{r^2}{1+r^2}\, dr,\ \int_0^\infty \frac{r^3}{(1+r^2)^2}\, dr,\ \frac{3\pi}{16}\right\}$$

```
FullSimplify[Bt[d]/At[d], Assumptions → d > 4]
```

$$\frac{4\ (-1+d)}{d\ (8-6\ d+d^2)}$$

```
Coef[d_] := 4 (-1 + d)/(d (8 - 6 d + d^2))
```

```
Table[Coef[d], {d, 5, 7}]
```

$$\left\{\frac{16}{15},\ \frac{5}{12},\ \frac{8}{35}\right\}$$

$$\mathbf{D}\left[\left(1+\mathbf{Coef}[\mathbf{d}]\ \mathbf{r}^{2}\right)^{-\frac{\mathbf{d}-2}{2}},\ \mathbf{r}\right]$$

$$\frac{4\ (2-d)\ (-1+d)\ r\ \left(1+\frac{4\ (-1+d)\ r^{2}}{d\ (8-6\ d+d^{2})}\right)^{-1+\frac{2-d}{2}}}{d\ (8-6\ d+d^{2})}$$

$$\mathbf{A1 = Integrate}\left[\mathbf{r}^{\mathbf{d}-1}\ \%^{2},\ \{\mathbf{r},\ 0,\ \infty\},\ \mathbf{Assumptions} \to \mathbf{d} > 4\right]$$

$$\frac{2^{1-d}\ \left(\frac{(-4+d)\ (-2+d)\ d}{-1+d}\right)^{d/2}\ \text{Gamma}\left[\frac{d}{2}\right]^{2}}{(-4+d)\ \text{Gamma}[-1+d]}$$

$$\left(1+\mathbf{Coef}[\mathbf{d}]\ \mathbf{r}^{2}\right)^{-\frac{\mathbf{d}-2}{2}}$$

$$\left(1+\frac{4\ (-1+d)\ r^{2}}{d\ (8-6\ d+d^{2})}\right)^{\frac{2-d}{2}}$$

$$\mathbf{B1 = Integrate}\left[\mathbf{r}^{\mathbf{d}-1}\ \%^{2},\ \{\mathbf{r},\ 0,\ \infty\},\ \mathbf{Assumptions} \to \mathbf{d} > 4\right]$$

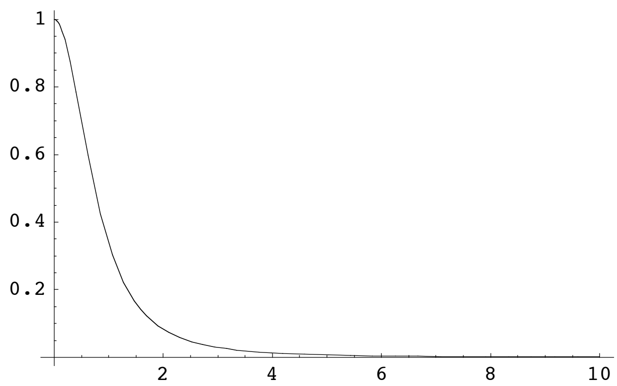
$$\frac{2^{-1-d}\ \left(\frac{-1+d}{d\ (8-6\ d+d^{2})}\right)^{-d/2}\ \text{Gamma}\left[-2+\frac{d}{2}\right]\ \text{Gamma}\left[\frac{d}{2}\right]}{\text{Gamma}[-2+d]}$$

$$\mathbf{Simplify}\left[\mathbf{PowerExpand}\left[\mathbf{FullSimplify}\left[\left(\frac{\mathbf{A1}}{\mathbf{B1}}\right)^{\frac{2}{\mathbf{d}}}\right]\right]\right]$$

1

```
Pref[d_, DF_] :=
  Plot[Evaluate[(1 + Coef[d] r^2)^(-(d-2)/2)], {r, 0, 10}, PlotRange → All, DisplayFunction → DF];
```

```
Pref[5, $DisplayFunction];
```



$$\mathbf{ww[r\_] := }\left(1+\mathbf{Coef}[\mathbf{d}]\ \mathbf{r}^{2}\right)^{-\frac{\mathbf{d}-2}{2}}$$

$$\mathbf{FullSimplify}\left[\mathbf{PowerExpand}\left[\frac{\mathbf{ww}''[\mathbf{r}]+\frac{\mathbf{d}-1}{\mathbf{r}}\ \mathbf{ww}'[\mathbf{r}]}{\mathbf{ww}[\mathbf{r}]^{\frac{\mathbf{d}+2}{\mathbf{d}-2}}}\right],\ \mathbf{Assumptions} \to \mathbf{d} > 4\right]$$

$$-\frac{4\ (-1+d)}{-4+d}$$

```
Table[N[Abs[%]], {d, 5, 7}]
```

{16., 10., 8.}

Numerical computation of the best constant in the Gagliardo-Nirenberg inequality

```
F[a_, p_, d_, rmax_, ϵ_, DF_, PR_] := Module[
  {M = Evaluate[u[s] /. NDSolve[{v'[r] + (d - 1) v[r]/r + a/Theta[p, d] Abs[u[r]]^(p-2) u[r] -
        (1 - Theta[p, d])/Theta[p, d] u[r] == 0, u'[r] == v[r], v[ϵ] == -(a + Theta[p, d] - 1)/Theta[p, d] ϵ/d,
        u[ϵ] == 1 - (a + Theta[p, d] - 1)/Theta[p, d] ϵ^2/(2 d)}, {u, v}, {r, ϵ, rmax}]]},
  Plot[M, {s, ϵ, rmax}, DisplayFunction → DF, PlotRange → PR]]

H[a_, p_, d_, rmax_, ϵ_] := Log[1 + u[rmax]^2 + v[rmax]^2] /.
  NDSolve[{v'[r] + (d - 1) v[r]/r + a/Theta[p, d] Abs[u[r]]^(p-2) u[r] - (1 - Theta[p, d])/Theta[p, d] u[r] ==
      0, u'[r] == v[r], v[ϵ] == -(a + Theta[p, d] - 1)/Theta[p, d] ϵ/d,
      u[ϵ] == 1 - (a + Theta[p, d] - 1)/Theta[p, d] ϵ^2/(2 d)}, {u, v}, {r, ϵ, rmax}][[1]]

Iter[a_, h_, p_, d_, rmax_, ϵ_, b_, η_, j_, Nmax_] :=
 Module[{M = H[a + h, p, d, rmax, ϵ]}, If[Or[Abs[b - M] < η, j > Nmax],
   {j, N[a], M, M - b, N[h], IGN[p, d, a, rmax, ϵ], p, K[Theta[p, d], p]},
   If[M < b, Iter[a + h, h, p, d, rmax, ϵ, M, η, j + 1, Nmax],
    Iter[a + h, -h / 2, p, d, rmax, ϵ, M, η, j + 1, Nmax]]]]

Init[a_, h_, p_, d_, rmax_, ϵ_, η_, Nmax_] :=
 Iter[a, h, p, d, rmax, ϵ, H[a, p, d, rmax, ϵ], η, 1, Nmax]

Nrm[p_, d_, a_, rmax_, ϵ_] := {z[rmax], w2[rmax], w[rmax]} /.
  NDSolve[{v'[r] + (d - 1) v[r]/r + a/Theta[p, d] Abs[u[r]]^(p-2) u[r] - (1 - Theta[p, d])/Theta[p, d] u[r] ==
      0, u'[r] == v[r], w'[r] == r^(d-1) Abs[u[r]]^p, w2'[r] == r^(d-1) Abs[u[r]]^2,
     z'[r] == r^(d-1) Abs[v[r]]^2, z[ϵ] == ((a + Theta[p, d] - 1)/Theta[p, d])^2 ϵ^(d+2)/(d^2 (d + 2)),
     w[ϵ] == ϵ^d/d, w2[ϵ] == ϵ^d/d, v[ϵ] == -(a + Theta[p, d] - 1)/Theta[p, d] ϵ/d,
     u[ϵ] == 1 - (a + Theta[p, d] - 1)/Theta[p, d] ϵ^2/(2 d)}, {u, v, w, w2, z}, {r, ϵ, rmax}][[1]]

IGN[p_, d_, a_, rmax_, ϵ_] :=
 Module[{M = Nrm[p, d, a, rmax, ϵ]}, M[[1]]^Theta[p, d] M[[2]]^(1-Theta[p, d]) / M[[3]]^(2/p)]

Fnagn[p_, d_, x_] := ac[d] - Sqrt[(x K[Theta[p, d], p])^(d/((d-1) Theta[p, d]))]

Visualize = $DisplayFunction;

Visualize = Identity;
```

```
Conclusion[a_, h_, p_, d_, rmax_, ε_, η_, amin_, amax_, Nmax_] :=
 Module[{M = Init[a, h, p, d, rmax, ε, η, Nmax]},
   {Plot[H[aa, p, d, rmax, ε], {aa, amin, amax}, DisplayFunction → Visualize];
    {M, {Fnagn[p, d, M[[6]]], Theta[p, d]}},
    Show[F[M[[2]], p, d, rmax, ε, Visualize, Automatic],
     Pref[d, Visualize], DisplayFunction → Visualize] }][[1]]
```

$$\texttt{CoefNorm[d\_] := S[d]}^{1-\frac{2}{p}}\ \texttt{Sobolev[d] /. p ->} \frac{2\,d}{d-2}$$

The best constant in the Gagliardo-Nirenberg inequality when approaching Sobolev's inequality

$\texttt{Conclusion[1, 1, 3, 5, 10, }10^{-12}\texttt{, }10^{-15}\texttt{, 1, 20, 100]}$
`%[[1]][[6]] CoefNorm[5]`

$\{\{48, 4.39471, 6.53159\times10^{-8}, -8.88178\times10^{-16}, -0.0000305176, 3.48823, 3, \frac{2}{5}\left(\frac{3}{5}\right)^{1/3} 2^{2/3}\},$
$\{-0.0979087, \frac{5}{6}\}\}$

0.871168

$\texttt{Conclusion[1, 1, 3.2, 5, 10, }10^{-12}\texttt{, }10^{-15}\texttt{, 1, 20, 100]}$
`%[[1]][[6]] CoefNorm[5]`

$\{\{55, 7.56973, 1.17715\times10^{-7}, 2.22045\times10^{-16}, 0.0000152588, 3.85675, 3.2, 0.498414\},$
$\{-0.0459953, 0.9375\}\}$

0.963203

$\texttt{Conclusion[1, 1, 3.3, 5, 10, }10^{-12}\texttt{, }10^{-15}\texttt{, 1, 20, 100]}$
`%[[1]][[6]] CoefNorm[5]`

$\{\{51, 15.0776, 5.03035\times10^{-8}, -4.44089\times10^{-16}, 0.000244141, 3.98711, 3.3, 0.482684\},$
$\{-0.0150774, 0.984848\}\}$

0.99576

$\texttt{Conclusion[1, 1, 3.31, 5, 12, }10^{-12}\texttt{, }10^{-15}\texttt{, 1, 20, 100]}$
`%[[1]][[6]] CoefNorm[5]`

$\{\{51, 15.2175, 1.20816\times10^{-8}, -2.22045\times10^{-16}, 0.000244141, 3.99274, 3.31, 0.4812\},$
$\{-0.0105743, 0.989426\}\}$

0.997165

$\texttt{Conclusion}\left[1, 1, \frac{10}{3} - 0.01, 5, 20, 10^{-12}, 10^{-15}, 1, 20, 100\right]$
`%[[1]][[6]] CoefNorm[5]`

$\{\{44, 14.8994, 2.26859\times10^{-10}, 8.88178\times10^{-16}, 0.000976563, 3.99926, 3.32333, 0.479245\},$
$\{-0.00447864, 0.995486\}\}$

0.998793

```
Conclusion[1, 1, 20/8 - 0.1, 10, 10, 10^-12, 10^-15, 2, 5, 100]
%[[1]][[6]] CoefNorm[10]
```

{{54, 3.65588, 1.78315×10^{-8}, -2.22045×10^{-16}, 0.0000610352, 12.1611, 2.4, 0.681011},
{-0.093411, 0.833333}}

0.633808

```
Conclusion[1, 1, 20/8 - 0.01, 10, 10, 10^-12, 10^-15, 2, 5, 100]
%[[1]][[6]] CoefNorm[10]
```

{{37, 9.58398, 3.52008×10^{-10}, -4.44089×10^{-16}, 0.000976563, 18.4265, 2.49, 0.636511},
{-0.0153714, 0.983936}}

0.960345

```
Conclusion[8, 1, 14/5 - 0.01, 7, 15, 10^-12, 10^-15, 8, 9, 100]
%[[1]][[6]] CoefNorm[7]
```

{{27, 8.34766, 1.28692×10^{-10}, 4.44089×10^{-16}, 0.000976563, 8.58, 2.79, 0.555463},
{-0.00703126, 0.991039}}

0.985746

```
Conclusion[1, 1, 2.9, 6, 20, 10^-12, 10^-15, 1, 30, 100]
%[[1]][[6]] CoefNorm[6]
```

{{45, 5.86665, 3.8527×10^{-12}, -4.44089×10^{-16}, 0.0000152588, 5.61414, 2.9, 0.540453},
{-0.0447907, 0.931034}}

0.915775

The case d=5

```
Conclusion[1, 0.001, 2.01, 5, 8, 10^-12, 10^-15, 1, 1.25, 100]
ll = {%[[2]]}
```

{{78, 1.01258, 5.22775×10^{-11}, 8.88178×10^{-16}, -2.98023×10^{-11}, 1.02201, 2.01, 0.99022},
{-0.322331, 0.0124378}}

{{-0.322331, 0.0124378}}

```
Conclusion[1.1, 0.01, 2.1, 5, 10, 10^-12, 10^-15, 1, 1.25, 100]
ll = Append[ll, %[[2]]]
```

{{91, 1.13386, 3.4639×10^{-14}, 0., 9.31323×10^{-12}, 1.2284, 2.1, 0.910692},
{-0.301942, 0.119048}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}}

```
Conclusion[1.25, 0.01, 2.2, 5, 10, 10^-12, 10^-15, 1, 1.5, 100]
ll = Append[ll, %[[2]]]
```

{{86, 1.2878, 2.10321×10^{-12}, -8.88178×10^{-16}, 1.49012×10^{-10}, 1.4727, 2.2, 0.837338},
{-0.279488, 0.227273}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}}

```
Conclusion[1.4, 0.01, 2.3, 5, 10, 10^-12, 10^-15, 1, 2, 100]
ll = Append[ll, %[[2]]]
```

{{73, 1.46622, 3.13967×10^{-11}, -2.22045×10^{-16}, 2.38419×10^{-9}, 1.72868, 2.3, 0.776281}, {-0.257204, 0.326087}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}}

```
Conclusion[1.6, 0.01, 2.4, 5, 10, 10^-12, 10^-15, 1, 2, 100]
ll = Append[ll, %[[2]]]
```

{{72, 1.6749, 2.21541×10^{-10}, -2.22045×10^{-16}, -4.76837×10^{-9}, 1.99196, 2.4, 0.724861}, {-0.235013, 0.416667}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}}

```
Conclusion[1.8, 0.01, 2.5, 5, 10, 10^-12, 10^-15, 1.5, 2.5, 100]
ll = Append[ll, %[[2]]]
```

{{67, 1.92173, 9.89191×10^{-10}, 2.22045×10^{-16}, -1.90735×10^{-8}, 2.25818, 2.5, 0.681103}, {-0.212832, 0.5}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}}

```
Conclusion[2, 0.01, 2.6, 5, 10, 10^-12, 10^-15, 1.5, 2.5, 100]
ll = Append[ll, %[[2]]]
```

{{64, 2.21773, 3.2551×10^{-9}, 2.22045×10^{-16}, 1.52588×10^{-7}, 2.52308, 2.6, 0.643519}, {-0.190566, 0.576923}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}}

```
Conclusion[2.4, 0.01, 2.7, 5, 10, 10^-12, 10^-15, 2.2, 3, 100]
ll = Append[ll, %[[2]]]
```

{{64, 2.57891, 8.61174×10^{-9}, 0., -3.05176×10^{-7}, 2.7825, 2.7, 0.610967}, {-0.1681, 0.648148}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}, {-0.1681, 0.648148}}

```
Conclusion[2.8, 0.01, 2.8, 5, 10, 10^-12, 10^-15, 2, 4, 100]
ll = Append[ll, %[[2]]]
```

{{65, 3.02976, 1.92988×10^{-8}, -8.88178×10^{-16}, 6.10352×10^{-7}, 3.03245, 2.8, 0.582562}, {-0.145293, 0.714286}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286}}

```
Conclusion[3.5, 0.01, 2.9, 5, 10, 10^-12, 10^-15, 3, 4, 100]
ll = Append[ll, %[[2]]]
```

{{52, 3.61048, 3.77646×10^{-8}, 2.22045×10^{-16}, 6.10352×10^{-7}, 3.26901, 2.9, 0.557606}, {-0.121971, 0.775862}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286}, {-0.121971, 0.775862}}

```
Conclusion[4, 0.1, 3, 5, 10, 10^-12, 10^-15, 3, 5, 100]
ll = Append[ll, %[[2]]]
```

{{48, 4.3947, 6.53158×10^{-8}, 4.44089×10^{-16}, 6.10352×10^{-6}, 3.48823, 3, $\frac{2}{5}\left(\frac{3}{5}\right)^{1/3}2^{2/3}$}, {-0.097908, $\frac{5}{6}$}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286}, {-0.121971, 0.775862}, {-0.097908, $\frac{5}{6}$}}

```
Conclusion[4, 0.1, 3.1, 5, 10, 10^-12, 10^-15, 3, 10, 100]
ll = Append[ll, %[[2]]]
```

{{54, 5.54445, 9.84406×10^{-8}, -6.66134×10^{-16}, 6.10352×10^{-6}, 3.68593, 3.1, 0.515936}, {-0.0727829, 0.887097}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286}, {-0.121971, 0.775862}, {-0.097908, $\frac{5}{6}$}, {-0.0727829, 0.887097}}

```
Conclusion[4, 1, 3.2, 5, 12, 10^-12, 10^-15, 3, 10, 100]
ll = Append[ll, %[[2]]]
```

{{42, 7.35091, 1.88268×10^{-8}, 0., 0.0000152588, 3.85343, 3.2, 0.498414}, {-0.0451083, 0.9375}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286}, {-0.121971, 0.775862}, {-0.097908, $\frac{5}{6}$}, {-0.0727829, 0.887097}, {-0.0451083, 0.9375}}

```
Conclusion[10, 1, 3.3, 5, 20, 10^-12, 10^-15, 10, 15, 100]
ll = Append[ll, %[[2]]]
```

{{40, 11.2235, 2.15732×10^{-10}, 4.44089×10^{-16}, -0.00012207, 3.97883, 3.3, 0.482684}, {-0.0130789, 0.984848}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273}, {-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5}, {-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286}, {-0.121971, 0.775862}, {-0.097908, $\frac{5}{6}$}, {-0.0727829, 0.887097}, {-0.0451083, 0.9375}, {-0.0130789, 0.984848}}

```
Conclusion[10, 5, 3.33, 5, 20, 10^-12, 10^-15, 10, 25, 100]
ll = Append[ll, %[[2]]]
```

{{35, 20.9473, 1.08596×10^{-10}, 6.66134×10^{-16}, 0.00488281, 4.00349, 3.33, 0.478277},
{-0.00172117, 0.998498}}

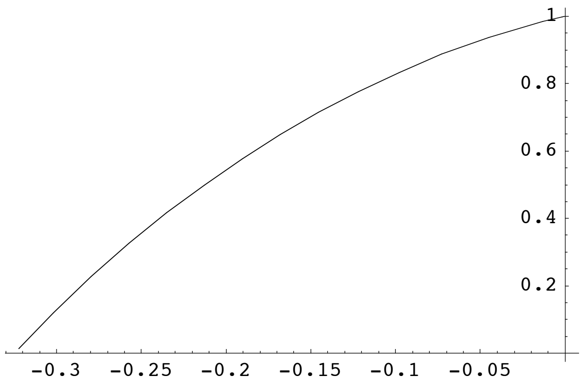
{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273},
{-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5},
{-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286},
{-0.121971, 0.775862}, {-0.097908, $\frac{5}{6}$}, {-0.0727829, 0.887097},
{-0.0451083, 0.9375}, {-0.0130789, 0.984848}, {-0.00172117, 0.998498}}

```
Conclusion[10, 5, 3.333, 5, 50, 10^-12, 10^-15, 10, 25, 100]
ll = Append[ll, %[[2]]]
```

{{23, 20.2734, 8.68194×10^{-14}, 0., -0.0390625, 4.00404, 3.333, 0.477844},
{-0.000173695, 0.99985}}

{{-0.322331, 0.0124378}, {-0.301942, 0.119048}, {-0.279488, 0.227273},
{-0.257204, 0.326087}, {-0.235013, 0.416667}, {-0.212832, 0.5},
{-0.190566, 0.576923}, {-0.1681, 0.648148}, {-0.145293, 0.714286},
{-0.121971, 0.775862}, {-0.097908, $\frac{5}{6}$}, {-0.0727829, 0.887097}, {-0.0451083, 0.9375},
{-0.0130789, 0.984848}, {-0.00172117, 0.998498}, {-0.000173695, 0.99985}}

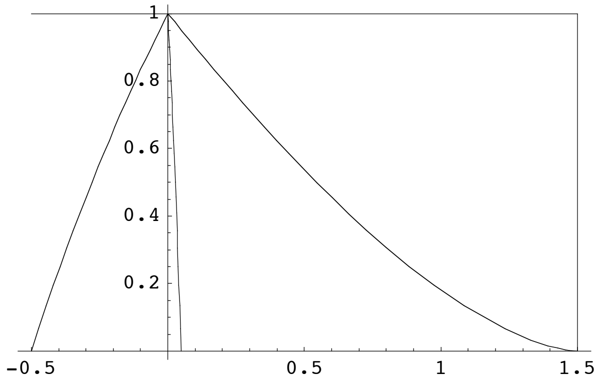
```
agn = ListPlot[ll, PlotJoined → True];
```

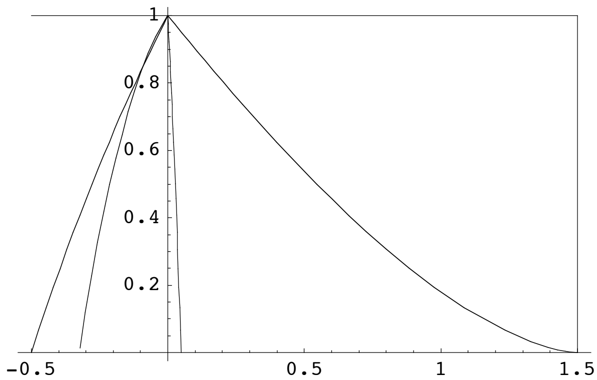


Recapitulation

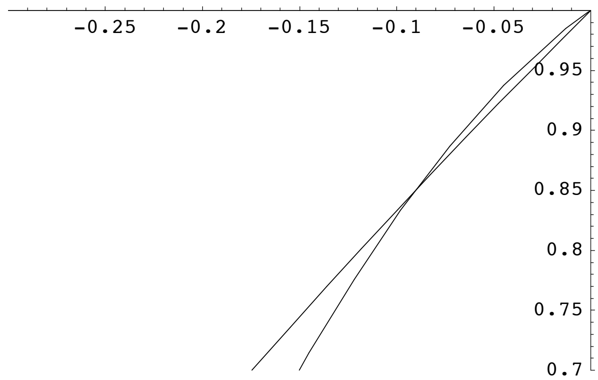
```
Recap[d_] := Show[s[d], fs[d], ae[d],
   ListPlot[{{ac[d], 0}, {ac[d], 1}, {-0.5, 1}}, PlotJoined → True,
    DisplayFunction → Identity], DisplayFunction -> $DisplayFunction];
```

```
P0 = Show[Recap[5], agn];
```






```
Show[P0, PlotRange → {{-0.3, 0}, {0.7, 1}}];
```



Existence

```
Nbre = 40;
ξ = 0.001;
```

```
ExistFS[d_] :=
 Table[{as[d, p], Theta[p, d]}, {p, 2 + ξ, (2 d)/(d - 2), (4/(d - 2) - 2 ξ) 1/Nbre}] /. d → 5
```

```
ExistenceFS = Join[ExistFS[5], {{0, 1}, {1.5, 1}, {1.5, 0}}];
```

```
P1 =
  Show[Graphics[{GrayLevel[0.9], Polygon[ExistenceFS]}], DisplayFunction → Visualize];
```

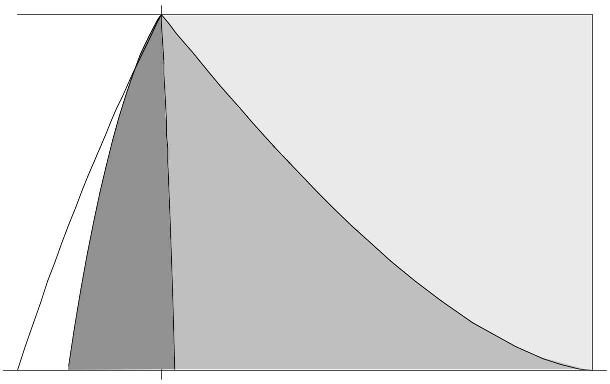
```
ExistAPriori[d_] :=
 Table[{aae[d, p], Theta[p, d]}, {p, (2 d)/(d - 2) - ξ, 2, -(4/(d - 2) - 2 ξ) 1/Nbre}] /. d → 5
```

```
ExistenceAP = Join[ExistAPriori[5], ExistFS[5]];
```

```
P2 =
  Show[Graphics[{GrayLevel[0.7], Polygon[ExistenceAP]}], DisplayFunction → Visualize];

ExistenceGN = Join[ExistAPriori[5], PlimExist, ll];
P3 =
  Show[Graphics[{GrayLevel[0.5], Polygon[ExistenceGN]}], DisplayFunction → Visualize];

Show[P1, P2, P3, P0, Ticks → None, Axes → True, DisplayFunction → $DisplayFunction];
```



Symmetry and symmetry breaking

```
SymmetryBreaking[d_] :=
 Table[{afs[d, p], Theta[p, d]}, {p, 2 + ξ, (2 d)/(d - 2), (4/(d - 2) - 2 ξ) 1/Nbre}]

SymmetryBFS = Join[SymmetryBreaking[5], {{0, 1}, {-0.6, 1}, {-0.6, 0}}];

P4 = Graphics[{GrayLevel[0.7], Polygon[SymmetryBFS]}];

Show[P4, DisplayFunction → Visualize];

SymmetryBreakingFive[d_] :=
 Table[{afs[d, p], Theta[p, d]}, {p, 2 + ξ, 3.030303, (4/(d - 2) - 2 ξ) 1/Nbre}]

SymmetryB = Join[SymmetryBreakingFive[5], {{afs[5, 3.030303], Theta[3.030303, 5]}},
   Table[ll[[k]], {k, 11, 1, -1}], PlimExist];


P5 = Graphics[{GrayLevel[0.5], Polygon[SymmetryB]}];

Show[P5, DisplayFunction → Visualize];

UnknownSymmetry = Join[PlimExist, Table[ll[[k]], {k, 1, 11, 1}],
   Table[{afs[5, p], Theta[p, 5]}, {p, 3.030303, 3.333, 0.01}], {{0, 1}},
   Table[ExistFS[5][[k]], {k, Length[ExistFS[5]], 1, -1}], {{1.5, 0}}];

P6 = Graphics[{GrayLevel[0.9], Polygon[UnknownSymmetry]}];
Show[P6, DisplayFunction → Visualize];
```

```
Show[P4, P5, P6, P0, Ticks → None, Axes → True, DisplayFunction → $DisplayFunction];
```