Optimality in critical and subcritical inequalities: flows, linearization and entropy methods

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, Université Paris-Dauphine

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Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Outline

▷ **Symmetry breaking and linearization**
- The critical Caffarelli-Kohn-Nirenberg inequality
- A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
- Linearization and spectrum

▷ **Without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows**
- Rényi entropy powers
- Self-similar variables and relative entropies
- The role of the spectral gap

▷ **With weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**
- Towards a parabolic proof
- Large time asymptotics and spectral gaps
- A discussion of optimality cases

Symmetry breaking and linearization
Entropy methods without weights
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Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and linearization for the evolution equations)

...and also

S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk ...

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Background references (partial)

- Rigidity methods, uniqueness in nonlinear elliptic PDE's:
  [B. Gidas, J. Spruck, 1981], [M.-F. Bidaut-Véron, L. Véron, 1991]
- Probabilistic methods (Markov processes), semi-group theory and
  carré du champ methods (Γ_2 theory): [D. Bakry, M. Emery,
  1984], [Bakry, Ledoux, 1996], [Demange, 2008], [JD, Esteban,
  Loss, 2014 & 2015] → *D. Bakry, I. Gentil, and M. Ledoux.
  Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs
  ▷ Entropy-entropy production inequalities: Arnold, Carrillo,
  Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani,
  Unterreiter, Villani..., [del Pino, JD, 2001], [Blanchet, Bonforte,
  JD, Grillo, Vázquez] → *A. Jüngel, Entropy Methods for Diffusive
  Partial Differential Equations (2016)*
  ▷ Mass transportation: [Otto] → *C. Villani, Optimal transport.
  Old and new (2009)*
  ▷ Rényi entropy powers (information theory) [Savaré, Toscani,
  2014], [Dolbeault, Toscani]

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Symmetry and symmetry breaking results

▷ The critical Caffarelli-Kohn-Nirenberg inequality

▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities

▷ Linearization and spectrum

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p \left(\mathbb{R}^d, |x|^{-b}\, dx \right) \, : \, |x|^{-a}\, |\nabla v| \in \mathrm{L}^2 \left(\mathbb{R}^d, dx \right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\, dx \right)^{2/p} \leq \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\, dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d - 2 + 2\,(b - a)} \qquad \text{(critical case)}$$

$\triangleright$ *An optimal function among radial functions:*

$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)} \right)^{-\frac{2}{p-2}} \quad and \quad \mathsf{C}_{a,b}^\star = \frac{\| \, |x|^{-b}\, v_\star \|_p^2}{\| \, |x|^{-a}\, \nabla v_\star \|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}_{a,b}^\star$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}_{a,b}^\star$ *(symmetry breaking) ?*

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx$$
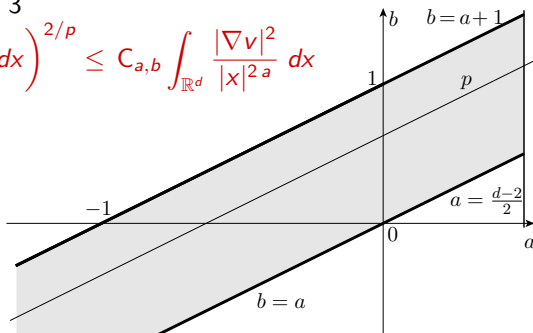
$a \leq b \leq a+1$ if $d \geq 3$

$a < b \leq a+1$ if $d = 2$, $a+1/2 < b \leq a+1$ if $d = 1$

and $a < a_c := (d-2)/2$

$$p = \frac{2d}{d-2+2(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]
[Caffarelli, Kohn, Nirenberg (1984)]
[F. Catrina, Z.-Q. Wang (2001)]

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$
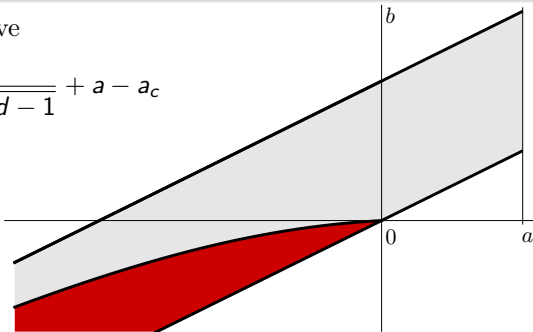
[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]
The functional

$$\mathsf{C}^{\star}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\ dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\ dx \right)^{2/p}$$

is linearly instable at $v = v_{\star}$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (Inventiones 2016)]
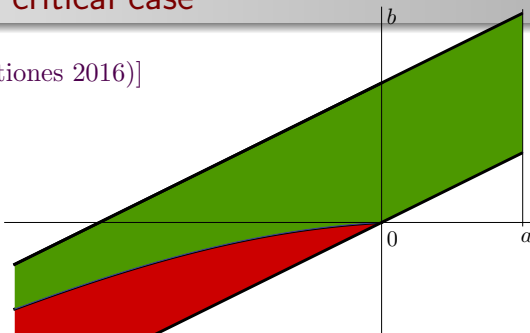
Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r, \omega) = r^{a-a_c} \, \varphi(s, \omega) \quad \text{with} \quad r = |x|, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s \varphi\|^2_{L^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{L^2(\mathcal{C})} + \Lambda \, \|\varphi\|^2_{L^2(\mathcal{C})} \geq \mu(\Lambda) \, \|\varphi\|^2_{L^p(\mathcal{C})} \quad \forall \, \varphi \in H^1(\mathcal{C})$$

where $\Lambda := (a_c - a)^2$, $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda) = \frac{1}{C_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s, \omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$H^1(\mathcal{C}) \ni \varphi \mapsto \|\partial_s \varphi\|_{L^2(\mathcal{C})}^2 + \|\nabla_\omega \varphi\|_{L^2(\mathcal{C})}^2 + \Lambda \|\varphi\|_{L^2(\mathcal{C})}^2$$

under a constraint on $\|\varphi\|_{L^p(\mathcal{C})}^2$

$\varphi_\star$ *is not* optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_\star^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*

Symmetry breaking and linearization   Critical Caffarelli-Kohn-Nirenberg inequality
Entropy methods without weights   **Subcritical Caffarelli-Kohn-Nirenberg inequalities**
Weighted nonlinear flows and CKN inequalities   Linearization and spectrum

Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: $\|w\|_{L^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$, $\|w\|_{L^q(\mathbb{R}^d)} := \|w\|_{L^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{L^{2p,\gamma}(\mathbb{R}^d)} \leq \mathsf{C}_{\beta,\gamma,p} \, \|\nabla w\|_{L^{2,\beta}(\mathbb{R}^d)}^{\vartheta} \, \|w\|_{L^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \geq 2\,, \quad \gamma - 2 < \beta < \tfrac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with } p_\star := \tfrac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma - p\,(d-\beta-2)\big)}$$
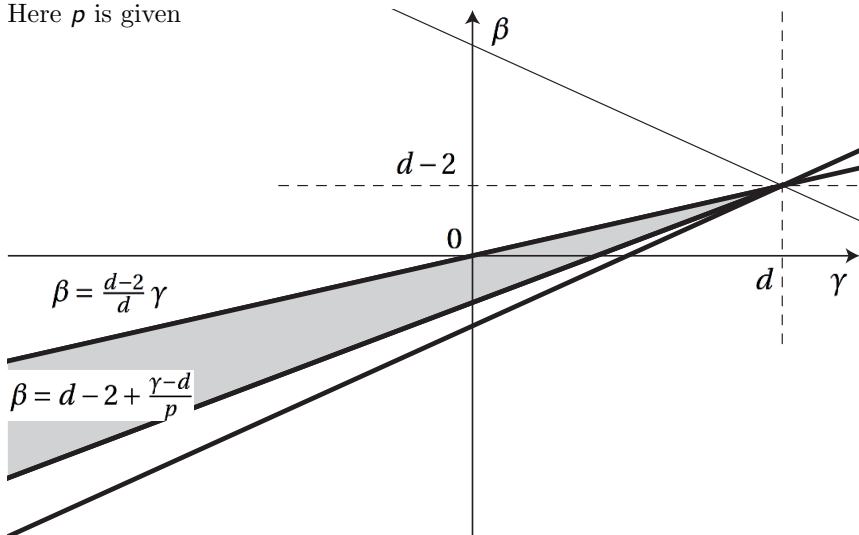
🟢 Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \big(1 + |x|^{2+\beta-\gamma}\big)^{-1/(p-1)} \qquad \forall\, x \in \mathbb{R}^d \quad ?$$

🟢 Do we know (*symmetry*) that the equality case is achieved among radial functions?

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Range of the parameters

Here p is given

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

○ Symmetry and symmetry breaking

[JD, Esteban, Loss, Muratori, 2016]

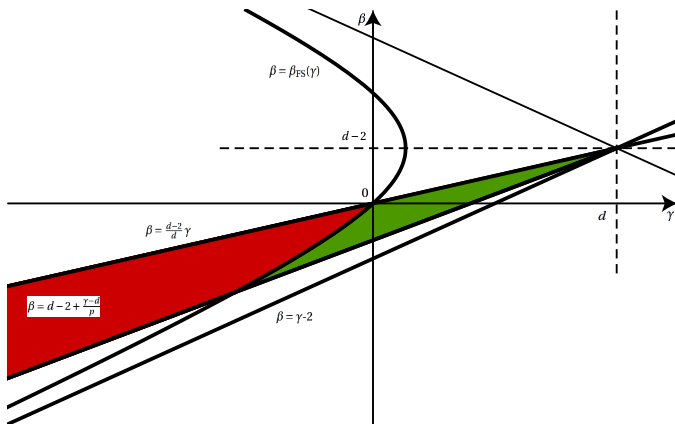Let us define $\beta_{\mathrm{FS}}(\gamma) := d - 2 - \sqrt{(d - \gamma)^2 - 4\,(d-1)}$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad and \quad \beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$$

is not optimal

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

The green area is the region of symmetry, while the red area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d - \gamma)^2 - (\beta - d + 2)^2 - 4\,(d - 1) = 0$$

Symmetry breaking and linearization   Critical Caffarelli-Kohn-Nirenberg inequality
Entropy methods without weights   Subcritical Caffarelli-Kohn-Nirenberg inequalities
Weighted nonlinear flows and CKN inequalities   **Linearization and spectrum**

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d - \gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha - 1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \leq \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \geq 2, \quad \alpha > 0, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n - 2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha < \alpha_{\mathrm{FS}} \quad \text{with} \quad \alpha_{\mathrm{FS}} := \sqrt{\frac{d - 1}{n - 1}}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

The second variation

$$\mathcal{J}[v] := \vartheta \log \left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)} \right) + (1 - \vartheta) \log \left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)} \right)$$
$$+ \log \mathsf{K}_{\alpha,n,p} - \log \left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\, dx$, where $\mu_\delta(x) := (1 + |x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^2 \, \mathcal{J}\left[v_\star + \varepsilon\, \mu_{\delta/2}\, f\right] = \tfrac{1}{2}\, \varepsilon^2\, \vartheta\, \mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\, |x|^{n-d}\, d\mu_\delta - \frac{4\, p\, \alpha^2}{p-1} \int_{\mathbb{R}^d} |f|^2\, |x|^{n-d}\, d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\, |x|^{n-d}\, d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Critical Caffarelli-Kohn-Nirenberg inequality
Subcritical Caffarelli-Kohn-Nirenberg inequalities
Linearization and spectrum

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \geq 2$, $\alpha \in (0, +\infty)$, $n > d$ and $\delta \geq n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \dfrac{(d-1)\,\delta^2}{n\,(2\,\delta - n)\,(\delta - 1)}$

$\mathcal{Q} \geq 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \leq \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \quad \Longleftrightarrow \quad \eta < 1$$

$$\Longleftrightarrow \quad \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \quad \Longleftrightarrow \quad \alpha > \alpha_{\mathrm{FS}}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Inequalities without weights and fast diffusion equations

▷ Rényi entropy powers

▷ Self-similar variables and relative entropies

▷ The role of the spectral gap

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Rényi entropy powers and fast diffusion

▷ Rényi entropy powers, the entropy approach without rescaling: [Savaré, Toscani]: scalings, nonlinearity and a concavity property inspired by information theory

▷ Faster rates of convergence: [Carrillo, Toscani], [JD, Toscani]

Symmetry breaking and linearization
Entropy methods without weights
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Rényi entropy powers
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The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t, x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m - 1), \quad \kappa := \left|\frac{2\,\mu\,m}{m - 1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m)\,\mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1)\,\mathsf{p}\,\Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \dfrac{\mu}{d\,(1-m)} = 1 + \dfrac{2}{1-m}\left(\dfrac{1}{d} + m - 1\right) = \dfrac{2}{d}\,\dfrac{1}{1-m} - 1$

has a linear growth asymptotically as $t \to +\infty$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

The concavity property

> ## Theorem
>
> [Toscani-Savaré] *Assume that* $m \geq 1 - \frac{1}{d}$ *if* $d > 1$ *and* $m > 0$ *if* $d = 1$.
> *Then* $F(t)$ *is increasing,* $(1 - m) F''(t) \leq 0$ *and*
>
> $$\lim_{t \to +\infty} \frac{1}{t} F(t) = (1 - m) \sigma \lim_{t \to +\infty} E^{\sigma-1} I = (1 - m) \sigma E_\star^{\sigma-1} I_\star$$

[Dolbeault-Toscani] The inequality

$$E^{\sigma-1} I \geq E_\star^{\sigma-1} I_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{L^2(\mathbb{R}^d)}^\theta \|w\|_{L^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq C_{GN} \|w\|_{L^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$. Hint: $v^{m-1/2} = \frac{w}{\|w\|_{L^{2q}(\mathbb{R}^d)}}$, $q = \frac{1}{2m-1}$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

The proof

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $\frac{1}{d} \leq m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx = -2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1) \, (\Delta \mathsf{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathsf{p}\|^2 - \frac{1}{d} \, (\Delta \mathsf{p})^2 = \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d} \, \Delta \mathsf{p} \, \mathrm{Id} \right\|^2$$

There are no boundary terms in the integrations by parts

Symmetry breaking and linearization
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Rényi entropy powers
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Remainder terms

$\mathsf{F}'' = -\sigma (1-m) \mathsf{R}[v]$. The *pressure variable* is $\mathsf{P} = \frac{m}{1-m} v^{m-1}$

$$\mathsf{R}[v] := (\sigma - 1)(1-m) \mathsf{E}^{\sigma-1} \int_{\mathbb{R}^d} v^m \left| \Delta\mathsf{P} - \frac{\int_{\mathbb{R}^d} v \, |\nabla\mathsf{P}|^2 \, dx}{\int_{\mathbb{R}^d} v^m \, dx} \right|^2 dx$$

$$+ \, 2\,\mathsf{E}^{\sigma-1} \int_{\mathbb{R}^d} v^m \left\| \mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id} \right\|^2 dx$$

Let

$$\mathsf{G}[v] := \frac{\mathsf{F}[v]}{\sigma(1-m)} = \left(\int_{\mathbb{R}^d} v^m \, dx \right)^{\sigma-1} \int_{\mathbb{R}^d} v \, |\nabla\mathsf{P}|^2 \, dx$$

The Gagliardo-Nirenberg inequality is equivalent to $\mathsf{G}[v_0] \geq \mathsf{G}[v_\star]$

> ### Proposition
>
> $$\mathsf{G}[v_0] \geq \mathsf{G}[v_\star] + \int_0^\infty \mathsf{R}[v(t,\cdot)] \, dt$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type *Barenblatt solutions*

$$v_\star(t,x) := \frac{1}{\kappa^d (\mu \, t)^{d/\mu}} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, (\mu \, t)^{1/\mu}}\right) \quad \text{where} \quad \mu := 2 + d \, (m-1)$$

where $\mathcal{B}_\star$ is the Barenblatt profile (with appropriate mass)

$$\mathcal{B}_\star(x) := \left(1 + |x|^2\right)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t,x) = \frac{1}{\kappa^d \, R^d} \, u\left(\tau, \frac{x}{\kappa \, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \tfrac{1}{2} \, \log\left(\frac{R(t)}{R_0}\right)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[u \left(\nabla u^{m-1} - 2 \, x\right)\right] = 0$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Free energy and Fisher information

● The function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[u \left(\nabla u^{m-1} - 2 \, x \right) \right] = 0$$

● [Ralston, Newman, 1984] Lyapunov functional:
Generalized entropy or *Free energy*

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left(-\frac{u^m}{m} + |x|^2 u \right) \, dx - \mathcal{E}_0$$

● Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt} \mathcal{E}[u] = -\mathcal{I}[u] \, , \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2 \, x \right|^2 \, dx$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|u_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$u_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{E}_0$ so that $\mathcal{E}[u_\infty] = 0$

- *Entropy – entropy production inequality* [del Pino, J.D.]

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[u] \geq 4\,\mathcal{E}[u]$$

Corollary

[del Pino, J.D.] *A solution u with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\, u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\mathcal{E}[u(t, \cdot)] \leq \mathcal{E}[u_0]\, e^{-4\,t}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[u \left(\nabla u^{m-1} - 2x \right) \right] = 0 \quad \tau > 0, \quad x \in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R > 0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1} - 2x \right) \cdot \frac{x}{|x|} = 0 \quad \tau > 0, \quad x \in \partial B_R.$$

With $z(\tau, x) := \nabla Q(\tau, x) := \nabla u^{m-1} - 2x$, the *relative Fisher information* is such that

$$\frac{d}{d\tau} \int_{B_R} u \, |z|^2 \, dx + 4 \int_{B_R} u \, |z|^2 \, dx$$

$$+ 2 \, \frac{1-m}{m} \int_{B_R} u^m \left(\left\| D^2 Q \right\|^2 - (1-m) \, (\Delta Q)^2 \right) dx$$

$$= \int_{\partial B_R} u^m \left(\omega \cdot \nabla |z|^2 \right) d\sigma \leq 0 \text{ (by Grisvard's lemma)}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Another improvement of the GN inequalities

Let us define the *relative entropy*

$$\mathcal{E}[u] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(u^m - \mathcal{B}_\star^m - m\,\mathcal{B}_\star^{m-1}\,(u - \mathcal{B}_\star) \right)\, dx$$

the *relative Fisher information*

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|z|^2\, dx = \int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - 2\,x\right|^2\, dx$$

and $\quad \mathcal{R}[u] := 2\,\frac{1-m}{m} \int_{\mathbb{R}^d} u^m \left(\left\|\mathrm{D}^2\mathrm{Q}\right\|^2 - (1-m)\,(\Delta\mathrm{Q})^2 \right)\, dx$

Proposition

If $1 - 1/d \le m < 1$ and $d \ge 2$, then

$$\mathcal{I}[u_0] - 4\,\mathcal{E}[u_0] \ge \int_0^\infty \mathcal{R}[u(\tau, \cdot)]\, d\tau$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Entropy – entropy production, Gagliardo-Nirenberg ineq.

$$4\,\mathcal{E}[u] \leq \mathcal{I}[u]$$

Rewrite it with $p = \frac{1}{2m-1}$, $u = w^{2p}$, $u^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \geq 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} u\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx\right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With $1 < p \leq \frac{d}{d-2}$ (fast diffusion case) and $d \geq 3$*

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq \mathcal{C}_{p,d}^{\mathrm{GN}} \|\nabla w\|_{L^2(\mathbb{R}^d)}^\theta \|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

$$\mathcal{C}_{p,d}^{\mathrm{GN}} = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}, \ \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \ y = \frac{p+1}{p-1}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities
Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \leq v_0 \leq V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \geq 3$ *and* $m \leq m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_* := \frac{d-4}{d-2}$, *the entropy decays according to*
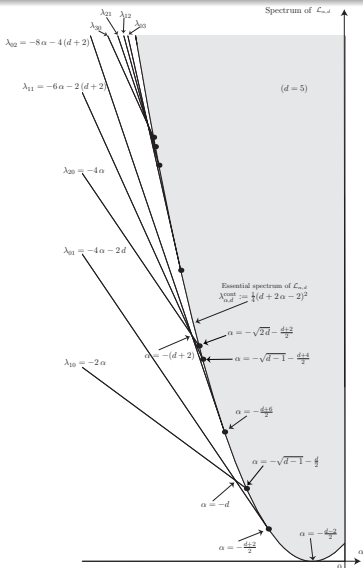
$$\mathcal{E}[v(t, \cdot)] \leq C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\, t} \quad \forall\, t \geq 0$$

where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$
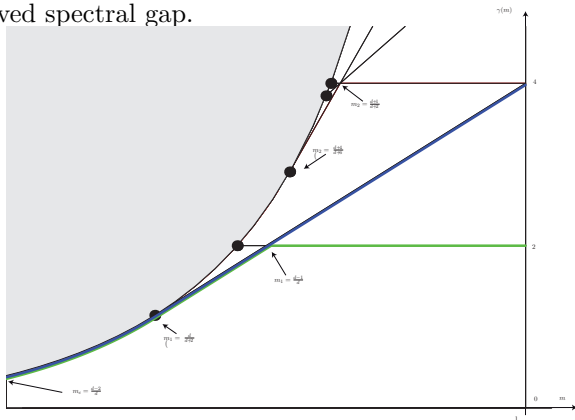
with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Plots ($d = 5$)

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
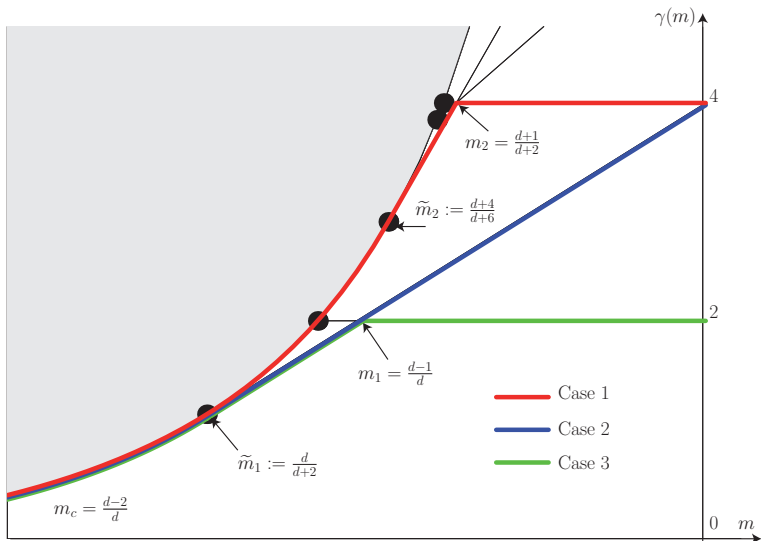**The role of the spectral gap**

Improved asymptotic rates

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$.
Under Assumption (H1), if v is a solution of the fast diffusion
equation with initial datum v_0 such that $\int_{\mathbb{R}^d} x \, v_0 \, dx = 0$, then the
asymptotic convergence holds with an improved rate corresponding to
the improved spectral gap.

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Spectral gaps and best constants

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Rényi entropy powers
Self-similar variables and relative entropies
The role of the spectral gap

Comments

- The spectral gap corresponding to the red curves relies on a refined notion of relative entropy with respect to *best matching Barenblatt profiles* [J.D., Toscani]

- A result by [Denzler, Koch, McCann] *Higher order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach*

- The constant C in

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \geq 0$$

can be made explicit, under additional restrictions on the initial data [Bonforte, J.D., Grillo, Vázquez]

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

▷ Entropy and Caffarelli-Kohn-Nirenberg inequalities

▷ Large time asymptotics and spectral gaps

▷ Optimality cases

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m} \left(2 + \beta - \gamma\right)^2 \mathcal{E}[v] \leq \mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{E}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left(v^m - \mathfrak{B}_{\beta,\gamma}^m - m \, \mathfrak{B}_{\beta,\gamma}^{m-1} \left(v - \mathfrak{B}_{\beta,\gamma} \right) \right) \frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathfrak{B}_{\beta,\gamma}^{m-1} \right|^2 \frac{dx}{|x|^\beta}.$$

If v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \nabla \cdot \left[|x|^{-\beta} \, v \, \nabla \left(v^{m-1} - |x|^{2+\beta-\gamma} \right) \right] = 0 \qquad \text{(WFDE-FP)}$$

then

$$\frac{d}{dt} \mathcal{E}[v(t, \cdot)] = - \frac{m}{1-m} \mathcal{I}[v(t, \cdot)]$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Proposition

Let $m = \frac{p+1}{2p}$ and consider a solution to (WFDE-FP) with nonnegative initial datum $u_0 \in \mathrm{L}^{1,\gamma}(\mathbb{R}^d)$ such that $\|u_0^m\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d} u_0 \, |x|^{2+\beta-2\gamma} \, dx$ are finite. Then

$$\mathcal{E}[v(t,\cdot)] \leq \mathcal{E}[u_0] \, e^{-(2+\beta-\gamma)^2 t} \quad \forall \, t \geq 0$$

if one of the following two conditions is satisfied:
(i) either u_0 is a.e. radially symmetric
(ii) or symmetry holds in (CKN)

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

A useful change of variables (reminder)

With
$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\frac{d - \gamma}{\beta + 2 - \gamma},$$

(CKN) can be rewritten for a function $v(|x|^{\alpha - 1} x) = w(x)$ as

$$\|v\|_{L^{2p, d-n}(\mathbb{R}^d)} \leq K_{\alpha, n, p} \|\mathfrak{D}_\alpha v\|_{L^{2, d-n}(\mathbb{R}^d)}^\vartheta \|v\|_{L^{p+1, d-n}(\mathbb{R}^d)}^{1-\vartheta}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha \frac{\partial v}{\partial s}, \frac{1}{s}\nabla_\omega v\right)$. Parameters are in the range

$$d \geq 2, \quad \alpha > 0, \quad n > d \quad \text{and} \quad p \in (1, p_\star], \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha > \alpha_{\mathrm{FS}} \quad \text{with} \quad \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Towards a parabolic proof

For any $\alpha \geq 1$, let $\quad \mathsf{D}_\alpha W = \left(\alpha \, \partial_r W, \, r^{-1} \, \nabla_\omega W \right)$ so that

$$\mathsf{D}_\alpha = \nabla + (\alpha - 1) \frac{x}{|x|^2} \, (x \cdot \nabla) = \nabla + (\alpha - 1) \, \omega \, \partial_r$$

and define the diffusion operator L_α by

$$\mathsf{L}_\alpha = - \mathsf{D}_\alpha^* \, \mathsf{D}_\alpha = \alpha^2 \left(\partial_r^2 + \frac{n-1}{r} \, \partial_r \right) + \frac{\Delta_\omega}{r^2}$$

where Δ_ω denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$
$\frac{\partial g}{\partial t} = \mathsf{L}_\alpha g^m$ is changed into

$$\frac{\partial u}{\partial \tau} = \mathsf{D}_\alpha^* \left(u \, z \right), \quad z := \mathsf{D}_\alpha \mathsf{q}, \quad \mathsf{q} := u^{m-1} - \mathcal{B}_\alpha^{m-1}, \quad \mathcal{B}_\alpha(x) := \left(1 + \frac{|x|^2}{\alpha^2} \right)^{\frac{1}{m-1}}$$

by the change of variables

$$g(t, x) = \frac{1}{\kappa^n \, R^n} \, u \left(\tau, \frac{x}{\kappa \, R} \right) \quad \text{where} \quad \begin{cases} \frac{dR}{dt} = R^{1-\mu}, \quad R(0) = R_0 \\[2mm] \tau(t) = \frac{1}{2} \, \log \left(\frac{R(t)}{R_0} \right) \end{cases}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

If the weight does not introduce any singularity at $x = 0$...

$$\frac{m}{1-m} \frac{d}{d\tau} \int_{B_R} u \, |z|^2 \, d\mu_n$$

$$= \int_{\partial B_R} u^m \left(\omega \cdot D_\alpha |z|^2 \right) |x|^{n-d} \, d\sigma \quad (\leq 0 \text{ by Grisvard's lemma})$$

$$- 2 \frac{1-m}{m} \left(m - 1 + \frac{1}{n} \right) \int_{B_R} u^m \, |\mathsf{L}_\alpha \mathsf{q}|^2 \, d\mu_n$$

$$- \int_{B_R} u^m \left(\alpha^4 \, m_1 \left| \mathsf{q}'' - \frac{\mathsf{q}'}{r} - \frac{\Delta_\omega \mathsf{q}}{\alpha^2 \, (n-1) \, r^2} \right|^2 + \frac{2 \, \alpha^2}{r^2} \left| \nabla_\omega \mathsf{q}' - \frac{\nabla_\omega \mathsf{q}}{r} \right|^2 \right) d\mu_n$$

$$- (n-2) \left(\alpha_{\mathrm{FS}}^2 - \alpha^2 \right) \int_{B_R} \frac{|\nabla_\omega \mathsf{q}|^2}{r^4} \, d\mu_n$$

A formal computation that still needs to be justified
(singularity at $x = 0$?)
- Other potential application: the computation of Bakry, Gentil and Ledoux (chapter 6) for non-integer dimensions; weights on manifolds

[...]

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \nabla \cdot \left[|x|^{-\beta} v \nabla \left(v^{m-1} - |x|^{2+\beta-\gamma} \right) \right] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Relative uniform convergence

$\zeta := 1 - \left(1 - \frac{2-m}{(1-m)\,q}\right)\left(1 - \frac{2-m}{1-m}\,\theta\right)$

$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma}$ is in the range $0 < \theta < \frac{1-m}{2-m} < 1$
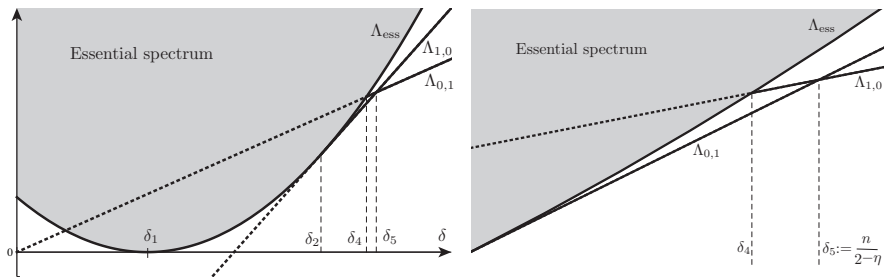
Theorem

For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \leq \mathcal{K}\,e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\,t \geq t_0$$

in the case $\gamma \in (0, d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \leq \mathcal{K}\,e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\,t \geq t_0$$

in the case $\gamma \leq 0$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\mathrm{ess}}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Main steps of the proof:

🌑 Existence of weak solutions, $L^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass

🌑 Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\, w_t = -\frac{1}{\mathfrak{B}}\, \nabla \cdot \left(|x|^{-\beta}\, \mathfrak{B}\, w\, \nabla \left((w^{m-1}-1)\, \mathfrak{B}^{m-1}\right)\right) & \text{in } \mathbb{R}^+ \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

🌑 *Regularity*: [Chiarenza, Serapioni], Harnack inequalities; relative uniform convergence (without rates) and asymptotic rates (linearization)

🌑 The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information

🌑 A Duhamel formula and a bootstrap

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Asymptotic rates of convergence

Corollary

Assume that $m \in (0, 1)$, with $m \neq m_* := \frac{n-4}{n-2}$. Under the relative mass condition, for any "good solution" v there exists a positive constant $\mathcal{C}$ such that

$$\mathcal{E}[v(t)] \leq \mathcal{C} \, e^{-2(1-m)\Lambda t} \quad \forall \, t \geq 0 \,.$$

🌑 With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $L^{1,\gamma}(\mathbb{R}^d)$

🌑 Improved estimates can be obtained using "best matching techniques"

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2 + \beta - \gamma)^2 \, \mathcal{E}[v] \leq \frac{m}{1-m} \, \mathcal{I}[v]$$

so that

$$\mathcal{E}[v(t)] \leq \mathcal{E}[v(0)] \, e^{-2(1-m)\Lambda_\star \, t} \quad \forall \, t \geq 0 \quad \text{with} \quad \Lambda_\star := \frac{(2 + \beta - \gamma)^2}{2(1-m)}$$

Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M) \, \mathcal{E}[v] \leq \mathcal{I}[v] \quad \forall \, v \in \mathrm{L}^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)} = M \, ,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M) := \frac{m}{2}(1-m)^{-2} \, \mathcal{K}(M)$

$$\mathcal{E}[v(t)] \leq \mathcal{E}[v(0)] \, e^{-2(1-m)\Lambda(M) \, t} \quad \forall \, t \geq 0$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- In the symmetry breaking range of (CKN), for any $M > 0$, we have
$$0 < \mathcal{K}(M) \leq \tfrac{2}{m}(1-m)^2 \Lambda_{0,1}$$

- If symmetry holds in (CKN) then
$$\mathcal{K}(M) \geq \tfrac{1-m}{m}(2+\beta-\gamma)^2$$

Corollary

Assume that $m \in [m_1, 1)$

(i) For any $M > 0$, if $\Lambda(M) = \Lambda_\star$ then $\beta = \beta_{\mathrm{FS}}(\gamma)$

(ii) If $\beta > \beta_{\mathrm{FS}}(\gamma)$ then $\Lambda_{0,1} < \Lambda_\star$ and $\Lambda(M) \in (0, \Lambda_{0,1}]$ for any $M > 0$

(iii) For any $M > 0$, if $\beta < \beta_{\mathrm{FS}}(\gamma)$ and if symmetry holds in (CKN), then
$$\Lambda(M) > \Lambda_\star$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon\, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathrm{D}_\alpha^*\left(|x|^{-\beta}\, \mathcal{B}_\star\, \mathrm{D}_\alpha f\right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathrm{D}_\alpha f_1 \cdot \mathrm{D}_\alpha f_2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

we compute

$$\frac{1}{2}\frac{d}{dt}\langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = \int_{\mathbb{R}^d} f\,(\mathcal{L}\, f)\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx = -\int_{\mathbb{R}^d} |\mathrm{D}_\alpha f|^2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx =$$

for any f smooth enough:

$$\frac{1}{2}\frac{d}{dt}\langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathrm{D}_\alpha f \cdot \mathrm{D}_\alpha(\mathcal{L}\, f)\, \mathcal{B}_\star\, |x|^{-\beta}\, dx = -\langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$- \mathcal{L} f_1 = \lambda_1 f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g, g \rangle\!\rangle = - \langle g, \mathcal{L} g \rangle \geq \lambda_1 \|g - \bar{g}\|^2, \quad \bar{g} := \langle g, 1 \rangle / \langle 1, 1 \rangle$$

$$- \langle\!\langle g, \mathcal{L} g \rangle\!\rangle \geq \lambda_1 \langle\!\langle g, g \rangle\!\rangle$$

Proof: expansion of the square :
$$- \langle\!\langle (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle\!\rangle = \langle \mathcal{L} (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle = \|\mathcal{L} (g - \bar{g})\|^2$$

- Key observation:

$$\lambda_1 \geq 4 \quad \Longleftrightarrow \quad \alpha \leq \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities
A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Symmetry breaking in CKN inequalities

- Symmetry holds in (CKN) if $\mathcal{J}[w] \geq \mathcal{J}[w_\star]$ with

$$\mathcal{J}[w] := \vartheta \, \log\left(\|D_\alpha \, w\|_{L^{2,\delta}(\mathbb{R}^d)}\right) + (1-\vartheta) \, \log\left(\|w\|_{L^{p+1,\delta}(\mathbb{R}^d)}\right) - \log\left(\|w\|_{L^{2p,\delta}(\mathbb{R}^d)}\right)$$

with $\delta := d - n$ and

$$\mathcal{J}[w_\star + \varepsilon \, g] = \varepsilon^2 \, \mathcal{Q}[g] + o(\varepsilon^2)$$

where

$$\frac{2}{\vartheta} \, \|D_\alpha \, w_\star\|^2_{L^{2,d-n}(\mathbb{R}^d)} \, \mathcal{Q}[g]$$

$$= \|D_\alpha \, g\|^2_{L^{2,d-n}(\mathbb{R}^d)} + \frac{p\,(2+\beta-\gamma)}{(p-1)^2} \left[d - \gamma - p\,(d-2-\beta)\right] \int_{\mathbb{R}^d} |g|^2 \, \frac{|x|^{n-d}}{1+|x|^2} \, dx$$

$$-p\,(2\,p-1) \, \frac{(2+\beta-\gamma)^2}{(p-1)^2} \int_{\mathbb{R}^d} |g|^2 \, \frac{|x|^{n-d}}{(1+|x|^2)^2} \, dx$$

is a nonnegative quadratic form if and only if $\alpha \leq \alpha_{FS}$

- Symmetry breaking holds if $\alpha > \alpha_{FS}$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Information – production of information inequality

Let $\mathcal{K}[u]$ be such that

$$\frac{d}{d\tau}\mathcal{I}[u(\tau,\cdot)] = -\mathcal{K}[u(\tau,\cdot)] = -\,(\text{sum of squares})$$

If $\alpha \leq \alpha_{\mathrm{FS}}$, then $\lambda_1 \geq 4$ and

$$u \mapsto \frac{\mathcal{K}[u]}{\mathcal{I}[u]} - 4$$

is a nonnegative functional

With $u_\varepsilon = \mathcal{B}_\star\left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right)$, we observe that

$$4 \leq \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \leq \lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_f \frac{\langle\!\langle f, \mathcal{L}\, f\rangle\!\rangle}{\langle\!\langle f, f\rangle\!\rangle} = \frac{\langle\!\langle f_1, \mathcal{L}\, f_1\rangle\!\rangle}{\langle\!\langle f_1, f_1\rangle\!\rangle} = \lambda_1$$

🌑 if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{\mathrm{FS}}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to \mathcal{B}_\star$ and determined by the spectral gap of $\mathcal{L}$

🌑 if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{\mathrm{FS}}$, then $\mathcal{K}/\mathcal{I} > 4$

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Symmetry in Caffarelli-Kohn-Nirenberg inequalities

If $\alpha \leq \alpha_{\text{FS}}$, the fact that $\mathcal{K}/\mathcal{I} \geq 4$ has an important consequence. Indeed we know that

$$\frac{d}{d\tau} \left(\mathcal{I}[u(\tau, \cdot)] - 4\,\mathcal{E}[u(\tau, \cdot)] \right) \leq 0$$

so that

$$\mathcal{I}[u] - 4\,\mathcal{E}[u] \geq \mathcal{I}[\mathcal{B}_\star] - 4\,\mathcal{E}[\mathcal{B}_\star] = 0$$

This inequality is equivalent to $\mathcal{J}[w] \geq \mathcal{J}[w_\star]$, which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for $\alpha \leq \alpha_{\text{FS}}$, the function

$$\tau \mapsto \mathcal{I}[u(\tau, \cdot)] - 4\,\mathcal{E}[u(\tau, \cdot)]$$

is monotone decreasing

🟢 This explains why the method based on nonlinear flows provides the *optimal range for symmetry*

Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

Entropy – production of entropy inequality

Using $\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - \mathcal{C}_2\,\mathcal{E}[u(\tau,\cdot)]\right) \leq 0$, we know that

$$\mathcal{I}[u] - \mathcal{C}_2\,\mathcal{E}[u] \geq \mathcal{I}[\mathcal{B}_\star] - \mathcal{C}_2\,\mathcal{E}[\mathcal{B}_\star] = 0$$

As a consequence, we have that

$$\mathcal{C}_1 := \inf_u \frac{\mathcal{I}[u]}{\mathcal{E}[u]} \geq \mathcal{C}_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]}$$

With $u_\varepsilon = \mathcal{B}_\star\left(1 + \varepsilon\,f\,\mathcal{B}_\star^{1-m}\right)$, we observe that

$$\mathcal{C}_1 \leq \lim_{\varepsilon \to 0}\inf_f \frac{\mathcal{I}[u_\varepsilon]}{\mathcal{E}[u_\varepsilon]} = \inf_f \frac{\langle f, \mathcal{L}\,f\rangle}{\langle f, f\rangle} = \frac{\langle f_1, \mathcal{L}\,f_1\rangle}{\langle f_1, f\rangle_1} = \lambda_1 = \lim_{\varepsilon \to 0}\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]}$$

- If $\lim_{\varepsilon \to 0}\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2 = \lambda_1$

This happens if $\alpha = \alpha_{FS}$ and in particular in the case without weights (Gagliardo-Nirenberg inequalities)

Symmetry breaking and linearization
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A parabolic proof ?
Large time asymptotics and spectral gaps
Linearization and optimality

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Thank you for your attention !