Construire des estimations de stabilité dans des inégalités de type Sobolev

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Introduction

- Sobolev inequality on $\mathbb{R}^d$ with $d \ge 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \mathscr{D}^{1,2}(\mathbb{R}^d) \qquad \text{(S)}$$

Equality holds on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

[Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \ge c_{\mathrm{BE}} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

- How do we estimate c_{BE} ? as $d \to +\infty$?
Stability & improved entropy – entropy production inequalities
Improved inequalities & faster decay rates for entropies

Outline

1. Results based on entropy methods and fast diffusion equations
   - Sobolev and HLS inequalities: duality and Yamabe flow
   - Stability, fast diffusion equation and entropy methods
2. Explicit stability for Sobolev and LSI on $\mathbb{R}^d$
   - Main results, optimal dimensional dependence; history
   - Sketch of the proof, definitions & preliminary results
   - The main steps of the proof
3. More stability results for LSI and related inequalities
   - Subcritical interpolation inequalities on the sphere
   - The large dimensional limit
   - More results on LSI and Gagliardo-Nirenberg inequalities

Results based on entropy methods and fast diffusion equations

Joint paper with G. Jankowiak
Sobolev and Hardy–Littlewood–Sobolev inequalities
J. Differential Equations, 257, 2014

Joint research project with with
M. Bonforte, B. Nazaret, N. Simonov

Sobolev and Hardy-Littlewood-Sobolev inequalities

- ▷ Stability in a weaker norm, with explicit constants
- ▷ From duality to improved estimates
- ▷ Fast diffusion equation with Yamabe's exponent
- ▷ Explicit stability constants

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d) \qquad \text{(S)}$$

and the Hardy-Littlewood-Sobolev inequality

$$\|g\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \mathsf{S}_d \int_{\mathbb{R}^d} g\,(-\Delta)^{-1} g\,dx \quad \forall\, g \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \qquad \text{(HLS)}$$

are dual of each other. Here S_d is the Aubin-Talenti constant, $2^* = \frac{2d}{d-2}$, $(2^*)' = \frac{2d}{d+2}$ and by the Legendre transform

$$\sup_{f \in \mathscr{D}^{1,2}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} f\,g\,dx - \frac{1}{2}\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right) = \frac{1}{2}\,\|g\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

$$\sup_{f \in \mathscr{D}^{1,2}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} f\,g\,dx - \frac{1}{2}\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right) = \frac{1}{2} \int_{\mathbb{R}^d} g\,(-\Delta)^{-1} g\,dx$$

Improved Sobolev inequality by duality

Theorem

[JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \in [\frac{d}{d+4}, 1)$ such that

$$\|f^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \mathsf{S}_d \int_{\mathbb{R}^d} f^q \, (-\Delta)^{-1} f^q \, dx$$

$$\leq \mathcal{C}\, \mathsf{S}_d^{-1} \, \|f\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)$$

for any $f \in \mathscr{D}^{1,2}(\mathbb{R}^d)$

The completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1} g|^2\,dx = \int_{\mathbb{R}^d} g\,(-\Delta)^{-1} g\,dx$$

and, if $g = f^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla f \cdot \nabla(-\Delta)^{-1} g\,dx = \int_{\mathbb{R}^d} f\,g\,dx = \int_{\mathbb{R}^d} g^q\,dx = \int_{\mathbb{R}^d} f^{2^*}\,dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}} \nabla f - \mathsf{S}_d\,\nabla(-\Delta)^{-1} g \right|^2 dx$$

shows that (with $\mathcal{C} = 1$)

$$0 \le \|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left(\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \mathsf{S}_d\,\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2 \right) - \mathsf{S}_d \left(\mathsf{S}_d\,\|f^q\|_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} f^q\,(-\Delta)^{-1} f^q\,dx \right)$$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \,, \quad t > 0 \,, \quad x \in \mathbb{R}^d \tag{Y}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, where

$$\mathsf{H}_d[v] := \mathsf{S}_d \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} \le 0$$

is, up to the sign, the deficit in (HLS), then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\,\mathsf{S}_d \int_{\mathbb{R}^d} v^{m+1}\,dx + \left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\,dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\,dx$$

where $v = v(t,\cdot)$ is a solution of (Y). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2} = q$, which corresponds to the Yamabe flow

A simple observation

Proposition

Assume that $d \geq 3$ *and* $m = \frac{d-2}{d+2}$*. If* v *is a solution of* (Y) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$*, then*

$$\frac{1}{2}\,\frac{d}{dt}\left(\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d^{-1}\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right)$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}} \left(\mathsf{S}_d^{-1}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right) \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$

Notice that $u = v^m$ is an optimal function for (S) if v is optimal for (HLS)

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[del Pino, Saez], [Vázquez, Esteban, Rodriguez] For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot)-1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

A convexity improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

Theorem

[JD, Jankowiak] Assume that $d \geq 3$. Then we have

$$0 \leq \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left(\mathsf{S}_d^{-1}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right)\right)$$

where $\varphi(x) := \sqrt{1+2\,x} - 1$ for any $x \geq 0$

Proof: with $\kappa_0 := \mathsf{H}_0'/\mathsf{J}_0$ and $\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t))$, consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \leq \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

Joint papers with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method

arXiv:2007.03674, to appear in *Memoirs of the AMS*

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion equations

DCDS, 43 (3&4): 10701089, 2023

Entropy – entropy production inequality

The fast diffusion equation on $\mathbb{R}^d$ in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big[v \left(\nabla v^{m-1} - 2\,x \right) \Big] = 0 \qquad \text{(FDE)}$$

admits a stationary Barenblatt solution $\mathcal{B}(x) := \left(1 + |x|^2\right)^{\frac{1}{m-1}}$

$$\frac{d}{dt} \mathcal{F}[v(t,\cdot)] = -\,\mathcal{I}[v(t,\cdot)]$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v - \mathcal{B}) \right)\,dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v\,\left|\nabla v^{m-1} - \nabla \mathcal{B}^{m-1}\right|^2\,dx$$

are such that $\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$ [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-\,4\,t}$$

Entropy growth rate

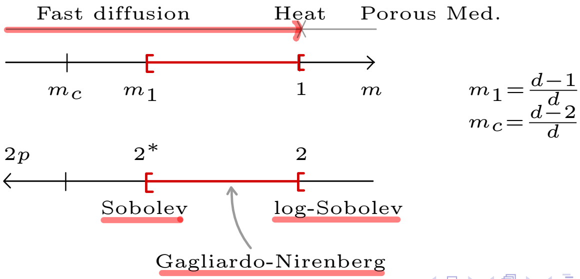
$\mathcal{I}[v] \geq 4\,\mathcal{F}[v] \iff$ *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with optimal constant. Under appropriate mass normalization

$v = f^{2p}$ so that $v^m = f^{p+1}$ and $v\,|\nabla v^{m-1}|^2 = (p-1)^2\,|\nabla f|^2$

$$p = \frac{1}{2m-1} \iff m = \frac{p+1}{2p} \in [m_1, 1)$$



Asymptotic regime as $t \to +\infty$

Take $f_\varepsilon := \mathcal{B}\,(1+\varepsilon\,\mathcal{B}^{1-m}\,w)$ and expand $\mathcal{F}[f_\varepsilon]$ and $\mathcal{I}[f_\varepsilon]$ at order $O(\varepsilon^2)$

linearized free energy and *linearized Fisher information*

$$\mathsf{F}[w] := \frac{m}{2}\int_{\mathbb{R}^d} w^2\,\mathcal{B}^{2-m}\,dx \quad\text{and}\quad \mathsf{I}[w] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla w|^2\,\mathcal{B}\,dx$$

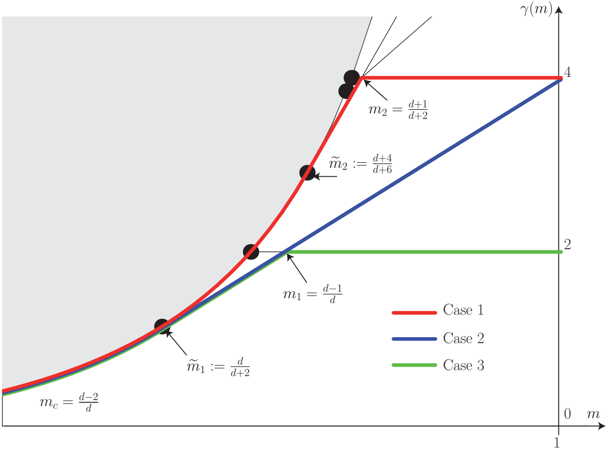
Proposition (Hardy-Poincaré inequality)

[BBDGV ,BDNS] Let $m\in[m_1,1)$ if $d\ge3$, $m\in(1/2,1)$ if $d=2$, and $m\in(1/3,1)$ if $d=1$. If $w\in\mathrm{L}^2(\mathbb{R}^d,\mathcal{B}^{2-m}\,dx)$ is such that $\nabla w\in\mathrm{L}^2(\mathbb{R}^d,\mathcal{B}\,dx)$, $\int_{\mathbb{R}^d} w\,\mathcal{B}^{2-m}\,dx=0$, then

$$\mathsf{I}[w]\ge 4\,\alpha\,\mathsf{F}[w]$$

with $\alpha=1$, or $\alpha=2-d\,(1-m)$. if $\int_{\mathbb{R}^d} x\,w\,\mathcal{B}^{2-m}\,dx=0$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathcal{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathcal{B}\,dx$$

Hardy-Poincaré inequality. Let $d \geq 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathcal{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathcal{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \geq 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and

$$(1-\varepsilon)\,\mathcal{B} \leq v \leq (1+\varepsilon)\,\mathcal{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathcal{I}[v] \geq (4+\eta)\,\mathcal{F}[v]$$

The initial time layer improvement: backward estimate

By the *carré du champ* method, we have
Away from the Barenblatt solutions, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q} - 4)$$

Lemma

Assume that $m > m_1$ *and* v *is a solution to* (FDE) *with nonnegative initial datum* v_0*. If for some* $\eta > 0$ *and* $t_\star > 0$*, we have* $\mathcal{Q}[v(t_\star,\cdot)] \ge 4 + \eta$*, then*

$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0, t_\star]$$

Uniform convergence in relative error: threshold time

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $t_\star \geq 0$ such that, if u is a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big[v \left(\nabla v^{m-1} - 2\,x \right) \Big] = 0 \qquad \text{(FDE)}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

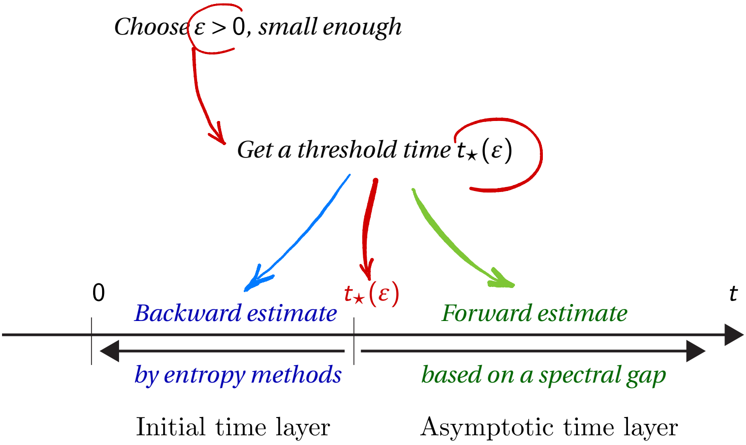
$$\mathrm{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \qquad (\mathrm{H}_A)$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq t_\star$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy



Two consequences (subcritical case)

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (FDE) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0]\,e^{-\,(4+\zeta)\,t} \quad \forall\,t \geq 0$$

▷ The ***stability of the entropy - entropy production inequality*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\;\; |x|\, f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Deficit of the Sobolev inequality: $\delta[f] := \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$

Theorem

Let $d \geq 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \quad \textit{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \leq A$$

we have

$$\delta[f] \geq \frac{\mathcal{C}_\star(A)}{4 + \mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2\,d)}\right)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on d*

Explicit stability results for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

arXiv: 2209.08651

A short review on improvements and stability for some interpolation inequalities

arXiv: 2402.08527

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\;\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\,\big(a + |x-b|^2\big)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\;\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d}\,\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d}\inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]
In the inequality $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, the optimal constant is

$$\mathsf{S}_d = \tfrac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\textit{if } \lim_{n\to\infty} \|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = \mathsf{S}_d\,,\ \textit{then } \lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb], 1985 a quantitative stability result ?
▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\rm BE} > 0$ such that

$$\|\nabla f\|_2^2 \geq \mathsf{S}_d\,\|f\|_{2^*}^2 + c_{\rm BE} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

- The strategy of Bianchi & Egnell involves two steps:
  – a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$
  – a local-to-global extension based on concentration-compactness : the *far away regime*
- The constant $c_{\rm BE}$ is not explicit

Stability for the logarithmic Sobolev inequality

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 69], [Costa, 85] *Cf.* [Villani, 08]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathscr{M} := \left\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a\cdot x} \quad \forall\, x \in \mathbb{R}^n$$

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance, in $\mathrm{W}^{1,1}$, *etc.*

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

Explicit stability results for the Sobolev inequality Proof

Sketch of the proof

Goal: prove that there is an ***explicit*** constant $\beta > 0$ such that for all $d \geq 3$ and all $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

$$\|\nabla f\|_2^2 \geq \mathsf{S}_d \|f\|_{2^*}^2 + \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

Part 1. We show the inequality for nonnegative functions far from $\mathcal{M}$

... the far away regime

Make it *constructive*

Part 2. We show the inequality for nonnegative functions close to $\mathcal{M}$

... the local problem

Get *explicit* estimates and remainder terms

Part 3. We show that the inequality for nonnegative functions implies the inequality for functions without a sign restriction, up to an acceptable loss in the constant

... dealing with sign-changing functions

Some definitions

What we want to minimize is

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathsf{d}(f,\mathcal{M})^2} \quad f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$$

where

$$\mathsf{d}(f,\mathcal{M})^2 := \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ up to a *conformal transformation*, we assume that $\mathsf{d}(f,\mathcal{M})^2 = \|\nabla f - \nabla g_*\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ with

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

▷ use the *inverse stereographic* projection

$$F(\omega) = \frac{f(x)}{g_*(x)} \quad x \in \mathbb{R}^d \text{ with } \begin{cases} \omega_j = \frac{2\,x_j}{1+|x|^2} & \text{if } 1 \le j \le d \\ \omega_{d+1} = \frac{1-|x|^2}{1+|x|^2} & \end{cases}$$

The problem on the unit sphere

Stability inequality on the unit sphere $\mathbb{S}^d$ for $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$

$$\int_{\mathbb{S}^d} \left(|\nabla F|^2 + \mathrm{A}\,|F|^2\right) d\mu - \mathrm{A}\left(\int_{\mathbb{S}^d} |F|^{2^*}\,d\mu\right)^{2/2^*}$$
$$\geq \frac{\beta}{d} \inf_{G \in \mathscr{M}} \left\{\|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|F - G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right\}$$

with $\mathrm{A} = \frac{1}{4}\,d\,(d-2)$ and a manifold $\mathscr{M}$ of optimal functions made of

$$G(\omega) = c\,(a + b\cdot\omega)^{-\frac{d-2}{2}} \quad \omega \in \mathbb{S}^d \quad (a, b, c) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}$$

- make the reduction of a ***far away problem*** to a local problem *constructive*... on $\mathbb{R}^d$
- make the analysis of the ***local problem*** *explicit*... on $\mathbb{S}^d$

Competing symmetries

- ***Rotations on the sphere*** combined with stereographic and inverse stereographic projections. Let $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$

$$(Uf)(x) := \left(\frac{2}{|x - e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x - e_d|^2}, \dots, \frac{x_{d-1}}{|x - e_d|^2}, \frac{|x|^2 - 1}{|x - e_d|^2}\right)$$

$$\mathcal{E}(Uf) = \mathcal{E}(f)$$

- ***Symmetric decreasing rearrangement*** $\mathcal{R}f = f^*$

$$f \text{ and } f^* \text{ are equimeasurable}$$

$$\|\nabla f^*\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

The method of ***competing symmetries***

Theorem (Carlen, Loss, 1990)

Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$. The sequence $f_n = (\mathcal{R}U)^n f$ is such that $\lim_{n\to+\infty} \|f_n - g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 0$. If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $\left(\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)}\right)_{n\in\mathbb{N}}$ is a non-increasing sequence

Useful preliminary results

- $\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$ where $h_f = \|f\|_{2^*}\, g_*/\|g_*\|_{2^*} \in \mathcal{M}$
- $(\|\nabla f_n\|_2^2)_{n\in\mathbb{N}}$ is a nonincreasing sequence

Lemma

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - \mathsf{S}_d \sup_{g\in\mathcal{M},\, \|g\|_{2^*}=1} \left(f, g^{2^*-1}\right)^2$$

Corollary

$\left(\mathrm{d}(f_n,\mathcal{M})\right)_{n\in\mathbb{N}}$ is strictly decreasing, $n\mapsto \sup_{g\in\mathcal{M}_1}\left(f_n, g^{2^*-1}\right)$ is strictly increasing, and

$$\lim_{n\to\infty} \mathrm{d}(f_n,\mathcal{M})^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - \mathsf{S}_d \|h_f\|_{2^*}^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - \mathsf{S}_d \|f\|_{2^*}^2$$

but no monotonicity for $n\mapsto \mathcal{E}(f_n) = \dfrac{\|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \|f_n\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathrm{d}(f_n,\mathcal{M})^2}$

Part 1: Global to local reduction

The ***local problem***

$$\mathscr{I}(\delta) := \inf\left\{\mathcal{E}(f) \,:\, f \geq 0\,,\ \mathsf{d}(f,\mathcal{M})^2 \leq \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\right\}$$

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the ***far away regime***

$$\mathsf{d}(f,\mathcal{M})^2 = \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

for some $\delta \in (0,1)$

Let $f_n = (\mathcal{R}U)^n f$. There are two cases:

- (Case 1) $\mathsf{d}(f_n,\mathcal{M})^2 \geq \delta\,\|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$
- (Case 2) for some $n \in \mathbb{N}$, $\mathsf{d}(f_n,\mathcal{M})^2 < \delta\,\|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Global to local reduction – Case 1

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the far away regime

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

Lemma

Let $f_n = (\mathcal{R}U)^n f$ and $\delta \in (0,1)$. If $\mathrm{d}(f_n, \mathcal{M})^2 \geq \delta \, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$, then

$$\mathcal{E}(f) \geq \delta$$

$$\lim_{n\to+\infty} \|\nabla f_n\|_2^2 \leq \frac{1}{\delta} \lim_{n\to+\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta} \left(\lim_{n\to+\infty} \|\nabla f_n\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2 \right)$$

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \geq \frac{\|\nabla f\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \geq \frac{\|\nabla f_n\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \underset{n\to+\infty}{\geq} \delta$$

Global to local reduction – Case 2

$$\mathscr{I}(\delta) := \inf \left\{ \mathcal{E}(f) \,:\, f \geq 0\,,\ \mathsf{d}(f,\mathcal{M})^2 \leq \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right\}$$

Lemma

$$\mathcal{E}(f) \geq \delta\, \mathscr{I}(\delta)$$

if $\inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\, \|\nabla f_{n_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ *and* $\inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} < \delta\, \|\nabla f_{n_0+1}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Adapt a strategy due to Christ: build a (semi-) *continuous* rearrangement *flow* $(\mathsf{f}_\tau)_{n_0\leq\tau<n_0+1}$ with $\mathsf{f}_{n_0} = U f_n$ such that $\|f_\tau\|_{2^*} = \|f\|_2$, $\tau \mapsto \|\nabla f_\tau\|_2$ is nonincreasing, and $\lim_{\tau\to n_0+1} \mathsf{f}_\tau = f_{n_0+1}$

$$\mathcal{E}(f) \geq 1 - \mathsf{S}_d \frac{\|f\|^2_{2^*}}{\|\nabla f\|^2_2} \geq 1 - \mathsf{S}_d \frac{\|\mathsf{f}_{\tau_0}\|^2_{2^*}}{\|\nabla \mathsf{f}_{\tau_0}\|^2_2} = \delta\, \mathcal{E}(f_{\tau_0}) \geq \delta\, \mathscr{I}(\delta)$$

Altogether: if $\mathsf{d}(f,\mathcal{M})^2 > \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, then $\mathcal{E}(f) \geq \min\left\{\delta, \delta\, \mathscr{I}(\delta)\right\}$

Part 2: The (simple) Taylor expansion

Proposition

Let $(X, d\mu)$ *be a measure space and* u, $r \in \mathrm{L}^q(X, d\mu)$ *for some* $q \ge 2$ *with* $u \ge 0$, $u + r \ge 0$ *and* $\int_X u^{q-1}\, r\, d\mu = 0$

▷ *If* $q = 6$, *then*

$$\|u+r\|_q^2 \le \|u\|_q^2 + \|u\|_q^{2-q} \Big(5\int_X u^{q-2}\, r^2\, d\mu + \tfrac{20}{3}\int_X u^{q-3}\, r^3\, d\mu + 5\int_X u^{q-4}\, r^4\, d\mu + 2\int_X u^{q-5}\, r^5\, d\mu + \tfrac{1}{3}\int_X r^6\, d\mu\Big)$$

▷ *If* $3 \le q \le 4$, *then*

$$\|u+r\|_q^2 - \|u\|_q^2 \le \|u\|_q^{2-q} \left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{(q-1)\,(q-2)}{3}\int_X u^{q-3}\, r^3\, d\mu + \frac{2}{q}\int_X |r|^q\, d\mu\right)$$

▷ *If* $2 \le q \le 3$, *then*

$$\|u+r\|_q^2 \le \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{2}{q}\int_X r_+^q\, d\mu\right)$$

Corollary

For all $\nu>0$ *and for all* $r\in\mathrm{H}^1(\mathbb{S}^d)$ *satisfying* $r\ge-1$,

$$\left(\int_{\mathbb{S}^d}|r|^q\,d\mu\right)^{2/q}\le\nu^2\quad\text{and}\quad\int_{\mathbb{S}^d}r\,d\mu=0=\int_{\mathbb{S}^d}\omega_j\,r\,d\mu\quad\forall\,j=1,\ldots d+1$$

if $d\mu$ *is the uniform probability measure on* $\mathbb{S}^d$, *then*

$$\int_{\mathbb{S}^d}\left(|\nabla r|^2+\mathrm{A}\,(1+r)^2\right)d\mu-\mathrm{A}\left(\int_{\mathbb{S}^d}(1+r)^q\,d\mu\right)^{2/q}\ge\mathrm{m}(\nu)\int_{\mathbb{S}^d}\left(|\nabla r|^2+\mathrm{A}\,r^2\right)d\mu$$

$$\begin{aligned}
&\mathrm{m}(\nu):=\tfrac{4}{d+4}-\tfrac{2}{q}\,\nu^{q-2} && \text{if}\quad d\ge6\\
&\mathrm{m}(\nu):=\tfrac{4}{d+4}-\tfrac{1}{3}\,(q-1)\,(q-2)\,\nu-\tfrac{2}{q}\,\nu^{q-2} && \text{if}\quad d=4,\,5\\
&\mathrm{m}(\nu):=\tfrac{4}{7}-\tfrac{20}{3}\,\nu-5\,\nu^2-2\,\nu^3-\tfrac{1}{3}\,\nu^4 && \text{if}\quad d=3
\end{aligned}$$

An explicit expression of $\mathscr{I}(\delta)$ if $\nu>0$ is small enough so that $\mathrm{m}(\nu)>0$

Part 3: Removing the positivity assumption

Take $f = f_+ - f_-$ with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 1$ and define $m := \|f_-\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and $1 - m = \|f_+\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} > 1/2$. The positive concave function

$$h_d(m) := m^{\frac{d-2}{d}} + (1-m)^{\frac{d-2}{d}} - 1$$

satisfies

$$2\,h_d(1/2)\,m \le h_d(m)\,, \quad h_d(1/2) = 2^{2/d} - 1$$

With $\delta(f) = \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, one finds $g_+ \in \mathcal{M}$ such that

$$\delta(f) \ge C^{d,\mathrm{pos}}_{\mathrm{BE}}\,\|\nabla f_+ - \nabla g_+\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{2\,h_d(1/2)}{h_d(1/2)+1}\,\|\nabla f_-\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

and therefore

$$C^d_{\mathrm{BE}} \ge \tfrac{1}{2}\,\min\left\{\max_{0<\delta<1/2}\,\delta\,\mathscr{I}(\delta),\,\frac{2\,h_d(1/2)}{h_d(1/2)+1}\right\}$$

Part 2, refined: The (complicated) Taylor expansion

To get a dimensionally sharp estimate, we expand $(1+r)^{2^*}-1-2^*r$ with an accurate remainder term for all $r\ge-1$

$$r_1:=\min\{r,\gamma\}\,,\quad r_2:=\min\left\{(r-\gamma)_+,M-\gamma\right\}\quad\text{and}\quad r_3:=(r-M)_+$$

with $0<\gamma<M$. Let $\theta=4/(d-2)$

Lemma

Given $d\ge6$, $r\in[-1,\infty)$, and $\overline{M}\in[\sqrt{e},+\infty)$, we have

$$\begin{aligned}(1+r)^{2^*}&-1-2^*r\\ &\le\tfrac12\,2^*\,(2^*-1)\,(r_1+r_2)^2+2\,(r_1+r_2)\,r_3+\left(1+C_M\,\theta\,\overline{M}^{-1}\ln\overline{M}\right)r_3^{2^*}\\ &\qquad+\left(\tfrac32\,\gamma\,\theta\,r_1^2+C_{M,\overline{M}}\,\theta\,r_2^2\right)\mathbb{1}_{\{r\le M\}}+C_{M,\overline{M}}\,\theta\,M^2\,\mathbb{1}_{\{r>M\}}\end{aligned}$$

where all the constants in the above inequality are explicit

There are constants ϵ_1, ϵ_2, k_0, and $\epsilon_0 \in (0, 1/\theta)$, such that

$$\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathrm{A}\,\|1+r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \ge \frac{4\,\epsilon_0}{d-2}\left(\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) + \sum_{k=1}^{3} I_k$$

$$I_1 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_1|^2 + \mathrm{A}\,r_1^2\right)d\mu - \mathrm{A}\,(2^*-1+\epsilon_1\,\theta)\int_{\mathbb{S}^d} r_1^2\,d\mu + \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d}(r_2^2 \ldots$$

$$I_2 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_2|^2 + \mathrm{A}\,r_2^2\right)d\mu - \mathrm{A}\left(2^*-1+(k_0+C_{\epsilon_1,\epsilon_2})\,\theta\right)\int_{\mathbb{S}^d} r_2^2\,d\mu$$

$$I_3 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_3|^2 + \mathrm{A}\,r_3^2\right)d\mu - \tfrac{2}{2^*}\,\mathrm{A}\,(1+\epsilon_2\,\theta)\int_{\mathbb{S}^d} r_3^{2^*}\,d\mu - \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d} r_3^2\,d\mu$$

- spectral gap estimates : $I_1 \ge 0$
- Sobolev inequality : $I_3 \ge 0$
- improved spectral gap inequality using that $\mu\big(\{r_2 > 0\}\big)$ is small: $I_2 \ge 0$

More explicit stability results for the logarithmic Sobolev and Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$

Joint work with G. Brigati and N. Simonov

Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results

Annales IHP, Analyse non linéaire, 362, 2023

Subcritical interpolation inequalities on the sphere

- ***Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

[Bakry, Emery, 1984], [Bidaut-Véron, Véron, 1991], [Beckner, 1993]

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem

Let $d \ge 1$ and $p \in (1,2) \cup (2,2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F] \ge \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

▷ The same result holds true for the logarithmic Sobolev inequality, again with an explicit constant $\mathscr{S}_{d,2}$, for any finite dimension d

From $\mathbb{S}^d$ to $\mathbb{R}^d$: the large dimensional limit

Joint work with G. Brigati and N. Simonov
On Gaussian interpolation inequalities
C. R. Math. Acad. Sci. Paris 41, 2024

Carré du champ – admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

$$m_\pm(d,p) := \tfrac{1}{(d+2)\,p} \left(d\,p + 2 \pm \sqrt{d\,(p-1)\,(2\,d-(d-2)\,p)} \right)$$

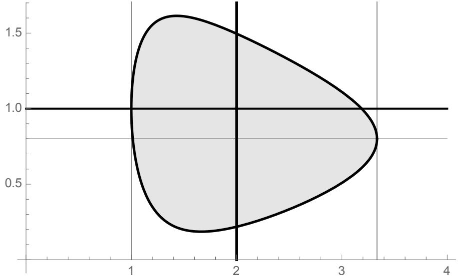


Figure: Case $d=5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)} \left(\mathcal{L} v + (m\,p-1)\,\frac{|\nabla v|^2}{v} \right) \quad \text{on} \quad \mathbb{R}^n$$

[JD, Brigati, Simonov] *Ornstein-Uhlenbeck operator:* $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_\pm(p) := \lim_{d \to +\infty} m_\pm(d,p) = 1 \pm \frac{1}{p} \sqrt{(p-1)\,(2-p)}$$

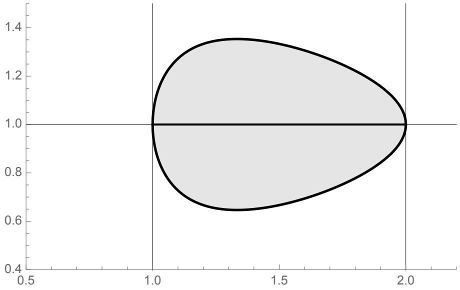


Figure: The admissible parameters $1 \le p \le 2$ and m are independent of n

Large dimensional limit

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem

Let $v \in \mathrm{H}^1(\mathbb{R}^n,dx)$ with compact support, $d \ge n$ and

$$u_d(\omega) = v\Big(\omega_1/r_d, \omega_2/r_d, \ldots, \omega_n/r_d\Big)\,, \quad r_d = \sqrt{\frac{d}{2\,\pi}}$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\,\pi)^{-n/2}\,e^{-\frac{1}{2}\,|y|^2}\,dy$,

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

L^2 stability of LSI: comments

[JD, Esteban, Figalli, Frank, Loss]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$
$$\geq \frac{\beta\,\pi}{2} \inf_{a\in\mathbb{R}^d,\, c\in\mathbb{R}} \int_{\mathbb{R}^n} |u - c\, e^{a\cdot x}|^2\, d\gamma$$

- One dimension is lost (for the manifold of invariant functions) in the limiting process
- Euclidean forms of the stability
- The $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ does not appear, it gets lost in the limit $d \to +\infty$
- $\int_{\mathbb{R}^n} |\nabla(u - c\, e^{a\cdot x})|^2\, d\gamma$? False, but makes sense under additional assumptions. Some results based on the Ornstein-Uhlenbeck flow and entropy methods: [Fathi, Indrei, Ledoux, 2016], [JD, Brigati, Simonov, 2023-24]
- Taking the limit is difficult because of the lack of compactness. Two proofs

Stability in the large dimensional limit

Let $u = f/g_\star$, $g_\star := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \big(\frac{2}{1+|x|^2}\big)^{(d-2)/2}$

$$\int_{\mathbb{R}^d} |\nabla u|^2\, g_\star^2\, dx + d\,(d-2) \int_{\mathbb{R}^d} |u|^2\, g_\star^{2^*}\, dx \\ - d\,(d-2)\, \|g_\star\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{2^*-2} \left(\int_{\mathbb{R}^d} |u|^{2^*}\, g_\star^{2^*}\, dx\right)^{2/2^*} \\ \ge \frac{\beta}{d} \left(\int_{\mathbb{R}^d} |\nabla u|^2\, g_\star^2\, dx + d\,(d-2) \int_{\mathbb{R}^d} |u - g_d/g_\star|^2\, g_\star^{2^*}\, dx\right),$$

where $g_d(x) = c_d\,(a_d + |x-b_d|^2)^{1-d/2}$ realizes the distance to $\mathcal{M}$

$$u(x) = v\,(r_d\,x)\,, \quad w_v^d\,(r_d\,x) = \frac{g_d(x)}{g_\star(x)}\,, \quad r_d = \sqrt{\frac{d}{2\,\pi}}$$

We consider a function $v(x)$ depending only on $y \in \mathbb{R}^N$ with $x = (y,z) \in \mathbb{R}^N \times \mathbb{R}^{d-N} \approx \mathbb{R}^d$, for some fixed integer N

$$\lim_{d\to+\infty} \left(1 + \frac{1}{r_d^2}\,|y|^2\right)^{-\frac{N+d}{2}} = e^{-\pi\,|y|^2}, \quad \lim_{d\to+\infty} \int_{\mathbb{R}^d} |v(y)|^2\, d\mu = \int_{\mathbb{R}^N} |v|^2\, d\gamma$$

$$\lim_{d\to+\infty} \int_{\mathbb{R}^d} |\nabla v|^2 \left(1 + \frac{1}{r_d^2}\,|x|^2\right)^2 d\mu = 4 \int_{\mathbb{R}^N} |\nabla v|^2\, d\gamma$$

Estimates on the parameters a_d, b_d and c_d !

More results on logarithmic Sobolev inequalities

Joint work with G. Brigati and N. Simonov
Stability for the logarithmic Sobolev inequality
Journal of Functional Analysis, 287, oct. 2024

▷ *Entropy methods, with constraints*

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} \mathrm{d}(u_\varepsilon, w)^\alpha \le \frac{1}{2}\,\varepsilon^4 + O\big(\varepsilon^6\big)$$

$\mathscr{M} := \big\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \big\}$ where $w_{a,c}(x) = c\, e^{-a\cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ and $\|x\, u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \frac{1}{2\,d} \left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \right)^2$$

and, with $\psi(s) := s - \frac{d}{4} \log\left(1 + \frac{4}{d}\, s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \psi\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \right)$$

Stability under log-concavity

Theorem

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ *such that* $u^2\,\gamma$ *is log-concave and such that*

$$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\gamma = (1,0) \quad \textit{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \le \mathsf{K}$$

we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,d\gamma \ge 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{432\,\mathsf{K}} \approx 1 + \frac{0.00231481}{\mathsf{K}}$$

Self-improving Poincaré inequality and stability for LSI:
[Fathi, Indrei, Ledoux, '16]

Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x)\, |u|^2\, d\gamma = (1, 0)\,, \quad \int_{\mathbb{R}^d} |x|^2\, |u|^2\, d\gamma \leq d\,, \quad \int_{\mathbb{R}^d} |u|^2\, e^{\,\varepsilon\,|x|^2}\, d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \geq \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2\, \log |u|^2\, d\gamma$$

with $\mathscr{C} = 1 + \frac{\mathscr{C}_\star(\mathsf{K}_\star) - 1}{1 + R^2\, \mathscr{C}_\star(\mathsf{K}_\star)}$, $\mathsf{K}_\star := \max\left(d, \frac{(d+1)\, R^2}{1+R^2}\right)$ *if* $\mathrm{supp}(u) \subset B(0, R)$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

More related papers can be found at

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For final versions, use `Dolbeault` as login and `Jean` as password

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Thank you for your attention !