Sobolev type inequalities, logarithmic Sobolev inequalities: stability in strong norms by entropy methods or direct methods of the calculus of variations

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Explicit stability for Sobolev and LSI on $\mathbb{R}^d$
Results based on entropy methods and fast diffusion equations
Sphere, Gaussian measure, interpolation and log-Sobolev inequalities

Introduction

- Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{L^2(\mathbb{R}^d)} \geq \mathsf{S}_d \|f\|^2_{L^{2^*}(\mathbb{R}^d)} \quad \forall f \in \mathscr{D}^{1,2}(\mathbb{R}^d) \tag{S}$$

Equality holds on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c \left(a + |x - b|^2\right)^{-\frac{d-2}{2}}, \quad a \in (0, \infty), \quad b \in \mathbb{R}^d, \quad c \in \mathbb{R}$$

[Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|^2_2 - \mathsf{S}_d \|f\|^2_{2^*} \geq c_{\mathrm{BE}} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_2$$

- How do we estimate c_{BE} ? as $d \to +\infty$?
Stability & improved entropy – entropy production inequalities
Improved inequalities & faster decay rates for entropies

Explicit stability for Sobolev and LSI on $\mathbb{R}^d$
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Outline

Explicit stability for Sobolev and LSI on $\mathbb{R}^d$
Results based on entropy methods and fast diffusion equations
Sphere, Gaussian measure, interpolation and log-Sobolev inequalities

Main results, optimal dimensional dependence; history
Sketch of the proof, definitions & preliminary results
The main steps of the proof

Explicit stability results for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence
arXiv: 2209.08651, Cambridge J. Math. 2025

A short review on improvements and stability for some interpolation inequalities
arXiv: 2402.08527, Proc. ICIAM 2023

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An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c \left(a + |x - b|^2\right)^{-\frac{d-2}{2}}, \quad a \in (0, \infty), \quad b \in \mathbb{R}^d, \quad c \in \mathbb{R}$$

> ## Theorem (JD, Esteban, Figalli, Frank, Loss)
>
> *There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have*
>
> $$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

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A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - \frac{1}{4}\,d\,(d-2)\left(\|F\|_{L^{2*}(\mathbb{S}^d)}^2 - \|F\|_{L^2(\mathbb{S}^d)}^2\right)$$

$$\geq \frac{\beta}{d}\inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F - \nabla G\|_{L^2(\mathbb{S}^d)}^2 + \frac{1}{4}\,d\,(d-2)\,\|F-G\|_{L^2(\mathbb{S}^d)}^2\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d \to +\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

> ### Corollary (JD, Esteban, Figalli, Frank, Loss)
>
> *With $\beta > 0$ as in the result for the Sobolev inequality*
>
> $$\|\nabla u\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|_{L^2(\mathbb{R}^n, d\gamma)}^2}\right) d\gamma$$
>
> $$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^n,\, c\in\mathbb{R}}\int_{\mathbb{R}^n} |u - c\,e^{a\cdot x}|^2\,d\gamma$$

Explicit stability for Sobolev and LSI on $\mathbb{R}^d$ **Main results, optimal dimensional dependence; history**
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Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]

In the inequality $\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 \geq S_d \, \|f\|_{L^{2^*}(\mathbb{R}^d)}^2$, the optimal constant is

$$S_d = \tfrac{1}{4} \, d \, (d-2) \, |\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$if \; \lim_{n\to\infty} \|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = S_d \,, then \; \lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb, 1985] a quantitative stability result ?

▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{BE} > 0$ such that

$$\|\nabla f\|_2^2 \geq S_d \, \|f\|_{2^*}^2 + c_{BE} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

🔋 The strategy of Bianchi & Egnell involves two steps:

– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$

– a local-to-global extension based on concentration-compactness :

🔋 The constant c_{BE} is not explicit   the *far away regime*

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Stability for the logarithmic Sobolev inequality

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} \geq \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}}\right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 69], [Costa, 85] *Cf.* [Villani, 08]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathcal{M} := \left\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a \cdot x} \quad \forall\, x \in \mathbb{R}^n$$

[Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance, in $\mathrm{W}^{1,1}$, *etc.*

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

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Explicit stability results for the Sobolev inequality

Proof

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Sketch of the proof

Goal: prove that there is an ***explicit*** constant $\beta > 0$ such that for all $d \geq 3$ and all $f \in \dot{H}^1\left(\mathbb{R}^d\right)$

$$\|\nabla f\|_2^2 \geq \mathsf{S}_d \|f\|_{2^*}^2 + \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

Part 1. We show the inequality for nonnegative functions far from $\mathcal{M}$

... the far away regime

Make it *constructive*

Part 2. We show the inequality for nonnegative functions close to $\mathcal{M}$

... the local problem

Get *explicit* estimates and remainder terms

Part 3. We show that the inequality for nonnegative functions implies the inequality for functions without a sign restriction, up to an acceptable loss in the constant

... sign-changing functions

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Some definitions

What we want to minimize is

$$\mathcal{E}(f) := \frac{\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2}{d(f, \mathcal{M})^2} \quad f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$$

where

$$d(f, \mathcal{M})^2 := \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2$$

▷ up to an elementary *transformation*, we assume that
$d(f, \mathcal{M})^2 = \|\nabla f - \nabla g_*\|_{L^2(\mathbb{R}^d)}^2$ with

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2d}} \left(\frac{2}{1 + |x|^2} \right)^{\frac{d-2}{2}}$$

▷ use the *inverse stereographic* projection

$$F(\omega) = \frac{f(x)}{g_*(x)} \quad x \in \mathbb{R}^d \text{ with } \begin{cases} \omega_j = \frac{2 x_j}{1 + |x|^2} & \text{if } 1 \le j \le d \\ \omega_{d+1} = \frac{1 - |x|^2}{1 + |x|^2} \end{cases}$$

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The problem on the unit sphere

Stability inequality on the unit sphere $\mathbb{S}^d$ for $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$

$$\int_{\mathbb{S}^d} \left(|\nabla F|^2 + \mathrm{A}\,|F|^2 \right) d\mu - \mathrm{A} \left(\int_{\mathbb{S}^d} |F|^{2^*} d\mu \right)^{2/2^*}$$

$$\geq \frac{\beta}{d} \inf_{G \in \mathcal{M}} \left\{ \|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|F - G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right\}$$

with $\mathrm{A} = \frac{1}{4}\,d\,(d-2)$ and a manifold $\mathcal{M}$ of optimal functions made of

$$G(\omega) = c\,(a + b \cdot \omega)^{-\frac{d-2}{2}} \quad \omega \in \mathbb{S}^d \quad (a, b, c) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}$$

🔘 make the reduction of a ***far away problem*** to a local problem *constructive...* on $\mathbb{R}^d$

🔘 make the analysis of the ***local problem*** *explicit...* on $\mathbb{S}^d$

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Competing symmetries

● ***Rotations on the sphere*** combined with stereographic and inverse stereographic projections. Let $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$

$$(Uf)(x) := \left(\frac{2}{|x - e_d|^2} \right)^{\frac{d-2}{2}} f \left(\frac{x_1}{|x - e_d|^2}, \ldots, \frac{x_{d-1}}{|x - e_d|^2}, \frac{|x|^2 - 1}{|x - e_d|^2} \right)$$

$$\mathcal{E}(Uf) = \mathcal{E}(f)$$

● ***Symmetric decreasing rearrangement*** $\mathcal{R}f = f^*$
$$f \text{ and } f^* \text{ are equimeasurable}$$
$$\|\nabla f^*\|_{L^2(\mathbb{R}^d)} \leq \|\nabla f\|_{L^2(\mathbb{R}^d)}$$

The method of ***competing symmetries***

Theorem (Carlen, Loss, 1990)

Let $f \in L^{2^*}(\mathbb{R}^d)$ be a non-negative function with $\|f\|_{L^{2^*}(\mathbb{R}^d)} = \|g_*\|_{L^{2^*}(\mathbb{R}^d)}$. The sequence $f_n = (\mathcal{R}U)^n f$ is such that $\lim_{n \to +\infty} \|f_n - g_*\|_{L^{2^*}(\mathbb{R}^d)} = 0$. If $f \in \dot{H}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_{L^2(\mathbb{R}^d)})_{n \in \mathbb{N}}$ is a non-increasing sequence

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Useful preliminary results

- $\lim_{n\to\infty} \|f_n - g_*\|_{2^*} = 0$ if $\|f\|_{2^*} = \|g_*\|_{2^*}$

- $(\|\nabla f_n\|_2^2)_{n\in\mathbb{N}}$ is a nonincreasing sequence

Lemma

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - \mathsf{S}_d \sup_{g\in\mathcal{M},\, \|g\|_{2^*}=1} \left(f, g^{2^*-1}\right)^2$$

Corollary

$\left(\mathsf{d}(f_n, \mathcal{M})\right)_{n\in\mathbb{N}}$ is strictly decreasing, $n \mapsto \sup_{g\in\mathcal{M}_1} \left(f_n, g^{2^*-1}\right)$ is strictly increasing, and

$$\lim_{n\to\infty} \mathsf{d}(f_n, \mathcal{M})^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - \mathsf{S}_d \|g_*\|_{2^*}^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - \mathsf{S}_d \|f\|_{2^*}^2$$

but no monotonicity for $n \mapsto \mathcal{E}(f_n) = \dfrac{\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \mathsf{S}_d \|f_n\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2}{\mathsf{d}(f_n, \mathcal{M})^2}$

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Part 1: Global to local reduction

The ***local problem***

$$\mathscr{I}(\delta) := \inf\left\{\mathcal{E}(f) \,:\, f \geq 0 \,,\, \mathsf{d}(f,\mathcal{M})^2 \leq \delta \,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2\right\}$$

$f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the ***far away regime*** iff

$$\mathsf{d}(f,\mathcal{M})^2 = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 > \delta \,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

for some $\delta \in (0,1)$

Let $f_n = (\mathcal{R}\,U)^n f$. There are two cases:
- (Case 1) $\mathsf{d}(f_n,\mathcal{M})^2 \geq \delta \,\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$ for all $n \in \mathbb{N}$
- (Case 2) for some $n \in \mathbb{N}$, $\mathsf{d}(f_n,\mathcal{M})^2 < \delta \,\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$

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Global to local reduction – Case 1

$f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the far away regime

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 > \delta \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

Lemma

Let $f_n = (\mathcal{R}U)^n f$ and $\delta \in (0, 1)$
If $\mathsf{d}(f_n, \mathcal{M})^2 \geq \delta \|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$ for all $n \in \mathbb{N}$, then $\mathcal{E}(f) \geq \delta$

$$\lim_{n \to +\infty} \|\nabla f_n\|_2^2 \leq \frac{1}{\delta} \lim_{n \to +\infty} \inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta} \left(\lim_{n \to +\infty} \|\nabla f_n\|_2^2 - \mathsf{S}_d \|f\|_{2*}^2 \right)$$

$$1 - \frac{\mathsf{S}_d \|f\|_{2*}^2}{\lim_{n \to +\infty} \|\nabla f_n\|_2^2} \geq \delta$$

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - \mathsf{S}_d \|f\|_{2*}^2}{\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2} \geq \frac{\|\nabla f\|_2^2 - \mathsf{S}_d \|f\|_{2*}^2}{\|\nabla f\|_2^2} \geq \frac{\|\nabla f_n\|_2^2 - \mathsf{S}_d \|f\|_{2*}^2}{\|\nabla f_n\|_2^2} \underset{n \to +\infty}{\geq} \delta$$

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Global to local reduction – Case 2

$$\mathscr{I}(\delta) := \inf\left\{\mathcal{E}(f) \, : \, f \geq 0 \, , \, \mathsf{d}(f, \mathcal{M})^2 \leq \delta \, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2\right\} > 0 \text{ (to be proven)}$$

Lemma

$$\mathcal{E}(f) \geq \delta \, \mathscr{I}(\delta)$$

$$\text{if} \quad \inf_{g \in \mathcal{M}} \|\nabla f_{n_0} - \nabla g\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 > \delta \, \|\nabla f_{n_0}\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

$$\text{and} \quad \inf_{g \in \mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 \leq \delta \, \|\nabla f_{n_0+1}\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

▷ Adapt a strategy due to Christ: build a (semi-)*continuous*
rearrangement *flow* $(\mathsf{f}_\tau)_{n_0 \leq \tau < n_0+1}$ with $\mathsf{f}_{n_0} = Uf_n$ such that
$\|f_\tau\|_{2^*} = \|f\|_2$, $\tau \mapsto \|\nabla f_\tau\|_2$ is nonincreasing, and $\lim_{\tau \to n_0+1} \mathsf{f}_\tau = \mathsf{f}_{n_0+1}$

$$\mathcal{E}(f) \geq 1 - \mathsf{S}_d \frac{\|f\|_{2^*}^2}{\|\nabla f\|_2^2} \geq 1 - \mathsf{S}_d \frac{\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_{\tau_0}\|_2^2} = \delta \, \mathcal{E}(\mathsf{f}_{\tau_0}) \geq \delta \, \mathscr{I}(\delta)$$

Altogether: $\boxed{\text{if } \mathsf{d}(f, \mathcal{M})^2 > \delta \, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2, \text{ then } \mathcal{E}(f) \geq \min\left\{\delta, \delta \, \mathscr{I}(\delta)\right\}}$

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Part 2: The (simple) Taylor expansion

Proposition

Let $(X, d\mu)$ be a measure space and u, $r \in L^q(X, d\mu)$ for some $q \geq 2$ with $u \geq 0$, $u + r \geq 0$ and $\int_X u^{q-1} r \, d\mu = 0$

$\triangleright$ If $q = 6$, then

$$\|u + r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left(5 \int_X u^{q-2} r^2 \, d\mu + \frac{20}{3} \int_X u^{q-3} r^3 \, d\mu \right.$$
$$\left. + 5 \int_X u^{q-4} r^4 \, d\mu + 2 \int_X u^{q-5} r^5 \, d\mu + \frac{1}{3} \int_X r^6 \, d\mu \right)$$

$\triangleright$ If $3 \leq q \leq 4$, then

$$\|u + r\|_q^2 - \|u\|_q^2$$
$$\leq \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2} r^2 \, d\mu + \frac{(q-1)(q-2)}{3} \int_X u^{q-3} r^3 \, d\mu + \frac{2}{q} \int_X |r|^q \, d\mu \right)$$

$\triangleright$ If $2 < q \leq 3$ (take $q = 2^*$, $d \geq 6$), then

$$\|u + r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2} r^2 \, d\mu + \frac{2}{q} \int_X r_+^q \, d\mu \right)$$

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Corollary

For all $\nu > 0$ and for all $r \in \mathrm{H}^1(\mathbb{S}^d)$ satisfying $r \geq -1$,

$$\left(\int_{\mathbb{S}^d} |r|^q \, d\mu \right)^{2/q} \leq \nu^2 \quad \text{and} \quad \int_{\mathbb{S}^d} r \, d\mu = 0 = \int_{\mathbb{S}^d} \omega_j \, r \, d\mu \quad \forall j = 1, \ldots d+1$$

if $d\mu$ is the uniform probability measure on $\mathbb{S}^d$, then

$$\int_{\mathbb{S}^d} \left(|\nabla r|^2 + \mathsf{A} \, (1+r)^2 \right) d\mu - \mathsf{A} \left(\int_{\mathbb{S}^d} (1+r)^q \, d\mu \right)^{2/q}$$
$$\geq \mathsf{m}(\nu) \int_{\mathbb{S}^d} \left(|\nabla r|^2 + \mathsf{A} \, r^2 \right) d\mu$$

$$\mathsf{m}(\nu) := \tfrac{4}{d+4} - \tfrac{2}{q} \, \nu^{q-2} \quad \text{if} \quad d \geq 6$$
$$\mathsf{m}(\nu) := \tfrac{4}{d+4} - \tfrac{1}{3} \, (q-1)\,(q-2)\,\nu - \tfrac{2}{q} \, \nu^{q-2} \quad \text{if} \quad d = 4, 5$$
$$\mathsf{m}(\nu) := \tfrac{4}{7} - \tfrac{20}{3} \, \nu - 5 \, \nu^2 - 2 \, \nu^3 - \tfrac{1}{3} \, \nu^4 \qquad \text{if} \quad d = 3$$

An explicit expression of $\mathscr{I}(\delta)$ if $\nu > 0$ is small enough so that $\mathsf{m}(\nu) > 0$

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Part 3: Removing the positivity assumption

- Take $f = f_+ - f_-$ with $\|f\|_{L^{2^*}(\mathbb{R}^d)} = 1$

$$\|\nabla f_+\|^2_{L^{2^*}(\mathbb{R}^d)} + \|\nabla f_-\|^2_{L^{2^*}(\mathbb{R}^d)} = \|\nabla f\|^2_{L^{2^*}(\mathbb{R}^d)}$$

- Let $m := \|f_-\|^{2^*}_{L^{2^*}(\mathbb{R}^d)}$ and $1 - m = \|f_+\|^{2^*}_{L^{2^*}(\mathbb{R}^d)} > 1/2$

$$\|f_+\|^2_{L^{2^*}(\mathbb{R}^d)} + \|f_-\|^2_{L^{2^*}(\mathbb{R}^d)} - \|f\|^2_{L^{2^*}(\mathbb{R}^d)}$$
$$= h_d(m) := m^{\frac{d-2}{d}} + (1-m)^{\frac{d-2}{d}} - 1$$

where h_d is a positive concave function
▷ For some $g_+ \in \mathcal{M}$

$$\|\nabla f\|^2_{L^2(\mathbb{R}^d)} - \mathsf{S}_d \|f\|^2_{L^{2^*}(\mathbb{R}^d)} \geq C^{d,\text{pos}}_{\text{BE}} \|\nabla f_+ - \nabla g_+\|^2_{L^2(\mathbb{R}^d)} + \frac{2\, h_d(1/2)}{h_d(1/2)+1} \|\nabla f_-\|^2_{L^2(\mathbb{R}^d)}$$

$$C^d_{\text{BE}} \geq \frac{1}{2} \min \left\{ \max_{0 < \delta < 1/2} \delta\, \mathscr{I}(\delta),\, \frac{2\, h_d(1/2)}{h_d(1/2)+1} \right\}$$

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Main results, optimal dimensional dependence; history
Sketch of the proof, definitions & preliminary results
The main steps of the proof

Part 2, refined: The (complicated) Taylor expansion

To get a dimensionally sharp estimate, we expand $(1 + r)^{2^*} - 1 - 2^* r$ with an accurate remainder term for all $r \geq -1$

$$r_1 := \min\{r, \gamma\}, \quad r_2 := \min\{(r - \gamma)_+, M - \gamma\} \quad \text{and} \quad r_3 := (r - M)_+$$

with $0 < \gamma < M$. Let $\theta = 4/(d - 2)$

Lemma

Given $d \geq 6$, $r \in [-1, \infty)$, and $\overline{M} \in [\sqrt{e}, +\infty)$, we have

$(1 + r)^{2^*} - 1 - 2^* r$

$\leq \frac{1}{2} 2^* (2^* - 1) (r_1 + r_2)^2 + 2 (r_1 + r_2) r_3 + \left(1 + C_M \theta \overline{M}^{-1} \ln \overline{M}\right) r_3^{2^*}$

$\quad + \left(\frac{3}{2} \gamma \theta r_1^2 + C_{M,\overline{M}} \theta r_2^2\right) \mathbb{1}_{\{r \leq M\}} + C_{M,\overline{M}} \theta M^2 \mathbb{1}_{\{r > M\}}$

where all the constants in the above inequality are explicit

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There are constants ϵ_1, ϵ_2, k_0, and $\epsilon_0 \in (0, 1/\theta)$, such that

$$\|\nabla r\|^2_{L^2(\mathbb{S}^d)} + A \, \|r\|^2_{L^2(\mathbb{S}^d)} - A \, \|1 + r\|^2_{L^{2^*}(\mathbb{S}^d)}$$

$$\geq \frac{4\,\epsilon_0}{d-2} \left(\|\nabla r\|^2_{L^2(\mathbb{S}^d)} + A \, \|r\|^2_{L^2(\mathbb{S}^d)} \right) + \sum_{k=1}^{3} I_k$$

$I_1 := (1 - \theta\,\epsilon_0) \int_{\mathbb{S}^d} \left(|\nabla r_1|^2 + A\, r_1^2 \right) d\mu - A\,(2^* - 1 + \epsilon_1\,\theta) \int_{\mathbb{S}^d} r_1^2 \, d\mu + A\, k_0\,\theta \int_{\mathbb{S}^d} (r_2^2 \ldots$

$I_2 := (1 - \theta\,\epsilon_0) \int_{\mathbb{S}^d} \left(|\nabla r_2|^2 + A\, r_2^2 \right) d\mu - A\,(2^* - 1 + (k_0 + C_{\epsilon_1,\epsilon_2})\,\theta) \int_{\mathbb{S}^d} r_2^2 \, d\mu$

$I_3 := (1 - \theta\,\epsilon_0) \int_{\mathbb{S}^d} \left(|\nabla r_3|^2 + A\, r_3^2 \right) d\mu - \frac{2}{2^*} A\,(1 + \epsilon_2\,\theta) \int_{\mathbb{S}^d} r_3^{2^*} \, d\mu - A\, k_0\,\theta \int_{\mathbb{S}^d} r_3^2 \, d\mu$

- spectral gap estimates : $I_1 \geq 0$
- Sobolev inequality : $I_3 \geq 0$
- improved spectral gap inequality using that $\mu(\{r_2 > 0\})$ is small: $I_2 \geq 0$

[Duoandikoetxea]

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Results based on entropy methods and fast diffusion equations

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Sobolev and Hardy-Littlewood-Sobolev inequalities

▷ Stability in a weaker norm, with explicit constants

▷ From duality to improved estimates

▷ Fast diffusion equation with Yamabe's exponent

▷ Explicit stability constants

Joint paper with G. Jankowiak
Sobolev and Hardy–Littlewood–Sobolev inequalities
J. Differential Equations, 257, 2014

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Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall \, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d) \qquad \text{(S)}$$

and the Hardy-Littlewood-Sobolev inequality

$$\|g\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \mathsf{S}_d \int_{\mathbb{R}^d} g \, (-\Delta)^{-1} g \, dx \quad \forall \, g \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \qquad \text{(HLS)}$$

are dual of each other. Here S_d is the Aubin-Talenti constant, $2^* = \frac{2d}{d-2}$, $(2^*)' = \frac{2d}{d+2}$ and by the Legendre transform

$$\sup_{f \in \mathscr{D}^{1,2}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} f \, g \, dx - \frac{1}{2} \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right) = \frac{1}{2} \|g\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

$$\sup_{f \in \mathscr{D}^{1,2}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} f \, g \, dx - \frac{1}{2} \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right) = \frac{1}{2} \int_{\mathbb{R}^d} g \, (-\Delta)^{-1} g \, dx$$

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Improved Sobolev inequality by duality

Theorem

[JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$
There exists a positive constant $\mathcal{C} \in [\frac{d}{d+4}, 1)$ such that

$$\|f^q\|^2_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} - S_d \int_{\mathbb{R}^d} f^q (-\Delta)^{-1} f^q \, dx$$

$$\leq \mathcal{C} \, S_d^{-1} \|f\|^{\frac{8}{d-2}}_{L^{2^*}(\mathbb{R}^d)} \left(\|\nabla f\|^2_{L^2(\mathbb{R}^d)} - S_d \|f\|^2_{L^{2^*}(\mathbb{R}^d)} \right)$$

for any $f \in \mathscr{D}^{1,2}(\mathbb{R}^d)$

$\mathcal{C} = 1$: "completion" of the square

$$0 \leq \int_{\mathbb{R}^d} \left| \|f\|^{\frac{4}{d-2}}_{L^{2^*}(\mathbb{R}^d)} \nabla f - S_d \nabla (-\Delta)^{-1} g \right|^2 dx$$

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Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m, \quad t > 0, \quad x \in \mathbb{R}^d \tag{Y}$$

Choice $m = \frac{d-2}{d+2}$ (Yamabe flow): $m + 1 = \frac{2d}{d+2}$

Proposition

Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If $u = v^m$ and v is a solution of (??)
with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2} \frac{d}{dt} \left(\mathsf{S}_d^{-1} \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \right)$$

$$= - \left(\int_{\mathbb{R}^d} v^{m+1} \, dx \right)^{\frac{2}{d}} \left(\mathsf{S}_d^{-1} \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right) \leq 0$$

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Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t, x) = c\,(T - t)^\alpha \left(F(x)\right)^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d - 2)\,F^{(d+2)/(d-2)}$$

Lemma

[del Pino, Saez] For any solution v with initial datum
$v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t \to T_-} (T - t)^{-\frac{1}{1-m}} \sup_{x \in \mathbb{R}^d} \left(1 + |x|^2\right)^{d+2} \left| \frac{v(t, x)}{\overline{v}(t, x)} - 1 \right| = 0$$

with $\overline{v}(t, x) = \lambda^{(d+2)/2}\,\overline{v}_T(t, (x - x_0)/\lambda)$

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A convexity improvement

$$\mathsf{J}[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}} \, dx \quad \text{and} \quad \mathsf{H}[v] := \mathsf{S}_d \, \|v\|_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx$$

Theorem

[JD, Jankowiak] Assume that $d \geq 3$. Then we have

$$0 \leq \mathsf{H}[v] + \mathsf{S}_d \, \mathsf{J}[v]^{1+\frac{2}{d}} \, \varphi \left(\mathsf{J}[v]^{\frac{2}{d}-1} \left(\mathsf{S}_d^{-1} \, \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2 \right) \right)$$

where $\varphi(x) := \sqrt{1 + 2x} - 1$ for any $x \geq 0$

Proof: with $\kappa_0 := -\mathsf{H}'_0/\mathsf{J}_0$ and $\mathsf{H} = \mathsf{Y}(\mathsf{J})$, consider the differential inequality

$$\mathsf{Y}' \left(\mathcal{C} \, \mathsf{S}_d \, s^{1+\frac{2}{d}} + \mathsf{Y} \right) \leq \frac{d+2}{2\,d} \, \mathcal{C} \, \kappa_0 \, \mathsf{S}_d^2 \, s^{1+\frac{4}{d}} \,, \quad \mathsf{Y}(0) = 0 \,, \quad \mathsf{Y}(\mathsf{J}_0) = \mathsf{H}_0$$

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Constructive stability results
in Gagliardo-Nirenberg-Sobolev
inequalities

Joint papers with M. Bonforte, B. Nazaret and N. Simonov
Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows,
regularity and the entropy method
arXiv:2007.03674, to appear in *Memoirs of the AMS*

Constructive stability results in interpolation inequalities
and explicit improvements of decay rates of fast diffusion
equations
DCDS, 43 (3&4): 1070–1089, 2023

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Entropy – entropy production inequality

The fast diffusion equation on $\mathbb{R}^d$ in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\, x \right) \right] = 0 \qquad \text{(FDE)}$$

admits a stationary Barenblatt solution $\mathcal{B}(x) := \left(1 + |x|^2 \right)^{\frac{1}{m-1}}$

$$\frac{d}{dt} \mathcal{F}[v(t, \cdot)] = - \mathcal{I}[v(t, \cdot)]$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\, \mathcal{B}^{m-1} \left(v - \mathcal{B} \right) \right)\, dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathcal{B}^{m-1} \right|^2\, dx$$

are such that $\mathcal{I}[v] \geq 4\, \mathcal{F}[v]$ [del Pino, JD, 2002] so that

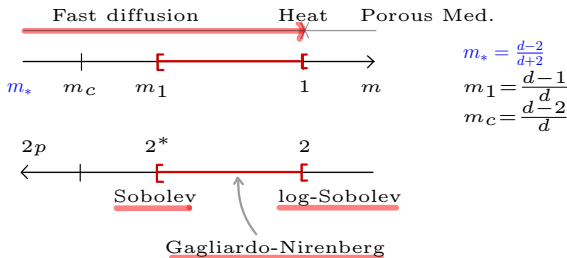$$\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0]\, e^{-4\, t}$$

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Entropy growth rate

$\mathcal{I}[v] \geq 4\,\mathcal{F}[v] \iff$ *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with optimal constant. Under appropriate mass normalization
$$v = f^{2p} \text{ so that } v^m = f^{p+1} \text{ and } v\,|\nabla v^{m-1}|^2 = (p-1)^2\,|\nabla f|^2$$

$$p = \frac{1}{2\,m-1} \qquad \iff \qquad m = \frac{p+1}{2\,p} \in [m_1, 1)$$

Fast diffusion   Heat   Porous Med.

$m_* \quad m_c \quad m_1 \qquad\qquad 1 \qquad\qquad m$

$m_* = \frac{d-2}{d+2}$

$m_1 = \dfrac{d-1}{d}$

$m_c = \dfrac{d-2}{d}$

$2p \qquad\qquad 2^* \qquad\qquad 2$

Sobolev   log-Sobolev

Gagliardo-Nirenberg

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Asymptotic regime as $t \to +\infty$

Take $f_\varepsilon := \mathcal{B} \left(1 + \varepsilon \, \mathcal{B}^{1-m} \, w \right)$ and expand $\mathcal{F}[f_\varepsilon]$ and $\mathcal{I}[f_\varepsilon]$ at order $O(\varepsilon^2)$

linearized free energy and *linearized Fisher information*

$$\mathsf{F}[w] := \frac{m}{2} \int_{\mathbb{R}^d} w^2 \, \mathcal{B}^{2-m} \, dx \quad \text{and} \quad \mathsf{I}[w] := m \, (1-m) \int_{\mathbb{R}^d} |\nabla w|^2 \, \mathcal{B} \, dx$$
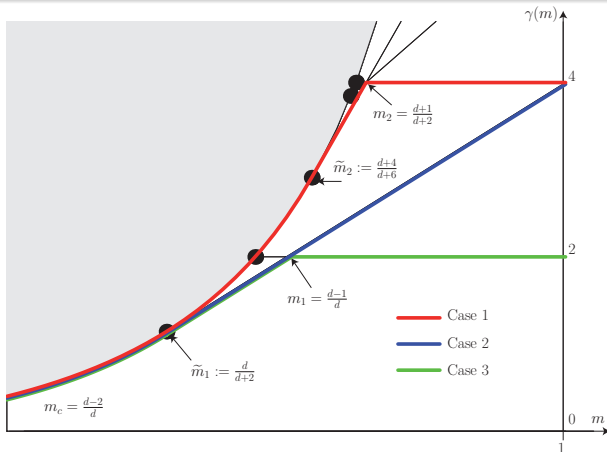

Proposition (Hardy-Poincaré inequality)

[BBDGV ,BDNS] Let $m \in [m_1, 1)$ if $d \geq 3$, $m \in (1/2, 1)$ if $d = 2$, and $m \in (1/3, 1)$ if $d = 1$. If $w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m} \, dx)$ is such that $\nabla w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B} \, dx)$, $\int_{\mathbb{R}^d} w \, \mathcal{B}^{2-m} \, dx = 0$, then

$$\mathsf{I}[w] \geq 4 \, \alpha \, \mathsf{F}[w]$$

with $\alpha = 1$, or $\alpha = 2 - d \, (1 - m)$ if $\int_{\mathbb{R}^d} x \, w \, \mathcal{B}^{2-m} \, dx = 0$

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Spectral gap

[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

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The asymptotic time layer improvement

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and

$$(1 - \varepsilon)\,\mathcal{B} \leq v \leq (1 + \varepsilon)\,\mathcal{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then
$$\mathcal{I}[v] \geq (4 + \eta)\,\mathcal{F}[v]$$

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Uniform convergence in relative error: threshold time

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $t_\star \geq 0$ such that, if u is a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2x \right) \right] = 0 \qquad \text{(FDE)}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r > 0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x| > r} u_0 \, dx \leq A < \infty \qquad (\mathsf{H}_A)$$

$\int_{\mathbb{R}^d} u_0 \, dx = \int_{\mathbb{R}^d} B \, dx = \mathcal{M}$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq t_\star$$

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The initial time layer improvement: backward estimate

By the *carré du champ* method, we have
Away from the Barenblatt solutions, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

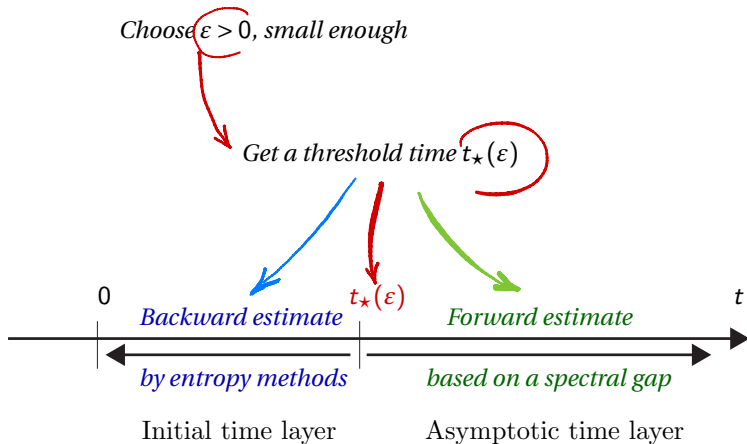$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}(\mathcal{Q} - 4)$$

Lemma

*Assume that $m > m_1$ and v is a solution to (**??**) with nonnegative initial datum v_0. If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star, \cdot)] \geq 4 + \eta$, then*

$$\mathcal{Q}[v(t, \cdot)] \geq 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0, t_\star]$$

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Our strategy

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Two consequences (subcritical case)

▷ Improved decay rate for the fast diffusion equation in rescaled variables

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Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of **(??)** with nonnegative initial datum $v_0 \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0 \, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x \, v_0 \, dx = 0$ and v_0 satisfies (H_A), then

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0] \, e^{-(4+\zeta)\, t} \quad \forall \, t \geq 0$$

▷ The **stability of the entropy - entropy production inequality** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4 + \zeta}\,\mathcal{I}[v]$$

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A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\; |x|\,f^{p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Deficit of the Sobolev inequality: $\delta[f] := \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \mathsf{S}_d^2\,\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2$

Theorem

Let $d \geq 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathsf{g}\, dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \leq A$$

we have

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \mathsf{S}_d^2\,\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2 \geq \frac{\mathcal{C}_\star(A)}{4 + \mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \frac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathsf{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\,\left(1 + A^{1/(2\,d)}\right)^{-1}$ and $\mathcal{C}_\star(0) > 0$ depends only on d

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Subcritical interpolation inequalities on the sphere
Gaussian interpolation inequalities on $\mathbb{R}^n$
More results on LSI and Gagliardo-Nirenberg inequalities

From interpolation inequalities on the sphere to Gaussian interpolation inequalities

Joint work with G. Brigati and N. Simonov
Gaussian interpolation inequalities
arXiv:2302.03926
C. R. Math. Acad. Sci. Paris 41, 2024

Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results
Annales IHP, Analyse non linéaire, 362, 2023
arXiv: 2211.13180

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Subcritical interpolation inequalities on the sphere

- ### *Gagliardo-Nirenberg-Sobolev inequality*

$$\|\nabla F\|^2_{L^2(\mathbb{S}^d)} \geq d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{L^p(\mathbb{S}^d)} - \|F\|^2_{L^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2, 2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \geq 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the *logarithmic Sobolev inequality*

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu \geq \frac{d}{2}\,\mathcal{E}_2[F] := \frac{d}{2}\int_{\mathbb{S}^d} F^2\, \log\left(\frac{F^2}{\|F\|^2_{L^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

[Bakry, Emery, 1984], [Bidaut-Véron, Véron, 1991], [Beckner, 1993]

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Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

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Theorem

Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in H^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d \, \mathcal{E}_p[F]$$

$$\geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|_{L^2(\mathbb{S}^d)}^4}{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \|F\|_{L^2(\mathbb{S}^d)}^2} + \|\nabla(\mathrm{Id} - \Pi_1) F\|_{L^2(\mathbb{S}^d)}^2 \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

▷ The result holds true for the logarithmic Sobolev inequality ($p = 2$), again with an explicit constant $\mathscr{S}_{d,2}$, for any finite dimension d

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Carré du champ – admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p(1-m)} \left(\Delta u + (m\,p - 1) \frac{|\nabla u|^2}{u} \right)$$

$$m_\pm(d, p) := \frac{1}{(d+2)\,p} \left(d\,p + 2 \pm \sqrt{d\,(p-1)\,(2\,d - (d-2)\,p)} \right)$$
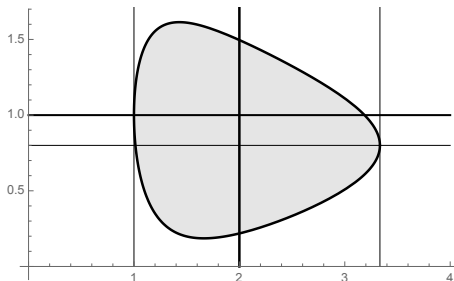

Figure: Case $d = 5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

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The "far away" regime and the "neighborhood" of $\mathcal{M}$

▷ If $\|\nabla F\|^2_{L^2(\mathbb{S}^d)} / \|F\|^2_{L^p(\mathbb{S}^d)} \geq \vartheta_0 > 0$, by the convexity of ψ

$$\|\nabla F\|^2_{L^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \geq d\,\|F\|^2_{L^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{L^2(\mathbb{S}^d)}}{\|F\|^2_{L^p(\mathbb{S}^d)}}\right)$$

$$\geq \frac{d}{\vartheta_0}\,\psi\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{L^2(\mathbb{S}^d)}$$

▷ From now on, we assume that $\|\nabla F\|^2_{L^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{L^p(\mathbb{S}^d)}$, take $\|F\|_{L^p(\mathbb{S}^d)} = 1$, learn that

$$\|\nabla F\|^2_{L^2(\mathbb{S}^d)} < \vartheta := \frac{d\,\vartheta_0}{d - (p-2)\,\vartheta_0} > 0$$

from the standard interpolation inequality and deduce from the Poincaré inequality that

$$\frac{d - \vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu\right)^2 \leq 1$$

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Partial decomposition on spherical harmonics

$\mathscr{M} = \Pi_0 F$ and $\Pi_1 F = \varepsilon \, \mathscr{Y}$ where $\mathscr{Y}(x) = \sqrt{\frac{d+1}{d}} \, x \cdot \nu$, $\nu \in \mathbb{S}^d$

$$F = \mathscr{M} \left(1 + \varepsilon \, \mathscr{Y} + \eta \, G \right)$$

Apply $c_{p,d}^{(-)} \, \varepsilon^6 \le \|1 + \varepsilon \, \mathscr{Y}\|_{\mathrm{L}^p(\mathbb{S}^d)}^p - \left(1 + a_{p,d} \, \varepsilon^2 + b_{p,d} \, \varepsilon^4 \right) \le c_{p,d}^{(+)} \, \varepsilon^6$
(with explicit constants) to $u = 1 + \varepsilon \, \mathscr{Y}$ and $r = \eta \, G$ the estimate

$$\|u + r\|_{\mathrm{L}^p(\mathbb{S}^d)}^2 - \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^2$$

$$\le \frac{2}{p} \, \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^{2-p} \, \left(p \int_{\mathbb{S}^d} u^{p-1} \, r \, d\mu + \frac{p}{2} \, (p-1) \int_{\mathbb{S}^d} u^{p-2} \, r^2 \, d\mu \right.$$

$$\left. + \sum_{2 < k < p} C_k^p \int_{\mathbb{S}^d} u^{p-k} \, |r|^k \, d\mu + K_p \int_{\mathbb{S}^d} |r|^p \, d\mu \right)$$

Estimate $\int_{\mathbb{S}^d} (1 + \varepsilon \, \mathscr{Y})^{p-1} \, G \, d\mu$, $\int_{\mathbb{S}^d} (1 + \varepsilon \, \mathscr{Y})^{p-k} \, |G|^k \, d\mu$, etc. to obtain (under the condition that $\varepsilon^2 + \eta^2 \sim \vartheta$)

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d \, \mathcal{E}_p[F] \ge \mathscr{M}^2 \left(A \, \varepsilon^4 - B \, \varepsilon^2 \, \eta + C \, \eta^2 - \mathcal{R}_{p,d} \left(\vartheta^p + \vartheta^{5/2} \right) \right)$$

$$\ge \mathscr{C} \left(\frac{\varepsilon^4}{\varepsilon^2 + \eta^2 + 1} + \eta^2 \right)$$

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Large dimensional limit

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d, d\mu_d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d, d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d, d\mu_d)}\right)$$

Theorem

Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ with compact support, $d \geq n$ and

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \ldots, \omega_n/\sqrt{d}\right)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\pi)^{-n/2}\, e^{-\frac{1}{2}|y|^2}\, dy$,

$$\lim_{d \to +\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d, d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d, d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d, d\mu_d)}\right)\right)$$

$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n, d\gamma)}\right)$$

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Gaussian interpolation inequalities on $\mathbb{R}^n$

- **Beckner interpolation inequalities**

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} \geq \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n, d\gamma)}\right)$$

▷ $1 \leq p < 2$ [Beckner, 1989], [Bakry, Emery, 1984]
▷ Poincaré inequality corresponding: $p = 1$
▷ *Gaussian logarithmic Sobolev inequality* $p \to 2$

- **Gaussian logarithmic Sobolev inequality $p \to 2$**

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} \geq \frac{1}{2}\int_{\mathbb{R}^n}|v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}}\right) d\gamma$$

$d\gamma(y) := (2\pi)^{-n/2} e^{-\frac{1}{2}|y|^2} dy$

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Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p(1-m)} \left(\mathcal{L}v + (m\,p-1)\frac{|\nabla v|^2}{v} \right) \quad \text{on} \quad \mathbb{R}^n$$

Ornstein-Uhlenbeck operator: $\mathcal{L} = \Delta - x \cdot \nabla$

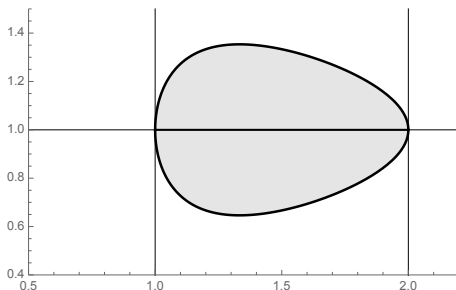$$m_{\pm}(p) := \lim_{d \to +\infty} m_{\pm}(d,p) = 1 \pm \frac{1}{p}\sqrt{(p-1)(2-p)}$$

Figure: The admissible parameters $1 \leq p \leq 2$ and m are independent of n

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A stability result for Gaussian interpolation inequalities

Theorem

For all $n \geq 1$, and all $p \in (1,2)$, there is an explicit constant $c_{n,p} > 0$ such that, for all $v \in \mathrm{H}^1(d\gamma)$,

$$\|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}^2 - \frac{1}{p-2} \left(\|v\|_{\mathrm{L}^p(\mathbb{R}^n, d\gamma)}^2 - \|v\|_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}^2 \right)$$

$$\geq c_{n,p} \left(\|\nabla(\mathrm{Id} - \Pi_1)v\|_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}^2 + \frac{\|\nabla \Pi_1 v\|_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}^4}{\|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}^2 + \|v\|_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}^2} \right)$$

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L² stability of LSI: comments

[JD, Esteban, Figalli, Frank, Loss]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} - \pi \int_{\mathbb{R}^n} u^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)}} \right) d\gamma$$

$$\geq \frac{\beta \pi}{2} \inf_{a \in \mathbb{R}^d, \, c \in \mathbb{R}} \int_{\mathbb{R}^n} |u - c \, e^{a \cdot x}|^2 \, d\gamma$$

- One dimension is lost (for the manifold of invariant functions) in the limiting process
- Euclidean forms of the stability
- The $\dot{H}^1(\mathbb{R}^n)$ does not appear, it gets lost in the limit $d \to +\infty$
- $\int_{\mathbb{R}^n} |\nabla(u - c \, e^{a \cdot x})|^2 \, d\gamma$? False, but makes sense under additional assumptions. Some results based on the Ornstein-Uhlenbeck flow and entropy methods: [Fathi, Indrei, Ledoux, 2016], [JD, Brigati, Simonov]
- Taking the limit is difficult because of the lack of compactness

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More results on
logarithmic Sobolev inequalities

Joint work with G. Brigati and N. Simonov
Stability for the logarithmic Sobolev inequality
Journal of Functional Analysis, 287, oct. 2024
arXiv: 2411.13271

▷ *Entropy methods, with constraints*

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Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon \, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{L^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} d(u_\varepsilon, w)^\alpha \leq \frac{1}{2} \, \varepsilon^4 + O(\varepsilon^6)$$

$\mathscr{M} := \big\{ w_{a,c} \, : \, (a,c) \in \mathbb{R}^d \times \mathbb{R} \big\}$ where $w_{a,c}(x) = c \, e^{-a \cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_{L^2(\mathbb{R}^d)} = 1$ and $\|x \, u\|^2_{L^2(\mathbb{R}^d)} \leq d$, we have

$$\|\nabla u\|^2_{L^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, d\gamma \geq \frac{1}{2 \, d} \left(\int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, d\gamma \right)^2$$

and, with $\psi(s) := s - \frac{d}{4} \, \log \big(1 + \frac{4}{d} \, s \big)$,

$$\|\nabla u\|^2_{L^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, d\gamma \geq \psi \left(\|\nabla u\|^2_{L^2(\mathbb{R}^d, d\gamma)} \right)$$

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Stability under log-concavity

> **Theorem**
>
> *For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that*
>
> $$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\gamma = (1,0) \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \leq \mathsf{K}$$
>
> *we have*
>
> $$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,d\gamma \geq 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{432\,\mathsf{K}} \approx 1 + \frac{0.00231481}{\mathsf{K}}$$

Self-improving Poincaré inequality and stability for LSI

[Fathi, Indrei, Ledoux, 2016]

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Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x) \, |u|^2 \, d\gamma = (1, 0), \quad \int_{\mathbb{R}^d} |u|^2 \, e^{\varepsilon |x|^2} \, d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}^2 \geq \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, d\gamma$$

with $\mathscr{C} = 1 + \frac{\mathscr{C}_\star(K_\star) - 1}{1 + R^2 \, \mathscr{C}_\star(K_\star)}$, $K_\star := \max\left(d, \frac{(d+1) R^2}{1 + R^2}\right)$ if $\mathrm{supp}(u) \subset B(0, R)$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

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Stability results for Sobolev, logarithmic Sobolev, and related inequalities

Proceedings of the Summer School "Direct and Inverse Problems with Applications, and Related Topics" August 19-23, 2024

arXiv: 2411.13271

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
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More related papers can be found at

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For final versions, use `Dolbeault` as login and `Jean` as password

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Thank you for your attention !