Stability in functional inequalities

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3. Sphere

Outline

Stability results on the sphere and on the Gaussian space seen as an infinite dimensional limit of spheres

Logarithmic Sobolev and Gagliardo-Nirenberg-Sobolev on the sphere

A joint work with G. Brigati and N. Simonov

Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results

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(Improved) logarithmic Sobolev inequality

On the sphere $\mathbb{S}^d$ with $d \geq 1$

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \tag{LSI}$$

$d\mu$: uniform probability measure; equality case: constant functions
Optimal constant: test functions $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$, $\nu \in \mathbb{S}^d$, $\varepsilon \to 0$
▷ *improved inequality* under an appropriate *orthogonality condition*

Theorem

Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that $\int_{\mathbb{S}^d} x\, F\, d\mu = 0$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \geq \frac{2}{d+2} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu$$

Improved ineq. $\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq \left(\frac{d}{2} + 1\right) \int_{\mathbb{S}^d} F^2 \log\left(F^2 / \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) d\mu$

Logarithmic Sobolev inequality: stability (1)

What if $\int_{\mathbb{S}^d} x\, F\, d\mu \neq 0$? Take $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$ and let $\varepsilon \to 0$

$$\|\nabla F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{2} \int_{\mathbb{S}^d} F_\varepsilon^2 \log\left(\frac{F_\varepsilon^2}{\|F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu = O(\varepsilon^4) = O\left(\|\nabla F_\varepsilon\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Such a behaviour is in fact optimal: *carré du champ* method

Proposition

Let $d \geq 1$, $\gamma = 1/3$ *if* $d = 1$ *and* $\gamma = (4\,d - 1)\,(d-1)^2/(d+2)^2$ *if* $d \geq 2$. *Then, for any* $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ *with* $\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$ *we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log F^2\, d\mu \geq \frac{1}{2}\, \frac{\gamma\, \|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\gamma\, \|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d}$$

In other words, if $\|\nabla F\|_{\mathrm{L}^2(\mathbb{S}^d)}$ is small

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - \tfrac{d}{2} \int_{\mathbb{S}^d} F^2 \log F^2\, d\mu \geq \tfrac{\gamma}{2\,d}\, \|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)} + o\left(\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Logarithmic Sobolev inequality: stability (2)

Let $\Pi_1 F$ denote the orthogonal projection of a function $F \in \mathrm{L}^2(\mathbb{S}^d)$ on the spherical harmonics corresponding to the first eigenvalue on $\mathbb{S}^d$

$$\Pi_1 F(x) = \frac{x}{d+1} \cdot \int_{\mathbb{S}^d} y\, F(y)\, d\mu(y) \quad \forall\, x \in \mathbb{S}^d$$

▷ a global (and detailed) stability result

Theorem

Let $d \ge 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu$$
$$\ge \mathscr{S}_d \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{2}\, \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_d > 0$

Gagliardo-Nirenberg(-Sobolev) inequalities

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \tag{GNS}$$

for any $p \in [1,2) \cup (2, 2^*)$, with $d\mu$: uniform probability measure
$2^* := 2\,d/(d-2)$ if $d \geq 3$ and $2^* = +\infty$ otherwise
Optimal constant: test functions $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$, $\nu \in \mathbb{S}^d$, $\varepsilon \to 0$
logarithmic Sobolev inequality: obtained by taking the limit as $p \to 2$

Theorem

Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that $\int_{\mathbb{S}^d} x\, F \, d\mu = 0$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \geq \mathscr{C}_{d,p} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu$$

with $\mathscr{C}_{d,p} = \frac{2\,d - p\,(d-2)}{2\,(d+p)}$

With $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$, the deficit is of order ε^4 as $\varepsilon \to 0$

Proposition

Let $d \geq 1$ and $p \in (1,2) \cup (2, 2^*)$. There is a convex function ψ on $\mathbb{R}^+$ with $\psi(0) = \psi'(0) = 0$ such that, for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \geq \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \; \psi\left(\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}} \right)$$

This is also a consequence of the *carré du champ* method, with an explicit construction of ψ
There is no orthogonality constraint

Gagliardo-Nirenberg inequalities: stability (2)

As in the case of the logarithmic Sobolev inequality, the improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined

Theorem

Let $d \ge 1$ *and* $p \in (1,2) \cup (2,2^*)$*. For any* $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$*, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \ge \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1) F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

Generalized entropy functionals

$$\mathcal{E}_p[F] := \frac{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2} \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[F] := \frac{1}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

▷ The key idea is to evolve these quantities by a diffusion flow and prove the inequalities as a consequence of a monotonicity along the flow

Heat flow estimates: fixing parameters

Let us consider the constant γ given by

$$\gamma := \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p) \quad \text{if} \quad d \geq 2\,, \qquad \gamma := \frac{p-1}{3} \quad \text{if} \quad d = 1$$

and the *Bakry-Emery exponent*

$$2^\# := \frac{2\,d^2+1}{(d-1)^2}$$

Let us define

$$s_\star := \frac{1}{p-2} \quad \text{if} \quad p > 2 \quad \text{and} \quad s_\star := +\infty \quad \text{if} \quad p \leq 2$$

For any $s \in [0, s_\star)$, let

$$\varphi(s) = \frac{1-(p-2)\,s-(1-(p-2)\,s)^{-\frac{\gamma}{p-2}}}{2-p-\gamma} \qquad \text{if} \quad \gamma \neq 2-p \quad \text{and} \quad p \neq 2$$

$$\varphi(s) = \tfrac{1}{2-p}\,(1+(2-p)\,s)\log\left(1+(2-p)\,s\right) \qquad \text{if} \quad \gamma = 2-p \neq 0$$

$$\varphi(s) = \tfrac{1}{\gamma}\,(e^{\gamma\,s}-1) \qquad \text{if} \quad p = 2$$

Heat flow: stability estimates

[JD, Esteban, Kowalczyk, Loss], [JD, Esteban 2020]

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\varphi\left(\frac{\mathcal{E}_p[F]}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

Since $\varphi(0)=0$, $\varphi'(0)=1$, and φ is convex increasing, with an asymptote at $s=s_\star$ if $p\in(2,2^\#)$, we know that $\varphi:[0,s_\star)\to\mathbb{R}^+$ is invertible and $\psi:\mathbb{R}^+\to[0,s_\star)$, $s\mapsto\psi(s):=s-\varphi^{-1}(s)$, is convex increasing with $\psi(0)=\psi'(0)=0$, $\lim_{t\to+\infty}\big(t-\psi(t)\big)=s_\star$, and

$$\psi''(0)=\varphi''(0)=\tfrac{(d-1)^2}{(d+2)^2}\,(2^\#-p)\,(p-1)>0\quad\forall\,p\in(1,2^\#)$$

First stability estimates for Gagliardo-Nirenberg inequalities

Proposition

With the above notations, $d\ge1$ and $p\in(1,2^\#)$, we have

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[F]\ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\quad\forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

If $p=2$, notice that ψ is explicit and given by

A simpler reformulation

Let $d \geq 1$, $\gamma \neq 2-p$ as above

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{d}{2-p-\gamma}\left(\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|F\|^{2-\frac{2\gamma}{2-p}}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|F\|^{\frac{2\gamma}{2-p}}_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d)$$

[JD, Esteban 2020]
which is a refinement of the standard Gagliardo-Nirenberg inequality

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu \geq \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

... with the restriction $p < 2^\# := \frac{2\,d^2+1}{(d-1)^2} < 2^* := \frac{2\,d}{d-2}$ if $d \geq 3$

So far, we considered only the case $1 \leq p < 2^{\#}$. Our goal is to cover also the subcritical range $p \in [2^{\#}, 2^*)$

$$\varphi_{m,p}(s) := \int_0^s \exp\left[-\zeta\left(\left(1 - (p-2)\,z\right)^{1-\delta} - \left(1 - (p-2)\,s\right)^{1-\delta}\right)\right] dz$$

provided m is admissible, that is,

$$m \in \mathscr{A}_p := \mathscr{A}_p := \left\{ m \in [m_-(d,p), m_+(d,p)] \,:\, \tfrac{2}{p} \leq m < 1 \text{ if } p < 4 \right\}$$

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p + 2 \pm \sqrt{d\,(p-1)\left(2\,d - (d-2)\,p\right)}\right)$$

The parameters δ and ζ are defined by

$$\delta := 1 + \tfrac{(m-1)\,p^2}{4\,(p-2)}$$

$$\zeta := \frac{(d+2)^2\,p^2\,m^2 - 2\,p\,(d+2)\,(d\,p+2)\,m + d^2\left(5\,p^2 - 12\,p + 8\right) + 4\,d\,(3-2\,p)\,p + 4}{(1-m)\,(d+2)^2\,p^2}$$

Nonlinear diffusion flow: stability estimates

We consider the inverse function $\varphi_{m,p}^{-1}:\mathbb{R}^+\to[0,s_\star)$ and $\psi_{m,p}(s):=s-\varphi_{m,p}^{-1}(s)$. Exactly as in the case $m=1$, we have the *improved entropy – entropy production inequality*

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\varphi_{m,p}\left(\frac{\mathcal{E}_p[F]}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\quad\forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

Proposition

With above notations, $d\ge1$, $p\in(2,2^*)$ and $m\in\mathscr{A}_p$, we have

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[F]\ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi_{m,p}\left(\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\quad\forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

The function $\varphi_{m,p}$ can be expressed in terms of the *incomplete* Γ *function* while $\psi_{m,p}$ is known only implicitly

(Improved) logarithmic Sobolev inequality

Where is the flow ?

▷ The case of the logarithmic Sobolev inequality is a limit case corresponding to $p = 2$ of the Gagliardo-Nirenberg-Sobolev inequalities for $p \neq 2$

▷ We use the fast diffusion flow ($m < 1$), porous medium flow ($m > 1$) and as a limit case the heat flow ($m = 1$) given by

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

where Δ is the Laplace-Beltrami operator on $\mathbb{R}^d$

... how do we relate ρ and F ?

Algebraic preliminaries

$$\mathrm{L}v := \mathrm{H}v - \frac{1}{d}\,(\Delta v)\,g_d \quad \text{and} \quad \mathrm{M}v := \frac{\nabla v \otimes \nabla v}{v} - \frac{1}{d}\,\frac{|\nabla v|^2}{v}\,g_d$$

With $a : b = a^{ij}\,b_{ij}$ and $\|a\|^2 := a : a$, we have

$$\|\mathrm{L}v\|^2 = \|\mathrm{H}v\|^2 - \tfrac{1}{d}\,(\Delta v)^2\,, \quad \|\mathrm{M}v\|^2 = \left\|\tfrac{\nabla v \otimes \nabla v}{v}\right\|^2 - \tfrac{1}{d}\,\tfrac{|\nabla v|^4}{v^2} = \tfrac{d-1}{d}\,\tfrac{|\nabla v|^4}{v^2}$$

A first identity

$$\int_{\mathbb{S}^d} \Delta v\,\frac{|\nabla v|^2}{v}\,d\mu = \frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathrm{M}v\|^2\,d\mu - 2\int_{\mathbb{S}^d}\mathrm{L}v : \frac{\nabla v \otimes \nabla v}{v}\,d\mu\right)$$

Second identity (Bochner-Lichnerowicz-Weitzenböck formula

$$\int_{\mathbb{S}^d} (\Delta v)^2\,d\mu = \frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathrm{L}v\|^2\,d\mu + d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$$

An estimate

With $b=(\kappa+\beta-1)\,\frac{d-1}{d+2}$ and $c=\frac{d}{d+2}\,(\kappa+\beta-1)+\kappa\,(\beta-1)$

$$\mathscr{K}[v]:=\int_{\mathbb{S}^d}\left(\Delta v+\kappa\,\frac{|\nabla v|^2}{v}\right)\left(\Delta v+(\beta-1)\,\frac{|\nabla v|^2}{v}\right)d\mu$$
$$=\frac{d}{d-1}\,\|\mathrm{L}v-\,b\,\mathrm{M}v\|^2+\left(c-b^2\right)\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu+d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$$

Let $\kappa=\beta\,(p-2)+1$. The condition $\gamma:=c-b^2\ge 0$ amounts to

$$\gamma=\tfrac{d}{d+2}\,\beta\,(p-1)+\big(1+\beta\,(p-2)\big)\,(\beta-1)-\left(\tfrac{d-1}{d+2}\,\beta\,(p-1)\right)^2$$

Lemma

$$\mathscr{K}[v]\ge\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu+d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$$

Hence $\mathscr{K}[v]\ge d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$ if $\gamma\ge 0$, which is a condition on β

... and finally, here is the flow

$$\frac{\partial u}{\partial t}=u^{-p\,(1-m)}\left(\Delta u+(m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$

Check: if $m=1+\frac{2}{p}\left(\frac{1}{\beta}-1\right)$, then $\rho=u^{\beta p}$ solves $\frac{\partial\rho}{\partial t}=\Delta\rho^m$

$$\frac{d}{dt}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}=0\,,\quad\frac{d}{dt}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}=2\,(p-2)\int_{\mathbb{S}^d}u^{-\,p\,(1-m)}\,|\nabla u|^2\,d\mu\,,$$

$$\frac{d}{dt}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}=-\,2\int_{\mathbb{S}^d}\left(\beta\,v^{\beta-1}\,\frac{\partial v}{\partial t}\right)\left(\Delta v^\beta\right)d\mu=-\,2\,\beta^2\,\mathscr{K}[v]$$

Lemma

Assume that $p\in(1,2^*)$ and $m\in[m_-(d,p),m_+(d,p)]$. Then

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[u]\right)\le-\,\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu\le0$$

Admissible parameters

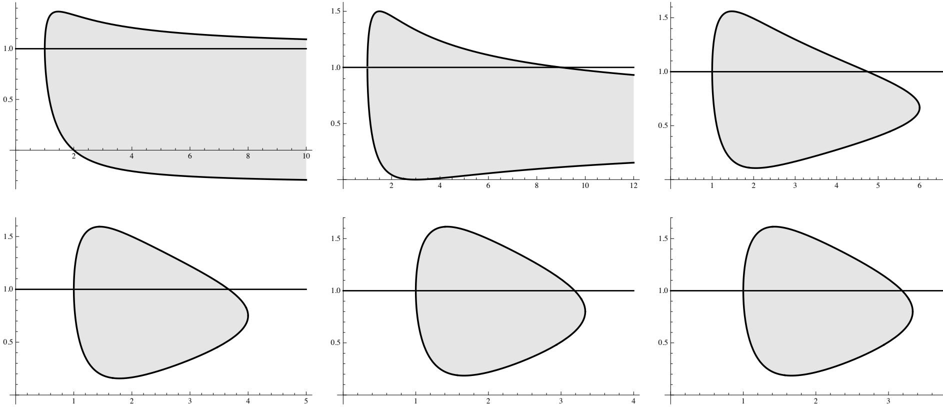

Figure: $d = 1, 2, 3$ (first line) and $d = 4, 5$ and 10 (second line): the curves $p \mapsto m_\pm(p)$ determine the admissible parameters (p, m) [JD, Esteban, 2019]

Inequalities and improved inequalities

From $\frac{1}{2\,\beta^2}\,\frac{d}{dt}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u]\right) \le -\,\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu \le 0$ and $\lim_{t\to+\infty}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u]\right) = 0$, we deduce the inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[u]$$

[Bakry-Emery, 1984], [Bidaut-Véron, Véron, 1991], [Beckner,1993]

... but we can do better

[Demange, 2008], [JD, Esteban, Kowalczyk, Loss]

Improved inequalities: flow estimates

With $\|u\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, consider the *entropy* and the *Fisher information*

$$\mathsf{e} := \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \text{and} \quad \mathsf{i} := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

Lemma

With $\delta := \frac{2-(4-p)\,\beta}{2\,\beta\,(p-2)}$ *if* $p > 2$, $\delta := 1$ *if* $p \in [1,2]$

$$(\mathsf{i} - d\,\mathsf{e})' \le \frac{\gamma\,\mathsf{i}\,\mathsf{e}'}{\big(1-(p-2)\,\mathsf{e}\big)^{\delta}}$$

$$\Longrightarrow \; \|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\psi\left(\tfrac{1}{d}\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d) \text{ s.t. } \|F\|_{\mathrm{L}^p(\mathbb{S}^d)}$$

With $\overline{F} := \int_{\mathbb{S}^d} F\,d\mu$, this improves upon [Frank, 2022]

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge \mathsf{c}_\star(d,p)\,\frac{\left(\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F-\overline{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

Improved interpolation inequalities under orthogonality

Decomposition of $\mathrm{L}^2(\mathbb{S}^d, d\mu)$ into spherical harmonics

$$\mathrm{L}^2(\mathbb{S}^d, d\mu) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell$$

Let Π_k be the orthogonal projection onto $\bigoplus_{\ell=1}^{k} \mathcal{H}_\ell$

Theorem

Assume that $d \geq 1$, $p \in (1, 2^*)$ and $k \in \mathbb{N} \setminus \{0\}$ be an integer. For some $\mathscr{C}_{d,p,k} \in (0,1)$ with $\mathscr{C}_{d,p,k} \leq \mathscr{C}_{d,p,1} = \frac{2\,d - p\,(d-2)}{2\,(d+p)}$

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - d\,\mathcal{E}_p[F] \geq \mathscr{C}_{d,p,k} \int_{\mathbb{S}^d} \left|\nabla(\mathrm{Id} - \Pi_k)\,F\right|^2 d\mu$$

Proof

Using the Funk-Hecke formula as in [Lieb, 1983] and following [Beckner,1993], we learn that

$$\mathcal{E}_p[F] \leq \sum_{j=1}^{\infty} \zeta_j(p) \int_{\mathbb{S}^d} |F_j|^2 \, d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

hold for any $p \in (1,2) \cup (2,2^*)$ with

$$\zeta_j(p) := \frac{\gamma_j\left(\frac{d}{p}\right) - 1}{p-2} \quad \text{and} \quad \gamma_j(x) := \frac{\Gamma(x)\,\Gamma(j+d-x)}{\Gamma(d-x)\,\Gamma(x+j)}$$

▷ Use convexity estimates and monotonicity properties of the coefficients

Proof of the stability results

It remains to combine the *improved entropy – entropy production inequality* (carré du champ method) and the *improved interpolation inequalities under orthogonality constraints*

Theorem

Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\, \mathcal{E}_p[F] \\ \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

N.B. This relies on the computations of [Frank, 2022] (Bianchi-Egnell) made quantitative

The "far away" regime and the "neighborhood" of $\mathcal{M}$

▷ If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \,/\, \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \vartheta_0 > 0$, by the convexity of $\psi_{m,p}$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi_{m,p}\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)$$
$$\ge \frac{d}{\vartheta_0}\,\psi_{m,p}\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

▷ From now on, we assume that $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$, take $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, learn that

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta := \frac{d\,\vartheta_0}{d-(p-2)\,\vartheta_0} > 0$$

from the standard interpolation inequality and deduce from the Poincaré inequality that

$$\frac{d-\vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu\right)^2 \le 1$$

Partial decomposition on spherical harmonics

With $\mathscr{M} = \Pi_0 F$ and $\Pi_1 F = \varepsilon\,\mathscr{Y}$ where $\mathscr{Y}(x) = \sqrt{\frac{d+1}{d}}\,x\cdot\nu$ for some given $\nu\in\mathbb{S}^d$

$$F = \mathscr{M}\left(1+\varepsilon\,\mathscr{Y}+\eta\,G\right)$$

For some explicit constants $a_{p,d}$, $b_{p,d}$ and $c_{p,d}^{(\pm)}$

$$c_{p,d}^{(-)}\,\varepsilon^6 \le \|1+\varepsilon\,\mathscr{Y}\|_{\mathrm{L}^p(\mathbb{S}^d)}^p - \left(1+a_{p,d}\,\varepsilon^2+b_{p,d}\,\varepsilon^4\right) \le c_{p,d}^{(+)}\,\varepsilon^6$$

We apply to $u = 1+\varepsilon\,\mathscr{Y}$ and $r = \eta\,G$ the estimate

$$\begin{aligned}\|u+r\|_{\mathrm{L}^p(\mathbb{S}^d)}^2 \le \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^2 &\\ +\tfrac{2}{p}\,\|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^{2-p}\Big(p\int_{\mathbb{S}^d}u^{p-1}\,r\,d\mu+\tfrac{p}{2}\,(p-1)\int_{\mathbb{S}^d}u^{p-2}\,r^2\,d\mu&\\ +\textstyle\sum_{2<k<p}C_k^p\int_{\mathbb{S}^d}u^{p-k}\,|r|^k\,d\mu+K_p\int_{\mathbb{S}^d}|r|^p\,d\mu\Big)&\end{aligned}$$

Estimate various terms like $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-1}\;G\,d\mu$, $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-2}\;|G|^2\,d\mu$, $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-k}\;|G|^k\,d\mu$, *etc.*

... conclusion

With explicit expressions for all constants we obtain

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d \, \mathcal{E}_p[F] \geq \mathscr{M}^2 \left(A \, \varepsilon^4 - B \, \varepsilon^2 \, \eta + C \, \eta^2 - \mathcal{R}_{p,d} \left(\vartheta^p + \vartheta^{5/2} \right) \right)$$

under the condition that $\varepsilon^2 + \eta^2 < \vartheta$...

Carré du champ – admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

$$m_\pm(d,p) := \frac{1}{(d+2)\,p} \left(d\,p + 2 \pm \sqrt{d\,(p-1)\,\big(2\,d-(d-2)\,p\big)} \right)$$

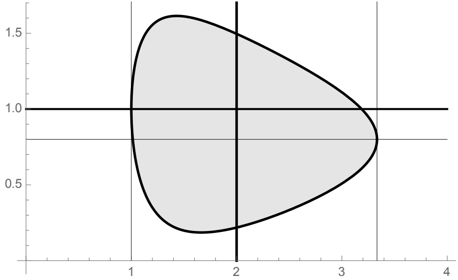

Figure: Case $d=5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)} \left(\mathcal{L} v + (m\,p-1)\,\frac{|\nabla v|^2}{v} \right) \quad \text{on} \quad \mathbb{R}^n$$

[JD, Brigati, Simonov] *Ornstein-Uhlenbeck operator:* $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_\pm(p) := \lim_{d \to +\infty} m_\pm(d,p) = 1 \pm \frac{1}{p} \sqrt{(p-1)\,(2-p)}$$

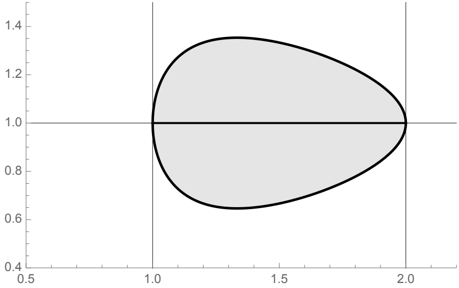

Figure: The admissible parameters $1 \leq p \leq 2$ and m are independent of n

Large dimensional limit

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem

Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ *with compact support,* $d \geq n$ *and*

$$u_d(\omega) = v\Big(\omega_1/r_d, \omega_2/r_d, \ldots, \omega_n/r_d\Big)\,, \quad r_d = \sqrt{\frac{d}{2\,\pi}}$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$*. With* $d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac12\,|y|^2}\,dy$*,*

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$