Relative equilibria in continuous stellar dynamics

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Relative equilibria [J. Campos, M. del Pino, and J. Dolbeault. Relative equilibria in continuous stellar dynamics. Communications in Mathematical Physics, 2010]

Flat systems [J. Dolbeault and J. Fernández, Localized minimizers of flat rotating gravitational systems, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis, 2008]

Numerical schemes for the computation of relative equilibria [J. Dolbeault, J. Fernández and J. Salomon, work in progress]

Symmetry breaking issues in functional inequalities, flows and rates of convergence

- [J. Dolbeault, M.J. Esteban, M. Loss, G. Tarantello, and A. Tertikas] About existence, symmetry and symmetry breaking for Caffarelli-Kohn-Nirenberg inequalities
- [J. Dolbeault, M.J. Esteban, A. Laptev, M. Loss] Spectral consequences
- [M. Bonforte, J. Dolbeault, G. Grillo, J.L. Vázquez] Fast diffusion equations
- [J. Dolbeault, M.J. Esteban, M. Kowalczyk, M. Loss] Flow methods
- [A. Blanchet, J. Dolbeault, M. Escobedo, J. Fernández] and [J. Campos, M. del Pino, and J. Dolbeault] Keller-Segel models in chemotaxis

Introduction: N-body problems and mean-field kinetic equations

- A first statement: there are non radially symmetric critical points
- Relative equilibria: examples and (partial) classification for systems of point particles
- Kinetic equations: extending the notion of relative equilibria to continuum mechanics
- Results for kinetic / diffusion equations
- The variational approach: heuristics
- The variational approach: a sketch of the proofs
- Flat systems: results and numerical computation
- Concluding remarks: symmetry breaking

Gravitational (non-relativistic) Vlasov-Poisson system in $\mathbb{R}^3$

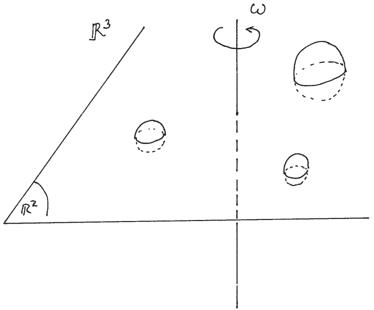
$$\begin{cases} \partial_t F + w \cdot \nabla_z F - \nabla_z \Phi \cdot \nabla_w F = 0 \\ \Delta \Phi = \int_{\mathbb{R}^3} F \, dw \end{cases} \tag{1}$$

Theorem

For any $N \geq 2$, any $p \in (1, 5)$, any positive numbers $\lambda_1, \lambda_2, \ldots \lambda_N$ and any $\omega > 0$ small enough, there is a solution F^ω of (1) which is a relative equilibrium with angular velocity ω whose support has N disjoint connected components, each of them with mass m_i^ω such that

$$\lim_{\omega \to 0_+} m_i^\omega = \lambda_i^{(3-p)/2} \, m_* =: m_i$$

for some positive constant m_*. The center of mass $z_i^\omega(t)$ of each component is such that $\lim_{\omega \to 0_+} \omega^{2/3} \, z_i^\omega(t) =: z_i(t)$ is a relative equilibrium of the N-body Newton's equations with gravitational interaction


$\mathbb{R}^3$
ω
$\mathbb{R}^2$

Systems of discrete particles: the N-body problem in gravitation

... many solutions are known

- No stationary (time independent) solutions
- Periodic solutions in Hamiltonian dynamics: [Ekeland et al.]
- Choregraphies: [Chenciner et al.], [Terracini et al.]

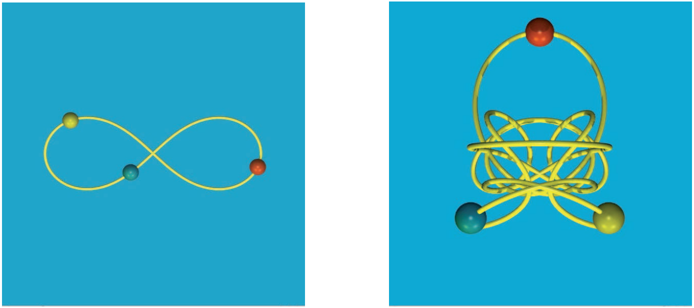


Figure: Choregraphies, pictures taken from S. Terracini's web page

http://www.matapp.unimib.it/∼suster/files/index.html

Consider N point particles with masses m_i located at $z_i(t) \in \mathbb{R}^3$ subject to Newton's equations

$$m_i \frac{d^2 z_i}{dt^2} = \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{4\pi} \frac{z_j - z_i}{|z_j - z_i|^3} \tag{2}$$

Ansatz: the system is stationary in a reference frame rotating at constant angular velocity $\Omega = \omega\, \mathrm{e}_3$

Notation: $x' = (x^1, x^2, 0) = x - (x \cdot \mathrm{e}_3)\, \mathrm{e}_3$, a change of coordinates

$$x^3 = z^3 \;, \quad x^1 + i\, x^2 = e^{i \omega t}(z^1 + i\, z^2)$$

provides Newton's equations in a rotating frame

$$\frac{d^2 x_i}{dt^2} = \sum_{i \neq j = 1}^{N} \frac{m_j}{4\pi} \frac{x_j - x_i}{|x_j - x_i|^3} + \omega^2\, x_i' + 2\, \Omega \wedge \frac{dx_i}{dt}$$

We look for stationary solutions in the rotating frame: relative equilibria

The configuration is *central* and planar: critical points of the function

$$\mathcal{V}_\omega(x_1', x_2', \dots x_N') := -\frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{|x_j' - x_i'|} - \frac{\omega^2}{2} \sum_{i=1}^{N} m_i\, |x_i'|^2$$

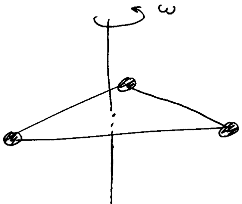
- All masses m_i are equal to some $m > 0$ and x'_i are located at the summits of a regular polygon, where $r = |x'_i|$ is adjusted so that

$$\frac{d}{dr}\left[\frac{a_N}{4\pi}\,\frac{m}{r}+\frac{1}{2}\,\omega^2\,r^2\right]=0 \quad \text{with} \quad a_N := \frac{1}{\sqrt{2}}\sum_{j=1}^{N-1}\frac{1}{\sqrt{1-\cos\left(2\pi j/N\right)}}$$

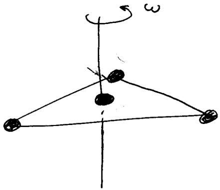
gives a *Lagrange solution* with $r = r(N,\omega) := \left(\frac{a_N\,m}{4\pi\,\omega^2}\right)^{1/3}$
[Perko-Walter]: all masses have to be equal if $N \geq 4$
Scale invariance:

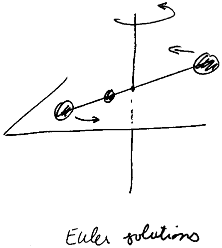
$$r(N,\varepsilon^{3/2}\,\omega)=\frac{1}{\varepsilon}\,r(N,\omega) \quad \forall\,\varepsilon>0$$

If $\nabla\mathcal{V}_\omega(x'_1,x'_2,\ldots x'_N)=0$
then $\nabla\mathcal{V}_{\varepsilon^{3/2}\,\omega}(\varepsilon^{-1}\,x'_1,\varepsilon^{-1}\,x'_2,\ldots\,\varepsilon^{-1}\,x'_N)=0$:
the study of the critical points
of $\mathcal{V}_\omega$ can be reduced to the case $\omega=1$



- $N - 1$ points particles of same mass are located at the summits of a regular centered polygon and one more point particle stands at the center (with not necessarily the same mass as the other ones). A solution is then found again by adjusting the size of the polygon
- The *Euler–Moulton solutions* are made of aligned points





Relative equilibria are critical points of the function $\mathcal{V}_\omega : (\mathbb{R}^2)^N \to \mathbb{R}$

$$\mathcal{V}_\omega(x_1', x_2', \ldots x_N') := -\frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{|x_j' - x_i'|} - \tfrac{1}{2}\,\omega^2 \sum_{i=1}^{N} m_i\, |x_i'|^2$$

Generic case: all masses are different

- $N = 2$:

$$|x_1 - x_2| = \left(\frac{M}{4\pi\,\omega^2}\right)^{1/3} \quad \text{and} \quad m_1\, x_1 + m_2\, x_2 = 0\,, \quad \text{with } M = m_1 + m_2$$

- $N = 3$:
  - *Lagrange solutions*: masses are located at the vertices of an equilateral triangle, and the distance between each point is $(M/(4\pi\,\omega^2))^{1/3}$ with $M = m_1 + m_2 + m_3$: two classes of solutions corresponding to the two orientations of the triangle when labeled by the masses
  - *Euler solutions* are made of aligned points and provide three classes of critical points, one for each ordering of the masses on the line

- $N \geq 4$: solutions made of aligned points are *Moulton's solutions*
- $N \geq 4$: Lagrange solutions (all particles are located at the vertices of a regular N-polygon) exists if and only if all masses are equal

- Standard variational setting [Smale]: for $m = (m_1, \ldots m_N) \in \mathbb{R}_+^N$, consider the manifold $(q_1, \ldots q_N) \in \mathbb{R}^{2N}$ such that

$$\sum_{i=1}^N m_i\, q_i = 0\,,\ \tfrac{1}{2}\sum_{i=1}^N m_i\, |q_i|^2 = 1\,,\ q_i \neq q_j \text{ if } i \neq j$$

quotiented by the equivalence classes associated to the invariances: rotations and scalings

$\dim(\mathcal{S}_m) = 2N - 3$, relative equilibria are critical points on $\mathcal{S}_m$ of the potential

$$U_m(q_1, \ldots q_N) = -\frac{1}{8\pi}\sum_{i\neq j=1}^N \frac{m_i\, m_j}{|q_j - q_i|}$$

Relative equilibria: classification (3/3)

For $N \geq 4$, various classification results have been achieved by [Palmore]

- For $N \geq 3$, the index of a relative equilibrium is always greater or equal than $N-2$. This bound is achieved by Moulton's solutions
- For $N \geq 3$, there are at least $\mu_i(N) := \binom{N}{i}(N-1-i)\,(N-2)\,!$ distinct relative equilibria in $\mathcal{S}_m$ of index $2N-4-i$ if U_m is a Morse function. As a consequence, there are at least
$$\sum_{i=0}^{N-2} \mu_i(N) = [2^{N-1}(N-2)+1]\,(N-2)\,!$$
distinct relative equilibria in $\mathcal{S}_m$ if U_m is a Morse function
- For every $N \geq 3$ and for almost all masses $m \in \mathbb{R}^N_+$, U_m is a Morse function
- There are only finitely many classes of relative equilibria for every $N \geq 3$ and for almost all masses $m \in \mathbb{R}^N_+$
- If $N \geq 4$, the set of masses for which there exist degenerate classes of relative equilibria has positive k-dimensional Hausdorff measure if $0 \leq k \leq N-1$

The kinetic problem

Gravitational Vlasov-Poisson system (with centrifugal and Coriolis forces):

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f - \omega^2 x' \cdot \nabla_v f + 2\,\Omega \wedge v \cdot \nabla_v f = 0$$

$$\Delta\phi = \rho := \int_{\mathbb{R}^3} f\, dv$$

Boundary conditions: $\phi = -\frac{1}{4\pi\,|\cdot|} * \rho$

Change of coordinates: $f(t,x,v) = F(t,z,w)$, $\phi(t,x) = \Phi(t,z)$

$$x = \exp(\omega\, t\, A)\, z\,, \quad v = \Omega \wedge x + \exp(\omega\, t\, A)\, w \quad \text{with} \quad A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For some arbitrary convex function β, critical points of the *free energy*

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\, dx\, dv + \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(|v|^2 - \omega^2\, |x'|^2\right) f\, dx\, dv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx$$

give stationary solutions under the constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\, dx\, dv = M$

For $\omega \neq 0$: no minimizers

[Binney-Tremaine] A typical example of such a function is

$$\beta(f) = \kappa\, f^q$$

A critical point of $\mathcal{F}$ such that $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\; dx\, dv = M$ is given by

$$f(x,v) = \gamma\left(\lambda + \frac{1}{2}\,|v|^2 + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)$$

where $\gamma(s) = (\beta')^{-1}(-s)$: $\gamma(s) = (-s)_+^{1/(q-1)}$
The problem is reduced to solve a non-linear Poisson equation

$$\Delta\phi = g\big(\lambda + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\big)\;\chi_{\mathrm{supp}(\rho)}$$

$$g(\mu) := \int_{\mathbb{R}^3} \gamma\big(\mu + \frac{1}{2}\,|v|^2\big)\; dv$$

Variational approach:

$$\mathcal{J}[\phi] := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\; dx + \int_{\mathbb{R}^3} G\left(\lambda + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)\, dx - \int_{\mathbb{R}^3} \lambda\,\rho\; dx$$

where $\lambda = \lambda[x,\phi]$ is now a functional which is constant on each connected component K_i of the support of $\rho(x)$

The total mass is $M = \sum_{i=1}^N m_i$

$$\int_{K_i} g\left(\lambda_i + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)\,dx = m_i$$

$$\mathcal{J}[\phi] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + \sum_{i=1}^N \left[\int_{K_i} G\left(\lambda_i + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)\,dx - m_i\,\lambda_i\right]$$

Heuristics. The various components K_i are far away from each other so that the dynamics of their center of mass is described by the N-body point particles system, at first order. On each component K_i, the solution is a perturbation of an isolated minimizer of $\mathcal{F}$ (without angular rotation) under the constraint that the mass is equal to m_i. Alternatively, we consider a critical point of $\mathcal{J}$ obtained as the perturbation of a superposition of single components critical points of $\mathcal{J}$ of mass m_i, provided the centers of mass x_i of each of the components are close enough of a critical point of $\mathcal{V}_\omega$, with $\omega > 0$, small

$\omega = 0$: [Guo et al.], [Lemou-Méhats-Raphaël], [Rein et al.], [Sánchez et al.], [Soler et al.], [Schaeffer], [Wolansky], [JD-Fernández] + [Kurth]

The first result

The spatial density $\rho^\omega := \int_{\mathbb{R}^3} f^\omega \, dv$ has exactly N disjoint connected components K_i^ω and

$$m_i^\omega = \int_{K_i^\omega} \rho^\omega \, dx \,, \quad z_i^\omega(t) = \exp\left(-\omega\, t\, A\right)\, x_i^\omega \quad \text{where} \quad x_i^\omega := \frac{1}{m_i^\omega} \int_{K_i^\omega} x\, \rho^\omega \, dx$$

$$\rho_i(x) := \frac{1}{\lambda_i^p}\, \rho^\omega \left(\lambda_i^{(1-p)/2}(x + x_i^\omega)\right) \chi_{K_i^\omega} \left(\lambda_i^{(1-p)/2}(x + x_i^\omega)\right)$$

converges to a density function $\rho_* = (w-1)_+^p$ given by

$$-\Delta w = (w-1)_+^p \quad \text{in } \mathbb{R}^3 \,, \qquad \lim_{|x|\to\infty} w(x) = 0$$

Theorem

For any $N \geq 2$, any $p \in (1,5)$, any positive numbers $\lambda_1, \lambda_2, \ldots \lambda_N$ and any $\omega > 0$ small enough, there is a relative equilibrium solution F^ω s.t.

$$\lim_{\omega \to 0_+} m_i^\omega = \lambda_i^{(3-p)/2}\, m_* =: m_i$$

for $m_* = \int_{\mathbb{R}^3} \rho_* \, dx$. The center of mass $z_i^\omega(t)$ of each component is such that $\lim_{\omega\to 0_+} \omega^{2/3}\, z_i^\omega(t) =: z_i(t)$ is a relative equilibrium of the N-body problem (Newton's equations)

Let $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, fix $\lambda_1, \dots \lambda_N$ and $\omega > 0$, small: the problem is

$$-\Delta u = \sum_{i=1}^{N} \rho_i \quad \text{in } \mathbb{R}^3 , \quad \rho_i := \left(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^p \chi_i \tag{3}$$

where χ_i denotes the characteristic function of K_i
Boundary condition $\lim_{|x|\to\infty} u(x) = 0$

Mass and center of mass associated to each component by

$$m_i := \int_{\mathbb{R}^3} \rho_i \, dx \quad \text{and} \quad x_i := \frac{1}{m_i} \int_{\mathbb{R}^3} x\, \rho_i \, dx$$

We shall say that two solutions u_1 and u_2 are equivalent if there is a rotation $R \in \mathrm{SO}(2) \times \{\mathrm{Id}\}$, *i.e.* a rotation in the plane orthogonal to the direction of rotation, such that $u_2(x) = u_1(R\,x)$ for any $x \in \mathbb{R}^3$. We shall say that u_1 and u_2 are distinct if they are not equivalent

Theorem

- *For ω small enough, and for almost every positive $(\lambda_1, \dots \lambda_N) \in (0, \infty)^N$, (3) has at least $[2^{N-1}(N-2)+1]\,(N-2)\,!$ distinct solutions which continuoulsy depend on ω*
- *If u^ω is such a solution, there are points $\xi_1^\omega, \dots \xi_N^\omega \in \mathbb{R}^3$ such that as $\omega \to 0$, $|\xi_j^\omega - \xi_i^\omega| \to \infty$ for any $j \neq i$ and $u^\omega(\cdot + \xi_i^\omega)$ locally converges to the unique radial nonnegative solution of*

$$-\Delta w = (w - \lambda_i)_+^p \quad \text{in } \mathbb{R}^3$$

- *For $\omega > 0$ small enough, the support of ρ^ω has N connected components*

$$\lim_{\omega \to 0} m_i^\omega = \lambda_i^{(3-p)/2}\, m_* =: m_i \quad \text{and} \quad \lim_{\omega \to 0} \omega^{2/3}\, \xi_i^\omega := \xi_i$$

and $(\xi_1, \dots \xi_N)$ is a relative equilibrium with masses $(m_i)_{1 \le i \le N}$

The scaling invariance is recovered only in the limit $\omega \to 0_+$

$$-\Delta w = (w-1)_+^p \quad \text{in } \mathbb{R}^3 \tag{4}$$

Lemma

Under the condition $\lim_{|x|\to\infty} w(x) = 0$*, Equation (4) has a unique solution, up to translations, which is positive and radially symmetric. It is a non-degenerate critical point of* $\frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2\, dx + \frac{1}{p+1}\int_{\mathbb{R}^3} (w-1)_+^{p+1}\, dx$

$$w_i(x) := w^{\lambda_i}(x-\xi_i)\ , \quad W_\xi := \sum_{i=1}^N w_i$$

Compatibility condition: for a large, fixed $\mu > 0$, and all small $\omega > 0$,

$$|\xi_i| < \mu\,\omega^{-\frac{2}{3}}\ , \quad |\xi_i - \xi_j| > \mu^{-1}\omega^{-\frac{2}{3}} \tag{5}$$

Ansatz: we look for a solution of (3) of the form

$$u = W_\xi + \phi$$

with $\operatorname{supp}(w^{\lambda_i} - \lambda_i)_+ \subset B(0,R)$ for all $i = 1, \dots N$ for $R > 0$ large

$$\Delta\phi + \sum_{i=1}^N p\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p-1}\chi_i\,\phi = -\mathrm{E} - \mathrm{N}[\phi]$$

We want to solve

$$\mathrm{L}[\phi] := \Delta\phi + \sum_{i=1}^{N} p\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p-1} \chi_i\,\phi = -\mathrm{E} - \mathrm{N}[\phi]$$

where

$$\mathrm{E} := \Delta W_\xi + \sum_{i=1}^{N} \left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p}\ \chi_i$$

$$\mathrm{N}[\phi] := \sum_{i=1}^{N} \Big[\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2 + \phi\right)_+^{p} - \left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p} - p\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p-1} \phi \Big]\,\chi_i$$

The variational scheme

- A linear theory
- The projected nonlinear problem (Lagrange multipliers)
- The variational reduction
- A variational approach in finite dimension

[Floer-Weinstein 1986] + many others...

$$\|\phi\|_* = \sup_{x\in\mathbb{R}^3} \left(\sum_{i=1}^N |x-\xi_i| + 1 \right) |\phi(x)| \,, \quad \|h\|_{**} = \sup_{x\in\mathbb{R}^3} \left(\sum_{i=1}^N |x-\xi_i|^4 + 1 \right) |h(x)|$$

Consider the *projected problem*

$$\mathrm{L}[\phi] = h + \sum_{i=1}^N \sum_{j=1}^3 c_{ij}\, Z_{ij}\, \chi_i$$

where $Z_{ij} := \partial_{x_j} w_i$, subject to orthogonality conditions

$$\int_{\mathbb{R}^3} \phi\, Z_{ij}\, \chi_i \, dx = 0 \quad \forall\, i\,,\ j = 1, 2 \ldots N$$

Lemma

Assume that (5) holds. Given h with $\|h\|_{**} < +\infty$, there is a unique solution $\phi =: \mathrm{T}[h]$ and there exists a positive constant C, which is independent of ξ such that, for $\omega > 0$ small enough,

$$\|\phi\|_* \le C\, \|h\|_{**}$$

The projected nonlinear problem

Find ϕ with $\|\phi\|_* < +\infty$, solution of

$$\mathrm{L}[\phi] = -\mathrm{E} - \mathrm{N}[\phi] + \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij}\, Z_{ij} \chi_i$$

such that $\phi(x) \to 0 \quad$ as $|x| \to +\infty$ and

$$\int_{\mathbb{R}^3} \phi\, Z_{ij}\, \chi_i\, dx = 0 \quad \text{for all } i\,,\ j\,.$$

To do this analysis we have to measure the size of the error E. We recall that

$$\mathrm{E} = \sum_{i=1}^{N} \left[(w_i + \sum_{j\neq i} w_j - \lambda_i + \tfrac{1}{2}\, \omega^2\, |x'|^2)_+^p - (w_i - \lambda_i)_+^p \right] \chi_i$$

$$= \sum_{i=1}^{N} p(w_i - \lambda_i + t\, (\sum_{j\neq i} w_j + \tfrac{1}{2}\, \omega^2\, |x'|^2))_+^{p-1} (\sum_{j\neq i} w_j + \tfrac{1}{2}\, \omega^2\, |x'|^2)\, \chi_i$$

for some $t \in (0, 1)$

$$|\mathsf{E}| \le C \sum_{i=1}^{N} \Big[\sum_{j\neq i} \frac{1}{|\xi_i - \xi_j|} + \frac{1}{2}\omega^2 |\xi_i|^2 \Big] \chi_i \ \le C\omega^{\frac{2}{3}} \sum_{i=1}^{N} \chi_i$$

Thus $\|\mathsf{E}\|_{**} \le C\omega^{\frac{2}{3}}$. Moreover, for $\|\phi\|_* \le 1$,

$$|\mathsf{N}[\phi]| \le C \sum_{i=1}^{N} |\phi|^{\gamma} \chi_i \ , \quad \gamma = \min\{p, 2\}$$

$\|\mathsf{N}[\phi]\|_{**} \le C\|\phi\|_*^{\gamma}$ and $\|N(\phi_1) - N(\phi_2)\|_{**} \le o(1)\|\phi_1 - \phi_2\|_*^{\gamma}$
We look for a fixed point $\phi = \mathcal{A}[\phi] := -\mathsf{T}\big[\mathsf{E} + \mathsf{N}[\phi]\big]$ on the region

$$\mathcal{B} = \Big\{ \phi \ : \ \|\phi\|_* \le K\omega^{\frac{2}{3}} \Big\}$$

Lemma

$\exists\,!\phi_\xi = \phi(\xi_1, \dots \xi_k)$ which depends continuously on its parameters for the $\|\ \|_*$-norm and $\|\phi_\xi\|_* \le C\omega^{\frac{2}{3}}$,

$$\phi_{e^{\theta A}\xi} = \phi_\xi(e^{-\theta A}\cdot) \quad \textit{and} \quad c_{i\cdot}(e^{\theta A}\xi) = e^{-\theta A} c_{i\cdot}$$

Lemma

We have that $c_{ij} = 0$ for all i , j if and only if the k-tuple $(\xi_1, \dots \xi_N)$ is a critical point of the functional

$$(\xi_1, \dots \xi_N) \mapsto \Lambda(\xi_1, \dots \xi_N) := J(W_\xi + \phi_\xi)$$

A Taylor expansion

$$J(W_\xi) = J(W_\xi + \phi_\xi) - DJ(W_\xi + \phi_\xi)[\phi_\xi] + \frac{1}{2} \int_0^1 D^2 J(W_\xi + (1-t)\phi_\xi)[\phi_\xi]^2 \, dt$$

$$D^2 J(W_\xi + (1-t)\phi_\xi)[\phi_\xi]^2 = O(\omega^{\frac{4}{3}})$$

$$\Lambda(\xi) = \sum_{i=1}^{N} \lambda_i^{5-p} e_* - \Big[\frac{1}{2} \sum_{i \neq j} \frac{m_i \, m_j}{|\xi_i - \xi_j|} + \tfrac{1}{2} \, \omega^2 \sum_{i=1}^{N} m_i \, |\xi_i|^2 \Big] + O(\omega^{\frac{4}{3}})$$

In the region

$$\mathfrak{B} = \left\{ (\xi_1, \dots \xi_k) \ : \ |\xi_i - \xi_j| > \rho\, \omega^{-\frac{2}{3}}, \quad |\xi_i| < \rho^{-1}\, \omega^{-\frac{2}{3}} \text{ for all } i\,,\ j \right\}$$

where $\rho > 0$ is chosen small enough and fixed, we have that

$$\sup_{\mathfrak{B}} \Lambda > \sup_{\partial \mathfrak{B}} \Lambda$$

so that this functional has a local maximum somewhere in $\mathfrak{B}$. Hence a critical point of this functional does exist in $\mathfrak{B}$ $\square$

Lemma

For any $\lambda_i > 0$, ξ_i, we have found a critical point of Λ, i.e. a solution of

$$\mathrm{L}[\phi] = -\mathrm{E} - \mathrm{N}[\phi]$$

$$U_m(q_1, \dots q_N) = -\frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{|q_j - q_i|}$$

is a Morse function on $\mathcal{S}_m$. Take a local system of coordinates $(\eta_1, ..., \eta_{2N-4})$ on a neighborhood of a critical point $\bar{q}$ and for $\alpha > 0$, let

$$\xi(\alpha, p, \eta) = (\alpha(q_1(\eta) + p), \dots \alpha(q_N(\eta) + p))$$

$$\Phi(\alpha, p, \eta) = \omega^{-\frac{2}{3}} \Lambda(\xi(\omega^{-\frac{2}{3}} \alpha, p, q(\eta)))$$

$$\nabla\Phi(\alpha, p, \eta) = \nabla\Big(-\frac{1}{2} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{\alpha |q_j(\eta) - q_i(\eta)|} - \frac{1}{2}\alpha^2 \sum_{i=1}^{N} m_i\, |q_i(\eta)|^2 + \frac{1}{2}\alpha^2 |p|^2 \sum_{i=1}^{N} m_i \Big) + O(\omega^{\frac{2}{3}})$$

$(\bar{\lambda}^{\frac{1}{3}}, 0, \bar{\eta})$ is nondegenerate, so the local degree $\deg(\nabla\tilde{\Phi}, \mathcal{U}, 0)$ is well defined and nonzero: there exists a critical point (α^*, p^*, η^*) as $\omega \to 0$

A flat model: theory and numerical results

[JD-Fernández] Written in cartesian coordinates, the equation is

$$\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\,\omega\, v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = 0$$
$$\phi = -\frac{1}{4\pi\,|x|} * \int_{\mathbb{R}^2} f\, dv$$

where $a \wedge b := a^{\perp} \cdot b = a_1\, b_2 - a_2\, b_1$ and $x,\ v \in \mathbb{R}^2$

Definition

A localized minimizer is a critical point ρ of $\mathcal{G}_\omega$ which is compactly supported in a ball $B(0, R-\varepsilon)$ for some $R > 0$ and $\varepsilon \in (0, R)$, and which is a minimizer of $\mathcal{G}_\omega$ restricted to the set

$$\{\rho \in L^1_+(\mathbb{R}^2)\ :\ \mathrm{supp}(\rho) \subset B(0,R) \text{ and } \int_{\mathbb{R}^2} \rho\, dx = M\}$$

Theorem

For any $M > 0$, there exists $\omega_*(M) = \omega_* > 0$ and $\omega^*(M) = \omega^* > 0$ with $\omega_* \leq \omega^*$ such that

(i) *If $\omega \in [-\omega_*, \omega_*]$, the reduced free energy functional*

$$\mathcal{G}_\omega[\rho] := \frac{\kappa}{m-1} \int_{\mathbb{R}^2} \rho^m \, dx - \frac{\omega^2}{2} \int_{\mathbb{R}^2} |x|^2 \rho \, dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\, \rho(y)}{|x-y|} \, dx \, dy$$

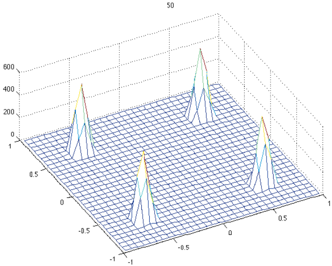
admits a localized minimizer

(ii) *If $|\omega| > \omega^*$, $\mathcal{G}_\omega[\rho]$ admits no localized minimizer*

More detailed results in the radial case. How does symmetry breaking occur ?

Goal: investigate the energy landscape [JD-Fernández-Salomon]

- Local minimizers (under appropriate constraints) have a very small basin of attraction
- Compact support has to be enforced at each step
- Iteration method inspired by mean-field models in quantum mechanics work, with similar difficulties: the Cancès-LeBris method of relaxation has to be introduced to achieve convergence



Trying to understand the symmetry breaking

Polytropic case: $\beta(f) = \kappa\, f^q$, without angular velocity: minimizing the free energy

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\, dx\, dv + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f\, dx\, dv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx$$

amounts to control the potential energy and find the optimal function in

$$\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx \leq C\, \|f\|_1^{\alpha} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\, dx\, dv\right)^{\beta(2-\alpha)} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2\, f\, dx\, dv\right)^{\beta(2-\alpha)}$$

- Explanation: With $\rho(x) = \int_{\mathbb{R}^3} f(x,v)\, dv$, HLS inequality

$$\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\,\rho(y)}{4\pi\,|x-y|}\, dx\, dy \leq C_{HLS}\, \|\rho\|_1^{\alpha}\, \|\rho\|_q^{2-\alpha}$$

$+$ interpolation $\|\rho\|_q \leq C_{p,q} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\, dx\, dv\right)^{\beta} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2\, f\, dx\, dv\right)^{\beta}$

- Relative equilibria: The constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x'|^2 f\, dx\, dv = Const$ (+ localization in x-space) may break the symmetry

Conclusion: symmetry breaking and stability

- In nonlinear PDEs, symmetry breaking usually occurs because of a competition between the nonlinearity and an external potential
- A classical example is the (PDE) Hénon problem
- The case covered by the theorem of [Gidas-Ni-Nirenberg] is the trivial one: the nonlinearity and an external potential cooperate
- Symmetry breaking is usually achieved by eigenvalue considerations, but here we have an example based on multiscale analysis

Some observations on symmetry

- A ground state is either a "minimizer" of a free energy / Casimir functional or a positive solutions with *ad hoc* properties at infinity...
- ... and it is (radially) symmetric
  - if Schwarz symmetrization can be applied
  - if uniqueness (rigidity) holds
  - or if the Gidas, Ni and Nirenberg (GNN) theorem applies (regularity, monotonicity, positivity properties are required)
- GNN, Maximum Principle and positivity of the lowest eigenvalue are intimately related
- Semi-linear elliptic form of the equation (for the potential) is mandatory

Related issues and open problems (1/2)

- Main issue (especially in gravitation) is dynamical / orbital stability Constrained (localized) minimization and mass transport methods [McCann] but new ideas are required. Dynamical (in)stability of (rotating) solutions: how to measure it ? how to characterize the threshold cases ?
- Building other examples of time-periodic solutions (choregraphies ?) would be an intermediate step. How can we give a variational characterization ? in which space ? equipped with which kind of distance ? (mass transportation / Wasserstein is a possible framework)
- Violent relaxation and Landau damping: which class of solutions ? Relative equilibria are a (probably rather non-generic) obstruction

Related issues and open problems (2/2)

- The semi-linear elliptic point of view: is there a cascade of symmetry breakings ? How to characterize the threshold for symmetry breaking ?
- Rigidity results: in which regimes do we get (local / global) uniqueness results ?
- In case of stationary solutions, the potential is related with the free energy by a duality relation (cf. chemotaxis). Can we extract some information of the potential for the evolution equation ?
- Kinetic models with relaxation terms: diffusion limits ? [Chavanis, Laurençot], [J.D., D. Oelz, P. Markowich, C. Schmeiser] hypocoercivity results ? [J.D., C. Mouhot, C. Schmeiser], [Villani] hypoellipticity ? rates of convergence ?

Thank you for your attention !