Stability in functional inequalities

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4. Sobolev and LSI

Outline

Constructive stability results for the Sobolev and the logarithmic Sobolev inequalities

1. Stability for Sobolev and LSI on $\mathbb{R}^d$
   - Main results, optimal dimensional dependence
2. Explicit stability result for the Sobolev inequality: proof
   - Sketch of the proof and definitions
   - Competing symmetries
   - The main steps of the proof
3. More results on logarithmic Sobolev inequalities

Explicit stability results for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

arXiv: 2209.08651

A short review on improvements and stability for some interpolation inequalities

arXiv: 2402.08527

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\,\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d}\inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F-\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

Stability for the logarithmic Sobolev inequality

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 69], [Costa, 85] *Cf.* [Villani, 08]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathscr{M} := \left\{w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R}\right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a\cdot x} \quad \forall\, x \in \mathbb{R}^n$$

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance, in $\mathrm{W}^{1,1}$, *etc.*

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

Explicit stability result for the Sobolev inequality

Proof

Sketch of the proof

Goal: prove that there is an ***explicit*** constant $\beta > 0$ such that for all $d \geq 3$ and all $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

$$\|\nabla f\|_2^2 \geq S_d \|f\|_{2^*}^2 + \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

Part 1. We show the inequality for nonnegative functions far from $\mathcal{M}$

... the far away regime

Make it *constructive*

Part 2. We show the inequality for nonnegative functions close to $\mathcal{M}$

... the local problem

Get *explicit* estimates and remainder terms

Part 3. We show that the inequality for nonnegative functions implies the inequality for functions without a sign restriction, up to an acceptable loss in the constant

... dealing with sign-changing functions

Some definitions

What we want to minimize is

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathrm{d}(f,\mathcal{M})^2} \quad f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$$

where

$$\mathrm{d}(f,\mathcal{M})^2 := \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ up to a *conformal transformation*, we assume that $\mathrm{d}(f,\mathcal{M})^2 = \|\nabla f - \nabla g_*\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ with

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

▷ use the *inverse stereographic* projection

$$F(\omega) = \frac{f(x)}{g_*(x)} \quad x \in \mathbb{R}^d \text{ with } \begin{cases} \omega_j = \frac{2\,x_j}{1+|x|^2} & \text{if } 1 \le j \le d \\ \omega_{d+1} = \frac{1-|x|^2}{1+|x|^2} & \end{cases}$$

The problem on the unit sphere

Stability inequality on the unit sphere $\mathbb{S}^d$ for $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$

$$\int_{\mathbb{S}^d} \left(|\nabla F|^2 + \mathsf{A}\,|F|^2\right) d\mu - \mathsf{A} \left(\int_{\mathbb{S}^d} |F|^{2^*}\, d\mu\right)^{2/2^*} \geq \frac{\beta}{d} \inf_{G \in \mathscr{M}} \left\{ \|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathsf{A}\,\|F - G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right\}$$

with $\mathrm{A} = \frac{1}{4}\, d\,(d-2)$ and a manifold $\mathscr{M}$ of optimal functions made of

$$G(\omega) = c\,(a + b \cdot \omega)^{-\frac{d-2}{2}} \quad \omega \in \mathbb{S}^d \quad (a, b, c) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}$$

- make the reduction of a ***far away problem*** to a local problem *constructive...* on $\mathbb{R}^d$
- make the analysis of the ***local problem*** *explicit...* on $\mathbb{S}^d$

Competing symmetries

- ***Rotations on the sphere*** combined with stereographic and inverse stereographic projections. Let $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$

$$(Uf)(x) := \left(\frac{2}{|x - e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x - e_d|^2}, \dots, \frac{x_{d-1}}{|x - e_d|^2}, \frac{|x|^2 - 1}{|x - e_d|^2}\right)$$

$$\mathcal{E}(Uf) = \mathcal{E}(f)$$

- ***Symmetric decreasing rearrangement*** $\mathcal{R}f = f^*$

$$f \text{ and } f^* \text{ are equimeasurable}$$
$$\|\nabla f^*\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

The method of ***competing symmetries***

Theorem (Carlen, Loss, 1990)

Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$. The sequence $f_n = (\mathcal{R}U)^n f$ is such that $\lim_{n\to+\infty} \|f_n - g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 0$. If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)})_{n\in\mathbb{N}}$ is a non-increasing sequence

Useful preliminary results

- $\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$ where $h_f = \|f\|_{2^*}\, g_*/\|g_*\|_{2^*} \in \mathcal{M}$
- $(\|\nabla f_n\|_2^2)_{n\in\mathbb{N}}$ is a nonincreasing sequence

Lemma

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g\in\mathcal{M},\,\|g\|_{2^*}=1} \left(f, g^{2^*-1}\right)^2$$

Corollary

$\big(\mathrm{d}(f_n,\mathcal{M})\big)_{n\in\mathbb{N}}$ is strictly decreasing, $n \mapsto \sup_{g\in\mathcal{M}_1} \left(f_n, g^{2^*-1}\right)$ is strictly increasing, and

$$\lim_{n\to\infty} \mathrm{d}(f_n,\mathcal{M})^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\, \|h_f\|_{2^*}^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\, \|f\|_{2^*}^2$$

but no monotonicity for $n \mapsto \mathcal{E}(f_n) = \frac{\|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\, \|f_n\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathrm{d}(f_n,\mathcal{M})^2}$

Part 1: Global to local reduction

The ***local problem***

$$\mathscr{I}(\delta) := \inf \left\{ \mathcal{E}(f) \,:\, f \geq 0\,,\ \mathsf{d}(f,\mathcal{M})^2 \leq \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right\}$$

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the ***far away regime***

$$\mathsf{d}(f,\mathcal{M})^2 = \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

for some $\delta \in (0,1)$

Let $f_n = (\mathcal{R}U)^n f$. There are two cases:

- (Case 1) $\mathsf{d}(f_n,\mathcal{M})^2 \geq \delta\, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$
- (Case 2) for some $n \in \mathbb{N}$, $\mathsf{d}(f_n,\mathcal{M})^2 < \delta\, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Global to local reduction – Case 1

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the far away regime

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

Lemma

Let $f_n = (\mathcal{R}U)^n f$ *and* $\delta \in (0,1)$*. If* $\mathrm{d}(f_n, \mathcal{M})^2 \ge \delta \, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ *for all* $n \in \mathbb{N}$*, then*

$$\mathcal{E}(f) \ge \delta$$

$$\lim_{n\to+\infty} \|\nabla f_n\|_2^2 \le \frac{1}{\delta} \lim_{n\to+\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta}\left(\lim_{n\to+\infty} \|\nabla f_n\|_2^2 - S_d \, \|f\|_{2^*}^2\right)$$

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \ge \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \ge \frac{\|\nabla f_n\|_2^2 - S_d \, \|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \underset{n\to+\infty}{\ge} \delta$$

Global to local reduction – Case 2

$$\mathscr{I}(\delta) := \inf \left\{ \mathcal{E}(f) \,:\, f \geq 0\,,\ \mathrm{d}(f,\mathcal{M})^2 \leq \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right\}$$

Lemma

$$\mathcal{E}(f) \geq \delta\, \mathscr{I}(\delta)$$

if $\inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\, \|\nabla f_{n_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ *and* $\inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} < \delta\, \|\nabla f_{n_0+1}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Adapt a strategy due to Christ: build a (semi-) *continuous* rearrangement *flow* $(\mathsf{f}_\tau)_{n_0\leq\tau<n_0+1}$ with $\mathsf{f}_{n_0} = U f_n$ such that $\|\mathsf{f}_\tau\|_{2^*} = \|f\|_2$, $\tau \mapsto \|\nabla \mathsf{f}_\tau\|_2$ is nonincreasing, and $\lim_{\tau\to n_0+1} \mathsf{f}_\tau = f_{n_0+1}$

$$\mathcal{E}(f) \geq 1 - S_d\, \frac{\|f\|^2_{2^*}}{\|\nabla f\|^2_2} \geq 1 - S_d\, \frac{\|\mathsf{f}_{\tau_0}\|^2_{2^*}}{\|\nabla \mathsf{f}_{\tau_0}\|^2_2} = \delta\, \mathcal{E}(f_{\tau_0}) \geq \delta\, \mathscr{I}(\delta)$$

Altogether: if $\mathrm{d}(f,\mathcal{M})^2 > \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, then $\mathcal{E}(f) \geq \min\left\{\delta, \delta\, \mathscr{I}(\delta)\right\}$

Part 2: The (simple) Taylor expansion

Proposition

Let $(X, d\mu)$ *be a measure space and* u, $r \in \mathrm{L}^q(X, d\mu)$ *for some* $q \geq 2$ *with* $u \geq 0$, $u + r \geq 0$ *and* $\int_X u^{q-1}\, r\, d\mu = 0$

▷ *If* $q = 6$, *then*

$$\|u+r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left(5\int_X u^{q-2}\, r^2\, d\mu + \tfrac{20}{3}\int_X u^{q-3}\, r^3\, d\mu \right.$$
$$\left. + 5\int_X u^{q-4}\, r^4\, d\mu + 2\int_X u^{q-5}\, r^5\, d\mu + \tfrac{1}{3}\int_X r^6\, d\mu\right)$$

▷ *If* $3 \leq q \leq 4$, *then*

$$\|u+r\|_q^2 - \|u\|_q^2$$
$$\leq \|u\|_q^{2-q}\left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{(q-1)\,(q-2)}{3}\int_X u^{q-3}\, r^3\, d\mu + \frac{2}{q}\int_X |r|^q\, d\mu\right)$$

▷ *If* $2 \leq q \leq 3$, *then*

$$\|u+r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q}\left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{2}{q}\int_X r_+^q\, d\mu\right)$$

Corollary

For all $\nu>0$ *and for all* $r\in\mathrm{H}^1(\mathbb{S}^d)$ *satisfying* $r\ge -1$,

$$\left(\int_{\mathbb{S}^d}|r|^q\,d\mu\right)^{2/q}\le\nu^2\quad\text{and}\quad\int_{\mathbb{S}^d}r\,d\mu=0=\int_{\mathbb{S}^d}\omega_j\,r\,d\mu\quad\forall\,j=1,\dots d+1$$

if $d\mu$ *is the uniform probability measure on* $\mathbb{S}^d$, *then*

$$\int_{\mathbb{S}^d}\left(|\nabla r|^2+\mathrm{A}\,(1+r)^2\right)d\mu-\mathrm{A}\left(\int_{\mathbb{S}^d}(1+r)^q\,d\mu\right)^{2/q}\ge\mathrm{m}(\nu)\int_{\mathbb{S}^d}\left(|\nabla r|^2+\mathrm{A}\,r^2\right)d\mu$$

$$\begin{aligned}
&\mathrm{m}(\nu):=\tfrac{4}{d+4}-\tfrac{2}{q}\,\nu^{q-2} && \textit{if}\quad d\ge 6\\
&\mathrm{m}(\nu):=\tfrac{4}{d+4}-\tfrac{1}{3}\,(q-1)\,(q-2)\,\nu-\tfrac{2}{q}\,\nu^{q-2} && \textit{if}\quad d=4,\,5\\
&\mathrm{m}(\nu):=\tfrac{4}{7}-\tfrac{20}{3}\,\nu-5\,\nu^2-2\,\nu^3-\tfrac{1}{3}\,\nu^4 && \textit{if}\quad d=3
\end{aligned}$$

An explicit expression of $\mathscr{I}(\delta)$ if $\nu>0$ is small enough so that $\mathrm{m}(\nu)>0$

Part 3: Removing the positivity assumption

Take $f = f_+ - f_-$ with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 1$ and define $m := \|f_-\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and $1 - m = \|f_+\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} > 1/2$. The positive concave function

$$h_d(m) := m^{\frac{d-2}{d}} + (1-m)^{\frac{d-2}{d}} - 1$$

satisfies

$$2\,h_d(1/2)\,m \le h_d(m)\,, \quad h_d(1/2) = 2^{2/d} - 1$$

With $\delta(f) = \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, one finds $g_+ \in \mathcal{M}$ such that

$$\delta(f) \ge C^{d,\mathrm{pos}}_{\mathrm{BE}}\,\|\nabla f_+ - \nabla g_+\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{2\,h_d(1/2)}{h_d(1/2)+1}\,\|\nabla f_-\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

and therefore

$$C^d_{\mathrm{BE}} \ge \tfrac{1}{2}\,\min\left\{\max_{0<\delta<1/2}\,\delta\,\mathscr{I}(\delta),\,\frac{2\,h_d(1/2)}{h_d(1/2)+1}\right\}$$

Part 2, refined: The (complicated) Taylor expansion

To get a dimensionally sharp estimate, we expand $(1+r)^{2^*}-1-2^*r$ with an accurate remainder term for all $r\ge-1$

$$r_1 := \min\{r,\gamma\}\,,\quad r_2 := \min\left\{(r-\gamma)_+, M-\gamma\right\}\quad\text{and}\quad r_3 := (r-M)_+$$

with $0<\gamma<M$. Let $\theta=4/(d-2)$

Lemma

Given $d\ge6$, $r\in[-1,\infty)$, and $\overline{M}\in[\sqrt{e},+\infty)$, we have

$$\begin{aligned}(1+r)^{2^*}&-1-2^*r\\ &\le\tfrac12\,2^*\,(2^*-1)\,(r_1+r_2)^2+2\,(r_1+r_2)\,r_3+\left(1+C_M\,\theta\,\overline{M}^{-1}\ln\overline{M}\right)r_3^{2^*}\\ &\qquad+\left(\tfrac32\,\gamma\,\theta\,r_1^2+C_{M,\overline{M}}\,\theta\,r_2^2\right)\mathbb{1}_{\{r\le M\}}+C_{M,\overline{M}}\,\theta\,M^2\,\mathbb{1}_{\{r>M\}}\end{aligned}$$

where all the constants in the above inequality are explicit

There are constants ϵ_1, ϵ_2, k_0, and $\epsilon_0 \in (0,1/\theta)$, such that

$$\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathrm{A}\,\|1+r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \\ \ge \frac{4\,\epsilon_0}{d-2}\left(\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) + \sum_{k=1}^3 I_k$$

$$I_1 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_1|^2 + \mathrm{A}\,r_1^2\right)d\mu - \mathrm{A}\,(2^*-1+\epsilon_1\,\theta)\int_{\mathbb{S}^d} r_1^2\,d\mu + \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d}(r_2^2\ldots$$

$$I_2 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_2|^2 + \mathrm{A}\,r_2^2\right)d\mu - \mathrm{A}\left(2^*-1+(k_0+C_{\epsilon_1,\epsilon_2})\,\theta\right)\int_{\mathbb{S}^d} r_2^2\,d\mu$$

$$I_3 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_3|^2 + \mathrm{A}\,r_3^2\right)d\mu - \tfrac{2}{2^*}\,\mathrm{A}\,(1+\epsilon_2\,\theta)\int_{\mathbb{S}^d} r_3^{2^*}\,d\mu - \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d} r_3^2\,d\mu$$

- spectral gap estimates : $I_1 \ge 0$
- Sobolev inequality : $I_3 \ge 0$
- improved spectral gap inequality using that $\mu\big(\{r_2>0\}\big)$ is small: $I_2 \ge 0$

L^2 stability of LSI: comments

[JD, Esteban, Figalli, Frank, Loss]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma \geq \frac{\beta\,\pi}{2} \inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}} \int_{\mathbb{R}^n} |u - c\,e^{a\cdot x}|^2\,d\gamma$$

- The $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ does not appear, it gets lost in the limit $d \to +\infty$
- Two proofs. Taking the limit is difficult because of the lack of compactness
- One dimension is lost (for the manifold of invariant functions) in the limiting process
- Euclidean forms of the stability
- $\int_{\mathbb{R}^n} |\nabla(u - c\,e^{a\cdot x})|^2\,d\gamma$? False, but makes sense under additioal assumptions. Some results based on the Ornstein-Uhlenbeck flow and entropy methods: [Fathi, Indrei, Ledoux, 2016], [JD, Brigati, Simonov, 2023-24]

More results on logarithmic Sobolev inequalities

Joint work with G. Brigati and N. Simonov
Stability for the logarithmic Sobolev inequality
Journal of Functional Analysis, 287 (8): 110562, Oct. 2024

▷ *Entropy methods, with constraints*

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} \mathsf{d}(u_\varepsilon, w)^\alpha \le \frac{1}{2}\,\varepsilon^4 + O\big(\varepsilon^6\big)$$

$\mathscr{M} := \big\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \big\}$ where $w_{a,c}(x) = c\, e^{-a\cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ *such that* $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ *and* $\|x\,u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, *we have*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log|u|^2\, d\gamma \ge \frac{1}{2\,d} \left(\int_{\mathbb{R}^d} |u|^2 \log|u|^2\, d\gamma \right)^2$$

and, with $\psi(s) := s - \frac{d}{4} \log\left(1 + \frac{4}{d}\, s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log|u|^2\, d\gamma \ge \psi\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \right)$$

Stability under log-concavity

Cheeger's inequality (for log-concave measures) and [Fathi, Indrei, Ledoux, 2016]

Theorem

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that

$$\int_{\mathbb{R}^d} (1, x)\,|u|^2\,d\gamma = (1, 0) \quad \textit{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \le \mathsf{K}$$

we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\,d\gamma \ge 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{432\,\mathsf{K}} \approx 1 + \frac{0.00231481}{\mathsf{K}}$$

Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\gamma = (1,0)\,,\quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \leq d\,,\quad \int_{\mathbb{R}^d} |u|^2\,e^{\,\varepsilon\,|x|^2}\,d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} \geq \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,d\gamma$$

with $\mathscr{C} = 1 + \frac{\mathscr{C}_\star(\mathsf{K}_\star)-1}{1+R^2\,\mathscr{C}_\star(\mathsf{K}_\star)}$, $\mathsf{K}_\star := \max\left(d, \frac{(d+1)\,R^2}{1+R^2}\right)$ *if* $\mathrm{supp}(u) \subset B(0,R)$

Compact support: [Lee, Vázquez, 2003]; [Chen, Chewi, Niles-Weed,2021]

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Thank you for your attention !