Stability results for Sobolev and logarithmic Sobolev inequalities

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October 3, 2024

Outline

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

Joint papers with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method

arXiv:2007.03674, to appear in *Memoirs of the AMS*

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion equations

DCDS, 43 (3&4): 1070–1089, 2023

Entropy – entropy production inequality

The fast diffusion equation on $\mathbb{R}^d$ in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0 \qquad \text{(FDE)}$$

admits a stationary Barenblatt solution $\mathcal{B}(x) := \left(1 + |x|^2\right)^{\frac{1}{m-1}}$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1} \left(v - \mathcal{B} \right) \right) dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 dx$$

are such that $\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$ [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\, e^{-4\,t}$$

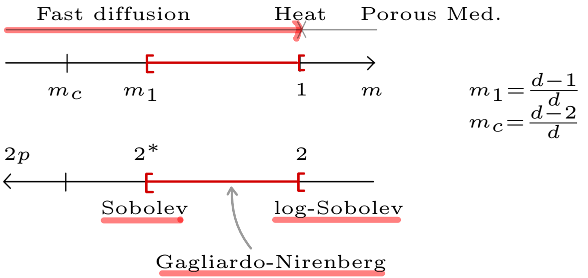
Entropy growth rate

$\mathcal{I}[v] \geq 4\,\mathcal{F}[v] \iff$ *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with optimal constant. Under appropriate mass normalization $v = f^{2\,p}$ so that $v^m = f^{p+1}$ and $v\,|\nabla v^{m-1}|^2 = (p-1)^2\,|\nabla f|^2$

$$p = \frac{1}{2\,m-1} \iff m = \frac{p+1}{2\,p} \in [m_1, 1)$$



Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha = 1/(m-1)<0$

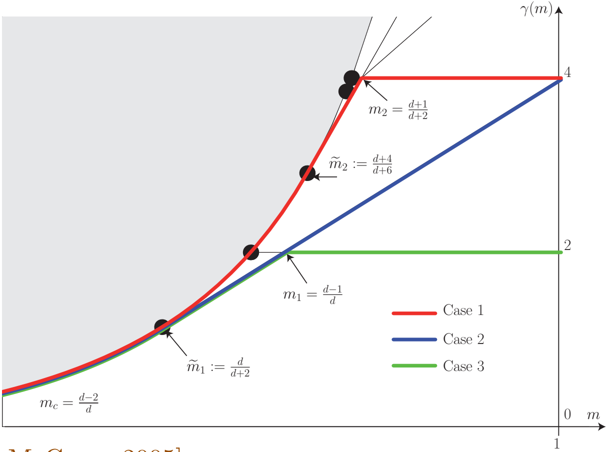
Lemma

Under assumption (**??**),

$$\mathcal{F}[v(t,\cdot)] \le C\,e^{-2\,\gamma(m)\,t} \quad \forall\, t\ge 0\,, \quad \gamma(m) := (1-m)\,\Lambda_{1/(m-1),d}$$

If $m_1 \le m < 1$, $\gamma(m)=2$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathcal{B}^{2-m}\,dx \quad\text{and}\quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathcal{B}\,dx$$

Hardy-Poincaré inequality. Let $d\ge 1$, $m\in(m_1,1)$ and $g\in\mathrm{L}^2(\mathbb{R}^d,\mathcal{B}^{2-m}\,dx)$ such that $\nabla g\in\mathrm{L}^2(\mathbb{R}^d,\mathcal{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathcal{B}^{2-m}\,dx=0$ and $\int_{\mathbb{R}^d} x\,g\,\mathcal{B}^{2-m}\,dx=0$

$$\mathsf{I}[g]\ge 4\,\alpha\,\mathsf{F}[g] \quad\text{where}\quad \alpha=2-d\,(1-m)$$

Proposition

Let $m\in(m_1,1)$ if $d\ge 2$, $m\in(1/3,1)$ if $d=1$, $\eta=2\,(d\,m-d+1)$ and $\chi=m/(266+56\,m)$. If $\int_{\mathbb{R}^d} v\,dx=\mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx=0$ and

$$(1-\varepsilon)\,\mathcal{B}\le v\le(1+\varepsilon)\,\mathcal{B}$$

for some $\varepsilon\in(0,\chi\,\eta)$, then

$$\mathcal{I}[v]\ge(4+\eta)\,\mathcal{F}[v]$$

The initial time layer improvement: backward estimate

For some strictly convex function ψ with $\psi(0) = 0$, $\psi'(0) = 1$

$$\mathcal{I} - 4\,\mathcal{F} \geq 4\,(\psi(\mathcal{F}) - \mathcal{F}) \geq 0$$

Away from the Barenblatt solutions, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}\,(\mathcal{Q} - 4)$$

by the *carré du champ* method

Lemma

Assume that $m > m_1$ and v is a solution to **(??)** *with nonnegative initial datum v_0. If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star, \cdot)] \geq 4 + \eta$, then*

$$\mathcal{Q}[v(t,\cdot)] \geq 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0, t_\star]$$

Uniform convergence in relative error: threshold time

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $t_\star \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \qquad (\mathrm{H}_A)$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$ *and* $\mathcal{F}[u_0] \leq G$*, then*

$$\sup_{x\in\mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq t_\star$$

Two consequences (subcritical case)

$$\zeta = \mathsf{Z}\big(A, \mathcal{F}[u_0]\big)\,,\quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\,\frac{2}{\alpha}} + G}\,,\quad \zeta_\star := \frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (**??**) *with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then*

$$\mathcal{F}[v(t,.)] \le \mathcal{F}[v_0]\,e^{-\,(4+\zeta)\,t} \quad \forall\,t \ge 0$$

▷ The ***stability of the entropy - entropy production inequality*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \ge \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \ge \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\;\; |x|\, f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Deficit of the Sobolev inequality: $\delta[f] := \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$

Theorem

Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathsf{g}\, dx \quad \textit{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \le A$$

we have

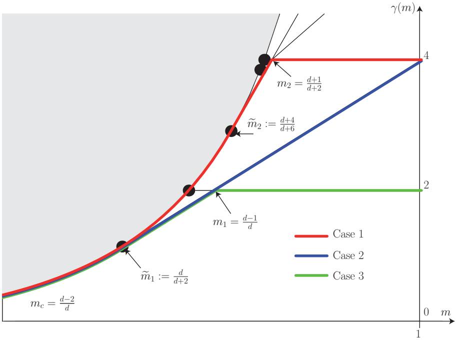
$$\delta[f] \ge \frac{\mathcal{C}_\star(A)}{4 + \mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathsf{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2\,d)}\right)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on d*

Extending the subcritical result to the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathcal{B}_\lambda(x) = \lambda^{-d/2}\,\mathcal{B}\left(x/\sqrt{\lambda}\right)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves (**??**) or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,, \quad \tau(0)=0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

Explicit stability results for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

arXiv: 2209.08651

A short review on improvements and stability for some interpolation inequalities

arXiv: 2402.08527

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d}\,\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d}\,\inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F-\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\,\inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]
In the inequality $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, the optimal constant is

$$\mathsf{S}_d = \tfrac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\textit{if}\ \lim_{n\to\infty} \|\nabla f_n\|^2_2 / \|f_n\|^2_{2^*} = \mathsf{S}_d\,,\ \textit{then}\ \lim_{n\to\infty}\ \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|^2_2 / \|\nabla f_n\|^2_2 = 0$$

▷ [Brezis, Lieb], 1985 a quantitative stability result ?

▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|^2_2 \geq \mathsf{S}_d\, \|f\|^2_{2^*} + c_{\mathrm{BE}} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_2$$

- The strategy of Bianchi & Egnell involves two steps:

– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$

– a local-to-global extension based on concentration-compactness : the *far away regime*

- The constant c_{BE} is not explicit

Stability for the logarithmic Sobolev inequality

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \pi \int_{\mathbb{R}^n} u^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}} \right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 69], [Costa, 85] *Cf.* [Villani, 08]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathscr{M} := \left\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a\cdot x} \quad \forall\, x \in \mathbb{R}^n$$

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance, in $\mathrm{W}^{1,1}$, *etc.*

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

Explicit stability result for the Sobolev inequality Proof

Sketch of the proof

Goal: prove that there is an ***explicit*** constant $\beta > 0$ such that for all $d \geq 3$ and all $f \in \dot{\mathrm{H}}^1\left(\mathbb{R}^d\right)$

$$\|\nabla f\|_2^2 \geq S_d \, \|f\|_{2^*}^2 + \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

Part 1. We show the inequality for nonnegative functions far from $\mathcal{M}$

... the far away regime

Make it *constructive*

Part 2. We show the inequality for nonnegative functions close to $\mathcal{M}$

... the local problem

Get *explicit* estimates and remainder terms

Part 3. We show that the inequality for nonnegative functions implies the inequality for functions without a sign restriction, up to an acceptable loss in the constant

... dealing with sign-changing functions

Some definitions

What we want to minimize is

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathsf{d}(f,\mathcal{M})^2} \quad f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$$

where

$$\mathsf{d}(f,\mathcal{M})^2 := \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ up to a *conformal transformation*, we assume that $\mathsf{d}(f,\mathcal{M})^2 = \|\nabla f - \nabla g_*\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ with

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

▷ use the *inverse stereographic* projection

$$F(\omega) = \frac{f(x)}{g_*(x)} \quad x \in \mathbb{R}^d \text{ with } \begin{cases} \omega_j = \frac{2\,x_j}{1+|x|^2} & \text{if } 1 \le j \le d \\ \omega_{d+1} = \frac{1-|x|^2}{1+|x|^2} \end{cases}$$

The problem on the unit sphere

Stability inequality on the unit sphere $\mathbb{S}^d$ for $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$

$$\int_{\mathbb{S}^d} \left(|\nabla F|^2 + \mathsf{A}\,|F|^2\right) d\mu - \mathsf{A}\left(\int_{\mathbb{S}^d} |F|^{2^*}\,d\mu\right)^{2/2^*} \\ \geq \frac{\beta}{d} \inf_{G \in \mathscr{M}} \left\{ \|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathsf{A}\,\|F - G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right\}$$

with $\mathrm{A} = \frac{1}{4}\,d\,(d-2)$ and a manifold $\mathscr{M}$ of optimal functions made of

$$G(\omega) = c\,(a + b\cdot\omega)^{-\frac{d-2}{2}} \quad \omega \in \mathbb{S}^d \quad (a, b, c) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}$$

- make the reduction of a ***far away problem*** to a local problem *constructive...* on $\mathbb{R}^d$
- make the analysis of the ***local problem*** *explicit...* on $\mathbb{S}^d$

Competing symmetries

- ***Rotations on the sphere*** combined with stereographic and inverse stereographic projections. Let $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$

$$(Uf)(x) := \left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x-e_d|^2}, \ldots, \frac{x_{d-1}}{|x-e_d|^2}, \frac{|x|^2-1}{|x-e_d|^2}\right)$$

$$\mathcal{E}(Uf) = \mathcal{E}(f)$$

- ***Symmetric decreasing rearrangement*** $\mathcal{R}f = f^*$

f and f^* are equimeasurable

$$\|\nabla f^*\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

The method of ***competing symmetries***

Theorem (Carlen, Loss, 1990)

Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$. The sequence $f_n = (\mathcal{R}U)^n f$ is such that $\lim_{n\to+\infty} \|f_n - g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 0$. If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $\left(\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)}\right)_{n\in\mathbb{N}}$ is a non-increasing sequence

Useful preliminary results

- $\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$ where $h_f = \|f\|_{2^*}\, g_*/\|g_*\|_{2^*} \in \mathcal{M}$
- $(\|\nabla f_n\|_2^2)_{n\in\mathbb{N}}$ is a nonincreasing sequence

Lemma

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g\in\mathcal{M},\, \|g\|_{2^*}=1} \left(f, g^{2^*-1}\right)^2$$

Corollary

$\big(\mathrm{d}(f_n,\mathcal{M})\big)_{n\in\mathbb{N}}$ is strictly decreasing, $n \mapsto \sup_{g\in\mathcal{M}_1} \left(f_n, g^{2^*-1}\right)$ is strictly increasing, and

$$\lim_{n\to\infty} \mathrm{d}(f_n,\mathcal{M})^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\,\|h_f\|_{2^*}^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2$$

but no monotonicity for $n \mapsto \mathcal{E}(f_n) = \dfrac{\|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f_n\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathrm{d}(f_n,\mathcal{M})^2}$

Part 1: Global to local reduction

The ***local problem***

$$\mathscr{I}(\delta) := \inf \left\{ \mathcal{E}(f) \,:\, f \geq 0 \,,\ \mathsf{d}(f, \mathcal{M})^2 \leq \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right\}$$

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the ***far away regime***

$$\mathsf{d}(f, \mathcal{M})^2 = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

for some $\delta \in (0, 1)$

Let $f_n = (\mathcal{R}U)^n f$. There are two cases:

- (Case 1) $\mathsf{d}(f_n, \mathcal{M})^2 \geq \delta \, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$
- (Case 2) for some $n \in \mathbb{N}$, $\mathsf{d}(f_n, \mathcal{M})^2 < \delta \, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Global to local reduction – Case 1

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the far away regime

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

Lemma

Let $f_n = (\mathcal{R}U)^n f$ and $\delta \in (0,1)$. If $\mathrm{d}(f_n, \mathcal{M})^2 \geq \delta \, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$, then

$$\mathcal{E}(f) \geq \delta$$

$$\lim_{n\to+\infty} \|\nabla f_n\|_2^2 \leq \frac{1}{\delta} \lim_{n\to+\infty} \inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta} \left(\lim_{n\to+\infty} \|\nabla f_n\|_2^2 - S_d \, \|f\|_{2^*}^2 \right)$$

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \geq \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \geq \frac{\|\nabla f_n\|_2^2 - S_d \, \|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \underset{n\to+\infty}{\geq} \delta$$

Global to local reduction – Case 2

$$\mathscr{I}(\delta) := \inf \left\{ \mathcal{E}(f) \,:\, f \geq 0\,,\ \mathrm{d}(f,\mathcal{M})^2 \leq \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right\}$$

Lemma

$$\mathcal{E}(f) \geq \delta \, \mathscr{I}(\delta)$$

if $\displaystyle \inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta \, \|\nabla f_{n_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ *and* $\displaystyle \inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} < \delta \, \|\nabla f_{n_0+1}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Adapt a strategy due to Christ: build a (semi-) *continuous* rearrangement *flow* $(\mathsf{f}_\tau)_{n_0\leq\tau<n_0+1}$ with $\mathsf{f}_{n_0} = U f_n$ such that $\|f_\tau\|_{2^*} = \|f\|_2$, $\tau \mapsto \|\nabla f_\tau\|_2$ is nonincreasing, and $\lim_{\tau\to n_0+1} \mathsf{f}_\tau = f_{n_0+1}$

$$\mathcal{E}(f) \geq 1 - S_d \frac{\|f\|^2_{2^*}}{\|\nabla f\|^2_2} \geq 1 - S_d \frac{\|\mathsf{f}_{\tau_0}\|^2_{2^*}}{\|\nabla \mathsf{f}_{\tau_0}\|^2_2} = \delta \, \mathcal{E}(f_{\tau_0}) \geq \delta \, \mathscr{I}(\delta)$$

Altogether: if $\mathrm{d}(f,\mathcal{M})^2 > \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, then $\mathcal{E}(f) \geq \min\{\delta, \delta \, \mathscr{I}(\delta)\}$

Part 2: The (simple) Taylor expansion

Proposition

Let $(X, d\mu)$ be a measure space and $u, r \in \mathrm{L}^q(X, d\mu)$ for some $q \geq 2$ with $u \geq 0$, $u + r \geq 0$ and $\int_X u^{q-1}\, r\, d\mu = 0$

▷ *If $q = 6$, then*

$$\|u+r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \Big(5\int_X u^{q-2}\, r^2\, d\mu + \tfrac{20}{3}\int_X u^{q-3}\, r^3\, d\mu + 5\int_X u^{q-4}\, r^4\, d\mu + 2\int_X u^{q-5}\, r^5\, d\mu + \tfrac{1}{3}\int_X r^6\, d\mu\Big)$$

▷ *If $3 \leq q \leq 4$, then*

$$\|u+r\|_q^2 - \|u\|_q^2 \leq \|u\|_q^{2-q} \left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{(q-1)\,(q-2)}{3}\int_X u^{q-3}\, r^3\, d\mu + \frac{2}{q}\int_X |r|^q\, d\mu\right)$$

▷ *If $2 \leq q \leq 3$, then*

$$\|u+r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{2}{q}\int_X r_+^q\, d\mu\right)$$

Corollary

For all $\nu > 0$ *and for all* $r \in \mathrm{H}^1(\mathbb{S}^d)$ *satisfying* $r \geq -1$,

$$\left(\int_{\mathbb{S}^d} |r|^q \, d\mu\right)^{2/q} \leq \nu^2 \quad \text{and} \quad \int_{\mathbb{S}^d} r \, d\mu = 0 = \int_{\mathbb{S}^d} \omega_j \, r \, d\mu \quad \forall \, j = 1, \ldots d+1$$

if $d\mu$ *is the uniform probability measure on* $\mathbb{S}^d$, *then*

$$\int_{\mathbb{S}^d} \left(|\nabla r|^2 + \mathrm{A}\,(1+r)^2\right) d\mu - \mathrm{A}\left(\int_{\mathbb{S}^d} (1+r)^q \, d\mu\right)^{2/q} \geq \mathrm{m}(\nu) \int_{\mathbb{S}^d} \left(|\nabla r|^2 + \mathrm{A}\, r^2\right) d\mu$$

$$\mathrm{m}(\nu) := \tfrac{4}{d+4} - \tfrac{2}{q}\,\nu^{q-2} \qquad \text{if} \quad d \geq 6$$

$$\mathrm{m}(\nu) := \tfrac{4}{d+4} - \tfrac{1}{3}\,(q-1)\,(q-2)\,\nu - \tfrac{2}{q}\,\nu^{q-2} \qquad \text{if} \quad d = 4,\ 5$$

$$\mathrm{m}(\nu) := \tfrac{4}{7} - \tfrac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \tfrac{1}{3}\,\nu^4 \qquad \text{if} \quad d = 3$$

An explicit expression of $\mathscr{I}(\delta)$ if $\nu > 0$ is small enough so that $\mathrm{m}(\nu) > 0$

Part 3: Removing the positivity assumption

Take $f = f_+ - f_-$ with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 1$ and define $m := \|f_-\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and $1 - m = \|f_+\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} > 1/2$. The positive concave function

$$h_d(m) := m^{\frac{d-2}{d}} + (1-m)^{\frac{d-2}{d}} - 1$$

satisfies

$$2\,h_d(1/2)\,m \le h_d(m)\,, \quad h_d(1/2) = 2^{2/d} - 1$$

With $\delta(f) = \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, one finds $g_+ \in \mathcal{M}$ such that

$$\delta(f) \ge C^{d,\mathrm{pos}}_{\mathrm{BE}}\,\|\nabla f_+ - \nabla g_+\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{2\,h_d(1/2)}{h_d(1/2)+1}\,\|\nabla f_-\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

and therefore

$$C^d_{\mathrm{BE}} \ge \tfrac{1}{2}\,\min\left\{\max_{0<\delta<1/2}\,\delta\,\mathscr{I}(\delta),\,\frac{2\,h_d(1/2)}{h_d(1/2)+1}\right\}$$

Part 2, refined: The (complicated) Taylor expansion

To get a dimensionally sharp estimate, we expand $(1+r)^{2^*}-1-2^*r$ with an accurate remainder term for all $r\ge -1$

$$r_1:=\min\{r,\gamma\}\,,\quad r_2:=\min\left\{(r-\gamma)_+,M-\gamma\right\}\quad\text{and}\quad r_3:=(r-M)_+$$

with $0<\gamma<M$. Let $\theta=4/(d-2)$

Lemma

Given $d\ge 6$, $r\in[-1,\infty)$, and $\overline{M}\in[\sqrt{e},+\infty)$, we have

$$\begin{aligned}(1+r)^{2^*}&-1-2^*r\\ &\le \tfrac12\,2^*\,(2^*-1)\,(r_1+r_2)^2+2\,(r_1+r_2)\,r_3+\left(1+C_M\,\theta\,\overline{M}^{-1}\ln\overline{M}\right)r_3^{2^*}\\ &\quad+\left(\tfrac32\,\gamma\,\theta\,r_1^2+C_{M,\overline{M}}\,\theta\,r_2^2\right)\mathbb{1}_{\{r\le M\}}+C_{M,\overline{M}}\,\theta\,M^2\,\mathbb{1}_{\{r>M\}}\end{aligned}$$

where all the constants in the above inequality are explicit

There are constants ϵ_1, ϵ_2, k_0, and $\epsilon_0 \in (0, 1/\theta)$, such that

$$\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\ \|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathrm{A}\ \|1+r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \geq \frac{4\,\epsilon_0}{d-2}\left(\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\ \|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) + \sum_{k=1}^{3} I_k$$

$$I_1 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_1|^2 + \mathrm{A}\,r_1^2\right)d\mu - \mathrm{A}\,(2^*-1+\epsilon_1\,\theta)\int_{\mathbb{S}^d} r_1^2\,d\mu + \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d}(r_2^2 \ldots$$

$$I_2 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_2|^2 + \mathrm{A}\,r_2^2\right)d\mu - \mathrm{A}\,\left(2^*-1+(k_0+C_{\epsilon_1,\epsilon_2})\,\theta\right)\int_{\mathbb{S}^d} r_2^2\,d\mu$$

$$I_3 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_3|^2 + \mathrm{A}\,r_3^2\right)d\mu - \frac{2}{2^*}\,\mathrm{A}\,(1+\epsilon_2\,\theta)\int_{\mathbb{S}^d} r_3^{2^*}\,d\mu - \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d} r_3^2\,d\mu$$

- spectral gap estimates : $I_1 \geq 0$
- Sobolev inequality : $I_3 \geq 0$
- improved spectral gap inequality using that $\mu\big(\{r_2 > 0\}\big)$ is small: $I_2 \geq 0$

More explicit stability result for the logarithmic Sobolev and Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$

Subcritical interpolation inequalities on the sphere

- ***Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem

Let $d \geq 1$ and $p \in (1,2) \cup (2, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F] \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

▷ The same result holds true for the logarithmic Sobolev inequality, again with an explicit constant $\mathscr{S}_{d,2}$, for any finite dimension d

From $\mathbb{S}^d$ to $\mathbb{R}^d$: the large dimensional limit

Carré du champ – admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

$$m_\pm(d,p) := \tfrac{1}{(d+2)\,p} \left(d\,p + 2 \pm \sqrt{d\,(p-1)\,\big(2\,d-(d-2)\,p\big)} \right)$$

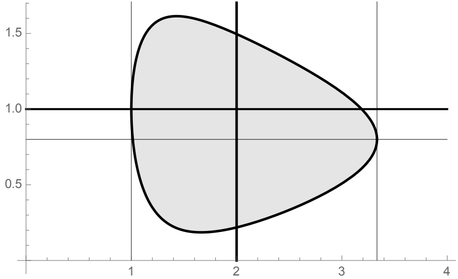


Figure: Case $d=5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)} \left(\mathcal{L} v + (m\,p-1)\,\frac{|\nabla v|^2}{v} \right) \quad \text{on} \quad \mathbb{R}^n$$

[JD, Brigati, Simonov] *Ornstein-Uhlenbeck operator:* $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_\pm(p) := \lim_{d \to +\infty} m_\pm(d,p) = 1 \pm \frac{1}{p} \sqrt{(p-1)\,(2-p)}$$

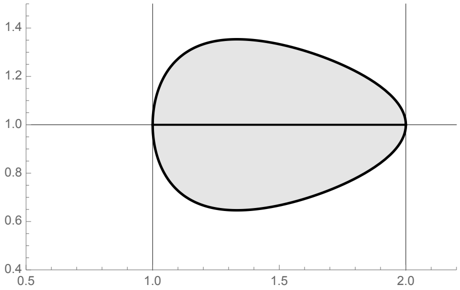


Figure: The admissible parameters $1 \leq p \leq 2$ and m are independent of n

Large dimensional limit

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem

Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ with compact support, $d \ge n$ and

$$u_d(\omega) = v\Big(\omega_1/r_d, \omega_2/r_d, \ldots, \omega_n/r_d\Big)\,, \quad r_d = \sqrt{\frac{d}{2\,\pi}}$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\,\pi)^{-n/2}\,e^{-\frac12\,|y|^2}\,dy$,

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

L^2 stability of LSI: comments

[JD, Esteban, Figalli, Frank, Loss]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma \geq \frac{\beta\,\pi}{2} \inf_{a\in\mathbb{R}^d,\, c\in\mathbb{R}} \int_{\mathbb{R}^n} |u - c\,e^{a\cdot x}|^2\, d\gamma$$

- One dimension is lost (for the manifold of invariant functions) in the limiting process
- Euclidean forms of the stability
- The $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ does not appear, it gets lost in the limit $d \to +\infty$
- $\int_{\mathbb{R}^n} |\nabla(u - c\,e^{a\cdot x})|^2\, d\gamma$? False, but makes sense under additional assumptions. Some results based on the Ornstein-Uhlenbeck flow and entropy methods: [Fathi, Indrei, Ledoux, 2016], [JD, Brigati, Simonov, 2023-24]
- Taking the limit is difficult because of the lack of compactness. Two proofs

Stability: large dimensional limit

Let $u = f/g_\star$, $g_\star := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{(d-2)/2}$

$$\int_{\mathbb{R}^d} |\nabla u|^2\, g_\star^2\, dx + d\,(d-2)\int_{\mathbb{R}^d} |u|^2\, g_\star^{2^*}\, dx$$
$$- d\,(d-2)\, \|g_\star\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{2^*-2} \left(\int_{\mathbb{R}^d} |u|^{2^*}\, g_\star^{2^*}\, dx\right)^{2/2^*}$$
$$\ge \frac{\beta}{d}\left(\int_{\mathbb{R}^d} |\nabla u|^2\, g_\star^2\, dx + d\,(d-2)\int_{\mathbb{R}^d} |u - g_d/g_\star|^2\, g_\star^{2^*}\, dx\right),$$

where $g_d(x) = c_d\,(a_d + |x-b_d|^2)^{1-d/2}$ realizes the distance to $\mathcal{M}$

$$u(x) = v\,(r_d\,x)\,,\quad w_v^d\,(r_d\,x) = \frac{g_d(x)}{g_\star(x)}\,,\quad r_d = \sqrt{\frac{d}{2\,\pi}}$$

We consider a function $v(x)$ depending only on $y \in \mathbb{R}^N$ with $x = (y,z) \in \mathbb{R}^N \times \mathbb{R}^{d-N} \approx \mathbb{R}^d$, for some fixed integer N

$$\lim_{d\to+\infty} \left(1 + \frac{1}{r_d^2}\,|y|^2\right)^{-\frac{N+d}{2}} = e^{-\pi\,|y|^2}, \quad \lim_{d\to+\infty} \int_{\mathbb{R}^d} |v(y)|^2\, d\mu = \int_{\mathbb{R}^N} |v|^2\, d\gamma$$

$$\lim_{d\to+\infty} \int_{\mathbb{R}^d} |\nabla v|^2 \left(1 + \frac{1}{r_d^2}\,|x|^2\right)^2 d\mu = 4\int_{\mathbb{R}^N} |\nabla v|^2\, d\gamma$$

Estimates on the parameters a_d, b_d and c_d !

More results on logarithmic Sobolev inequalities

Joint work with G. Brigati and N. Simonov

Stability for the logarithmic Sobolev inequality

arXiv:2303.12926

▷ *Entropy methods, with constraints*

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} \mathrm{d}(u_\varepsilon, w)^\alpha \le \frac{1}{2}\,\varepsilon^4 + O(\varepsilon^6)$$

$\mathscr{M} := \left\{ w_{a,c} \,:\, (a, c) \in \mathbb{R}^d \times \mathbb{R} \right\}$ where $w_{a,c}(x) = c\, e^{-a \cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ and $\|x\, u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \frac{1}{2\,d} \left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \right)^2$$

and, with $\psi(s) := s - \frac{d}{4} \log\left(1 + \frac{4}{d}\, s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \psi\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \right)$$

Stability under log-concavity

Theorem

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that

$$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\gamma = (1,0) \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \le \mathsf{K}$$

we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,d\gamma \ge 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{432\,\mathsf{K}} \approx 1 + \frac{0.00231481}{\mathsf{K}}$$

Self-improving Poincaré inequality and stability for LSI:
[Fathi, Indrei, Ledoux, '16]

Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x)\, |u|^2\, d\gamma = (1, 0)\,, \quad \int_{\mathbb{R}^d} |x|^2\, |u|^2\, d\gamma \leq d\,, \quad \int_{\mathbb{R}^d} |u|^2\, e^{\varepsilon\, |x|^2}\, d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \geq \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2\, \log |u|^2\, d\gamma$$

with $\mathscr{C} = 1 + \frac{\mathscr{C}_\star(\mathsf{K}_\star) - 1}{1 + R^2\, \mathscr{C}_\star(\mathsf{K}_\star)}$, $\mathsf{K}_\star := \max\left(d, \frac{(d+1)\, R^2}{1+R^2}\right)$ *if* $\operatorname{supp}(u) \subset B(0, R)$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

These slides can be found at

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Thank you for your attention !