Entropy methods for degenerate diffusions and weighted functional inequalities

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Outline

The *fast diffusion equation with weights*

$$u_t + |x|^\gamma \, \nabla \cdot \left(|x|^{-\beta} \, u \, \nabla u^{m-1} \right) = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

- Simple facts
- Large time asymptotics
- A symmetry result

Without weights, the FDE (fast diffusion equation) has been a key model for designing sharp entropy methods

With weights, we have a model for degenerate / singular diffusions, with good scaling properties, in which new phenomena appear: asymptotic rates (given by the linearization) are not necessarily global, the underlying functional inequalities may experience *symmetry breaking*, Barenblatt self-similar solution are not always optimal, etc.

Fast diffusion equations with weights: simple facts

- The equation and the self-similar solutions
- Without weights
- A perturbation result
- Symmetry breaking

New results: joint work with M. Bonforte, M. Muratori and B. Nazaret

Fast diffusion equations with weights: self-similar solutions

Let us consider the *fast diffusion equation with weights*

$$u_t + |x|^\gamma \, \nabla \cdot \left(|x|^{-\beta} \, u \, \nabla u^{m-1} \right) = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

Here β and γ are two real parameters, and $m \in [m_1, 1)$ with

$$m_1 := \frac{2\,d-2-\beta-\gamma}{2\,(d-\gamma)}$$

Generalized *Barenblatt self-similar solutions*

$$u_\star(\rho\,t,x) = t^{-\,\rho\,(d-\gamma)}\,\mathfrak{B}_{\beta,\gamma}\left(t^{-\rho}\,x\right)\,, \quad \mathfrak{B}_{\beta,\gamma}(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{\frac{1}{m-1}}$$

where $1/\rho = (d-\gamma)\,(m-m_c)$ with $m_c := \frac{d-2-\beta}{d-\gamma} < m_1 < 1$
Self-similar solutions are known to govern the asymptotic behavior of the solutions when $(\beta,\gamma) = (0,0)$

- Mass conservation

$$\frac{d}{dt}\int_{\mathbb{R}^d} u\,\frac{dx}{|x|^\gamma}=0$$

and self-similar solutions suggest to introduce the

- Time-dependent rescaling

$$u(t,x)=R^{\gamma-d}\,v\left((2+\beta-\gamma)^{-1}\log R,\frac{x}{R}\right)$$

with $R=R(t)$ defined by

$$\frac{dR}{dt}=(2+\beta-\gamma)\,R^{(m-1)(\gamma-d)-(2+\beta-\gamma)+1}\,,\quad R(0)=1$$

$$R(t)=\left(1+\tfrac{2+\beta-\gamma}{\rho}\,t\right)^\rho$$

with $1/\rho=(1-m)\,(\gamma-d)+2+\beta-\gamma=(d-\gamma)\,(m-m_c)$

- A *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0$$

with initial condition $v(t=0,\cdot)=u_0$

Without weights: time-dependent rescaling, free energy

- *Time-dependent rescaling*: Take $u(\tau, y) = R^{-d}(\tau)\, v\,(t, y/R(\tau))$ where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|t=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\,|x|^2 v \right) dx - \mathcal{F}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]\;, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \tfrac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[v_\infty] = 0$

- *Entropy – entropy production inequality*

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \geq 2\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\mathcal{F}[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \mathcal{F}_0 \le \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} \mathcal{I}[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \ge 0$$

$K = K_0 \left(\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx \right)^\gamma$, $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With* $1 < p \le \frac{d}{d-2}$ *(fast diffusion case) and* $d \ge 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}_{p,d}^{\mathrm{GN}} \|\nabla w\|_{L^2(\mathbb{R}^d)}^{\theta} \|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

Proofs: variational methods [Del Pino, J.D.], or *carré du champ method* (Bakry-Emery): [Carrillo, Toscani], [Carrillo, Vázquez], [CJMTU]

Entropy methods and linearization: sharp asymptotic rates

Generalized Barenblatt profiles: $V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}}$

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_* := \frac{d-4}{d-2}$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_*$*, the entropy decays according to*

$$\mathcal{F}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

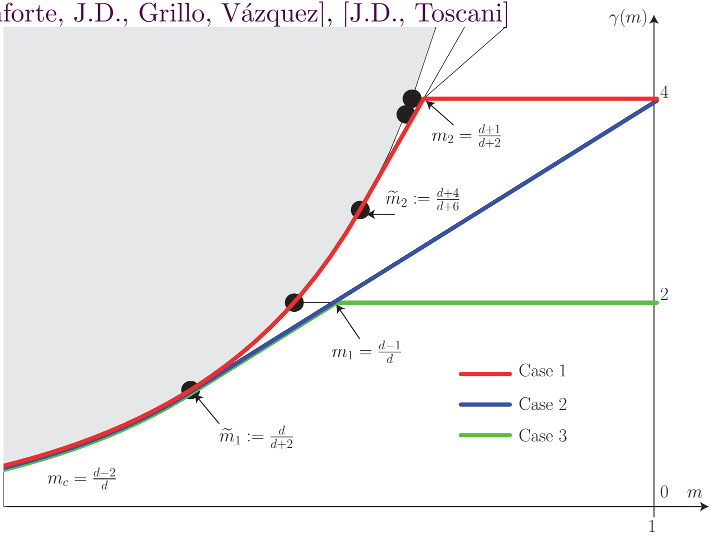
where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad f \in H^1(d\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$

1. Lowest eigenvalues

[Bonforte, J.D., Grillo, Vázquez], [J.D., Toscani]



2. Higher order matching asymptotics

[J.D., G. Toscani] For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

Without choosing R, we may define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \,, \quad R = R(\tau) \,, \quad t = \tfrac{1}{2} \log R \,, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

3. Best matching Barenblatt functions

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution (it plays the role of a *local Gibbs state*) but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1} (v - B_\sigma) \right] dx$$

The time derivative of this relative entropy is

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt} \left(\frac{d}{d\sigma} \mathcal{F}_\sigma[v] \right)_{|\sigma=\sigma(t)}}_{\text{choose it } =0} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left(v^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial v}{\partial t}\, dx$$

$\Longleftrightarrow$ Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$

4. Further results (still without weights)

- Improved asymptotic rates, *Improved entropy – entropy production inequalities*

$$\varphi(\mathcal{F}[v]) \le \mathcal{I}[v]$$

  for some convex function φ (and under appropriate normalization conditions), best matching and accelerated rates of convergence (initial time layer) [J.D., Toscani]
- *Rényi entropy powers:* concavity, asymptotic regime (self-similar solutions) and Gagliardo-Nirenberg inequalities in scale invariant form [Savaré, Toscani], [J.D., Toscani]
- Concavity of second moment estimates and *delays* [J.D., Toscani]
- *Stability* of entropy – entropy production inequalities (scaling methods), and improved rates of convergence [Carrillo, Toscani], [J.D., Toscani]

With one weight: a perturbation result

On the space of smooth functions on $\mathbb{R}^d$ with compact support

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_\gamma \, \|\nabla w\|^{\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)} \, \|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}$$

where $\vartheta := \frac{2^*_\gamma\,(p-1)}{2\,p\big(2^*_\gamma-p-1\big)} = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+2-2\,\gamma-p\,(d-2)\big)}$ and

$$\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q} \quad \text{and} \quad \|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$$

and $d \ge 3$, $\gamma \in (0,2)$, $p \in \big(1, 2^*_\gamma/2\big)$ with $2^*_\gamma := 2\,\frac{d-\gamma}{d-2}$

Theorem

[J.D., Muratori, Nazaret] *Let $d \ge 3$. For any $p \in (1, d/(d-2))$, there exists a positive γ^* such that equality holds for all $\gamma \in (0,\gamma^*)$ with*

$$w_\star(x) := \big(1+|x|^{2-\gamma}\big)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

Caffarelli-Kohn-Nirenberg inequalities (with two weights)

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p} \, \|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)} \, \|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,, \quad \gamma-2 < \beta < \frac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with } p_\star := \frac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

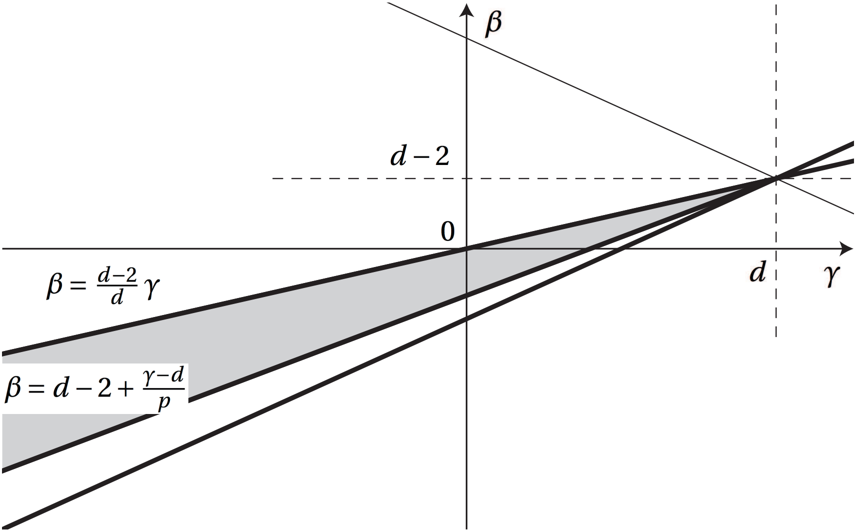
$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters



CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{F}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}\,.$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0\qquad\text{(WFDE-FP)}$$

then

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\,\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Proposition

Let $m=\frac{p+1}{2\,p}$ *and consider a solution to (WFDE-FP) with nonnegative initial datum* $u_0\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$ *such that* $\|u_0^m\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}$ *and* $\int_{\mathbb{R}^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ *are finite. Then*

$$\mathcal{F}[v(t,\cdot)]\le\mathcal{F}[u_0]\,e^{-(2+\beta-\gamma)^2t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:
(i) either u_0 *is a.e. radially symmetric*
(ii) or symmetry holds in (CKN)

With two weights: a symmetry breaking result

Let us define

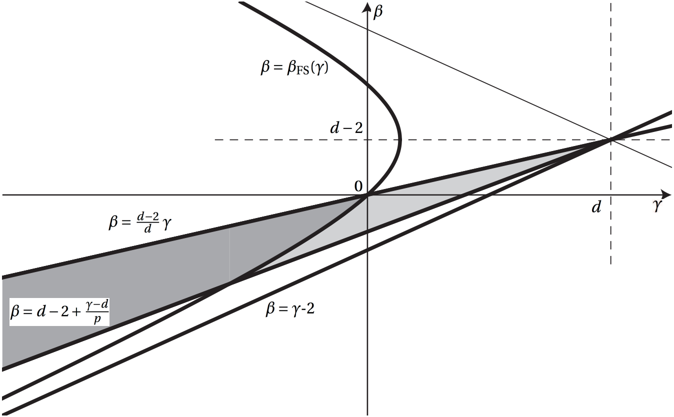
$$\beta_{\mathrm{FS}}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad \text{and} \quad \beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$, $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal.



The grey area corresponds to the admissible cone. The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d - \gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha < \alpha_{\mathrm{FS}} \quad \text{with} \quad \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta\,\log\left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta)\,\log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{J}\left[v_\star+\varepsilon\,\mu_{\delta/2}\,f\right] = \tfrac{1}{2}\,\varepsilon^2\,\vartheta\,\mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2\,p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \geq 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \geq n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta - n)\,(\delta - 1)}$.

$\mathcal{Q} \geq 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \leq \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \iff \eta < 1$$

$$\iff \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \iff \alpha > \alpha_{\rm FS}$$

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \, \nabla \cdot \Big[|x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$$\zeta := 1 - \left(1 - \tfrac{2-m}{(1-m)\,q}\right)\left(1 - \tfrac{2-m}{1-m}\,\theta\right)$$

$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma}$ is in the range $0 < \theta < \frac{1-m}{2-m} < 1$

Theorem

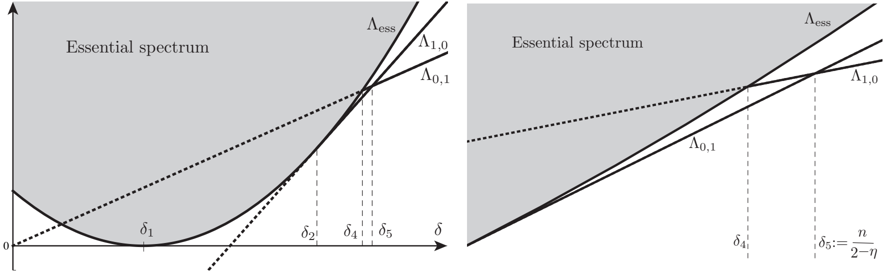
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-\,2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0,d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-\,2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum.

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\, w_t = -\,\frac{1}{\mathfrak{B}}\,\nabla\cdot\Big(|x|^{-\beta}\,\mathfrak{B}\,w\,\nabla\left(\left(w^{m-1}-1\right)\mathfrak{B}^{m-1}\right)\Big) & \text{in } \mathbb{R}^+\times\mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*, relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

Regularity (1/2): Harnack inequality and Hölder regularity

We change variables: $x \mapsto |x|^{\alpha-1}\,x$ and adapt the ideas of F. Chiarenza and R. Serapioni to

$$u_t + \mathsf{D}_\alpha^* \Big[\mathsf{a} \left(\mathsf{D}_\alpha \, u + \mathsf{B}\, u \right) \Big] = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d$$

Proposition (A parabolic Harnack inequality)

Let $d \geq 2$, $\alpha > 0$ and $n > d$. If u is a bounded positive solution, then for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $r > 0$ such that $Q_r(t_0, x_0) \subset \mathbb{R}^+ \times B_1$, we have

$$\sup_{Q_r^-(t_0,x_0)} u \leq H \inf_{Q_r^+(t_0,x_0)} u$$

The constant $H > 1$ depends only on the local bounds on the coefficients a, B *and on d, α, and n*

By adapting the classical method *à la De Giorgi* to our weighted

Regularity (1/2): from local to global estimates

Lemma

If w is a solution of the the Ornstein-Uhlenbeck equation with initial datum w_0 bounded from above and from below by a Barenblatt profile (+ relative mass condition) = "good solutions", then there exist $\nu \in (0,1)$ and a positive constant $\mathcal{K} > 0$, depending on d, m, β, γ, C, C_1, C_2 such that:

$$\|\nabla v(t)\|_{\mathrm{L}^\infty(B_{2\lambda}\setminus B_\lambda)} \le \frac{Q_1}{\lambda^{\frac{2+\beta-\gamma}{1-m}+1}} \quad \forall\, t \ge 1\,, \quad \forall\, \lambda > 1\,,$$

$$\sup_{t\ge1} \|w\|_{C^k((t,t+1)\times B_\varepsilon^c)} < \infty \quad \forall\, k \in \mathbb{N}\,,\ \forall\, \varepsilon > 0$$

$$\sup_{t\ge1} \|w(t)\|_{C^\nu(\mathbb{R}^d)} < \infty$$

$$\sup_{\tau\ge t} |w(\tau) - 1|_{C^\nu(\mathbb{R}^d)} \le \mathcal{K} \sup_{\tau\ge t} \|w(\tau) - 1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \quad \forall\, t \ge 1$$

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, with $m \neq m_*$ with $m_* :=$. Under the relative mass condition, for any "good solution" v there exists a positive constant $\mathcal{C}$ such that

$$\mathcal{F}[v(t)] \leq \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\,t} \quad \forall\, t \geq 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates cane be obtained using "best matching techniques"

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{F}[v] \le \frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{F}[v(t)] \le \mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t} \quad \forall\, t\ge 0 \quad \text{with} \quad \Lambda_\star := \tfrac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{F}[v] \le \mathcal{I}[v] \quad \forall\, v\in \mathrm{L}^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)} = M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M) := \frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{F}[v(t)] \le \mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t} \quad \forall\, t\ge 0$$

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any* $M > 0$, *we have*
$$0 < \mathcal{K}(M) \leq \frac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) then*
$$\mathcal{K}(M) > \frac{1-m}{m}\,(2+\beta-\gamma)^2$$

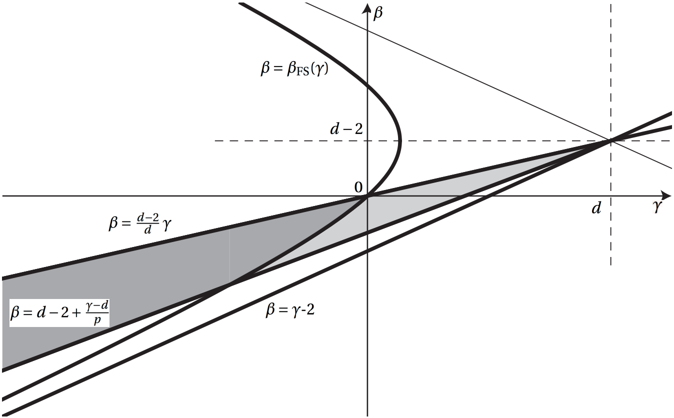
Corollary

Assume that $m \in [m_1, 1)$

(i) For any $M > 0$, *if* $\Lambda(M) = \Lambda_\star$ *then* $\beta = \beta_{\rm FS}(\gamma)$

(ii) If $\beta > \beta_{\rm FS}(\gamma)$ *then* $\Lambda_{0,1} < \Lambda_\star$ *and* $\Lambda(M) \in (0, \Lambda_{0,1}]$ *for any* $M > 0$

(iii) For any $M > 0$, *if* $\beta < \beta_{\rm FS}(\gamma)$ *and if symmetry holds in (CKN), then*
$$\Lambda(M) > \Lambda_\star$$


β
$\beta = \beta_{FS}(\gamma)$
$d-2$
0
d
γ
$\beta = \frac{d-2}{d}\,\gamma$
$\beta = d-2+\frac{\gamma-d}{p}$
$\beta = \gamma$-2

Fast diffusion equations with weights: a symmetry result

- Rényi entropy powers
- The symmetry result
- The strategy of the proof

Joint work with M.J. Esteban, M. Loss in the critical case $\beta = d - 2 + \frac{\gamma - d}{p}$

Joint work with M.J. Esteban, M. Loss and M. Muratori in the subcritical case $d - 2 + \frac{\gamma - d}{p} < \beta < \frac{d-2}{d}\,\gamma$

Rényi entropy powers

We consider the flow $\frac{\partial u}{\partial t} = \Delta u^m$ and the Gagliardo-Nirenberg inequalities (GN)

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}_{p,d}^{\rm GN}\,\|\nabla w\|_{L^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

where $u = w^{2p}$, that is, $w = u^{m-1/2}$ with $p = \frac{1}{2\,m-1}$. Straightforward computations show that (GN) can be brought into the form

$$\left(\int_{\mathbb{R}^d} u\,dx\right)^{(\sigma+1)\,m-1} \le C\,\mathcal{I}\,\mathcal{E}^{\sigma-1} \quad \text{where} \quad \sigma = \frac{2}{d\,(1-m)} - 1$$

where $\mathcal{E} := \int_{\mathbb{R}^d} u^m\,dx$ and $\mathcal{I} := \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$, $\mathsf{P} = \frac{m}{1-m}\,u^{m-1}$ is the *pressure variable*. If $\mathcal{F} = \mathcal{E}^\sigma$ is the *Rényi entropy power* and $\sigma = \frac{2}{d}\,\frac{1}{1-m} - 1$, then $\mathcal{F}''$ is proportional to

$$-2\,(1-m)\,\left\langle \mathrm{Tr}\left(\left(\mathrm{Hess}\,\mathsf{P} - \tfrac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right)^2\right)\right\rangle + (1-m)^2\,(1-\sigma)\,\left\langle\left(\Delta\mathsf{P} - \langle\Delta\mathsf{P}\rangle\right)^2\right\rangle$$

where we have used the notation $\langle A\rangle := \int_{\mathbb{R}^d} u^m\,A\,dx / \int_{\mathbb{R}^d} u^m\,dx$

The symmetry result

▷ critical case: [J.D., Esteban, Loss; Inventiones]
▷ subcritical case: [J.D., Esteban, Loss, Muratori]

Theorem

Assume that $\beta \le \beta_{\rm FS}(\gamma)$. *Then all positive solutions in* $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ *of*

$$-\,div\left(|x|^{-\beta}\,\nabla w\right) = |x|^{-\gamma}\left(w^{2p-1} - \,w^p\right) \quad in \quad \mathbb{R}^d\setminus\{0\}$$

are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$

The strategy of the proof (1/3)

The first step is based on a change of variables which amounts to rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise to consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,, \quad \alpha = 1 + \frac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta} \quad \forall\, v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d$$

The strategy of the proof (2/3)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathfrak{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

With $\mathcal{L}_\alpha = -\mathcal{D}_\alpha^* \mathfrak{D}_\alpha = \alpha^2 \left(u'' + \frac{n-1}{s}\, u' \right) + \frac{1}{s^2}\, \Delta_\omega\, u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\alpha\, u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\,\left(\textstyle\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma} \\
&\geq (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\,\left|\mathcal{L}_\alpha \mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathfrak{D}_\alpha \mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu \\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega \mathsf{P}' - \frac{\nabla_\omega \mathsf{P}}{s}\right|^2\right) u^m\,d\mu \\
&+2\int_{\mathbb{R}^d}\left((n-2)\,(\alpha_{\rm FS}^2-\alpha^2)\,|\nabla_\omega \mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \leq \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result.

The strategy of the proof (3/3)

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathcal{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathcal{L}_\alpha u^m\,d\mu\geq\mathcal{H}[u]\geq 0$$

where the last inequality holds because $\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term). In original variables: test by $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Concluding remarks

- The fast diffusion equation (without weights) has a rich structure: a lot has been done (for instance, with parabolic methods or gradient flow techniques) and this is a fundamental equation to explore qualitative behaviors, sharp rates, *entropy methods in PDEs*, etc.

- With *weights*, self-similar Barenblatt solutions attract all solutions (in good spaces) on the long time range, the linearization of the entropy determines the sharp asymptotic rates... but when *symmetry breaking* occurs, there are other critical points and Barenblatt solutions are not optimal for entropy – entropy production ineq.

- Entropy methods can be used *as a tool* to produce symmetry / uniqueness / rigidity results which go well beyond results of elliptic PDEs (rearrangement, moving planes), energy / calculus of variations methods (concentration-compactness methods) and methods of spectral theory (so far).

- An example of doubly defective / degenerate operator, which is waiting for extension in (non-homogenous) kinetic equations !

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Thank you for your attention !