Stability in functional inequalities

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2. Euclidean space

Outline

Stability results for Gagliardo-Nirenberg inequalities on the Euclidean space Stability results for Gagliardo-Nirenberg inequalities on the Euclidean space and extension (weights)

1. Stability, fast diffusion equation and entropy methods
   - Rényi entropy powers, fast diffusion and Gagliardo-Nirenberg-Sobolev inequalities
   - The threshold time and the improved entropy – entropy production inequality (subcritical case)
   - Stability results (subcritical and critical case)

2. Stability in Caffarelli-Kohn-Nirenberg inequalities ?

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

A joint project with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method
arXiv:2007.03674, to appear in *Memoirs of the AMS*

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion equations
DCDS, 43 (3&4): 10701089, 2023

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u(t,x)\,dx = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+\left(\mathbb{R}^d, (1+|x|^2)\,dx\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2\,u(t,x)\,dx = 2\,d \int_{\mathbb{R}^d} u^m(t,x)\,dx$$

(iii) *Entropy estimate.* With $m \geq m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+\left(\mathbb{R}^d, (1+|x|^2)\,dx\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u^m(t,x)\,dx = \frac{m^2}{1-m} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy growth rate

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\ge\mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}\in[m_1,1)$$

$u=f^{2\,p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathcal{M}=\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{2\,p}\,,\quad \mathsf{E}[u]=\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1}\,,\quad \mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$

$$\mathsf{E}'\ge\frac{p-1}{2\,p}\,(p+1)^2\,\left(\mathcal{C}_{\mathrm{GNS}(p)}\right)^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{-\frac{2\,(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Equality case: $u(t,x)=\frac{c_1}{R(t)^d}\,\mathcal{B}\left(\frac{c_2\,x}{R(t)}\right)\,,\ \mathcal{B}(x):=\left(1+|x|^2\right)^{\frac{1}{m-1}}$

Pressure variable and decay of the Fisher information

The t-derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla\mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla\mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla\mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta\mathsf{P} + |\nabla\mathsf{P}|^2\Big)\,dx$$
$$= -2\int_{\mathbb{R}^d} u^m\left(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta\mathsf{P})^2\right)dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares: with $m_1 = \frac{d-1}{d}$

$$
\begin{aligned}
&-\frac{\mathsf{I}}{2\,\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)\\
&=\int_{\mathbb{R}^d} u^m\,\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2 dx+(m-m_1)\int_{\mathbb{R}^d} u^m\,\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2 dx
\end{aligned}
$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$
\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^{\theta}\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\,\theta}{p}}}=\frac{\mathsf{I}[\mathcal{B}]^{\theta}\,\mathsf{E}[\mathcal{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{2\,\theta}{p}}}=(p+1)^{2\,\theta}\,\left(\mathcal{C}_{\mathrm{GNS}}(p)\right)^{2\,\theta}
$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$
\mathsf{I}[u]^{\theta}\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\geq(p+1)^{2\,\theta}\,\left(\mathcal{C}_{\mathrm{GNS}}(p)\right)^{2\,\theta}\,\mathcal{M}^{\frac{2\,\theta}{p}}
$$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\Big(\tau,\frac{x}{\kappa\,R}\Big)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\,\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial\tau}+\nabla\cdot\Big[v\,\big(\nabla v^{m-1}-\,2\,x\big)\Big]=0\qquad(r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal F[v]:=-\frac{1}{m}\int_{\mathbb R^d}\big(v^m-\mathcal B^m-m\,\mathcal B^{m-1}\,(v-\mathcal B)\big)\,dx$$

$$\mathcal I[v]:=\int_{\mathbb R^d}v\,\big|\nabla v^{m-1}+\,2\,x\big|^2\,dx$$

are such that $\mathcal I[v]\ge 4\,\mathcal F[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathcal F[v(t,\cdot)]\le\mathcal F[v_0]\,e^{-\,4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

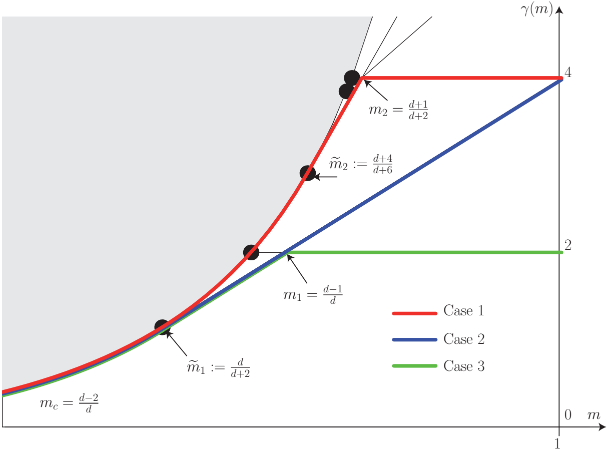
Lemma

Under assumption (H),

$$\mathcal{F}[v(t,\cdot)] \le C\,e^{-2\,\gamma(m)\,t} \quad \forall\, t\ge 0\,, \quad \gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m):=2$ *if* $\frac{d-1}{d}=m_1\le m<1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathcal{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathcal{B}\,dx$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathcal{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathcal{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and

$$(1-\varepsilon)\,\mathcal{B} \le v \le (1+\varepsilon)\,\mathcal{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathcal{I}[v] \ge (4+\eta)\,\mathcal{F}[v]$$

The initial time layer improvement: backward estimate

Hint: for some strictly convex function ψ with $\psi(0)=0$, $\psi'(0)=1$, we have

$$\mathcal{I}-4\,\mathcal{F}\geq\,4\,(\psi(\mathcal{F})-\mathcal{F})\geq 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement
Rephrasing the *carré du champ* method, $\mathcal{Q}[v]:=\frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt}\leq\mathcal{Q}\,(\mathcal{Q}-4)$$

Lemma

Assume that $m>m_1$ and v is a solution to $(r\,\mathrm{FDE})$ *with nonnegative initial datum v_0. If for some $\eta>0$ and $t_\star>0$, we have $\mathcal{Q}[v(t_\star,\cdot)]\geq 4+\eta$, then*

$$\mathcal{Q}[v(t,\cdot)]\geq\,4+\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}\quad\forall\,t\in[0,t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy

Choose $\varepsilon > 0$, *small enough*

Get a threshold time $t_\star(\varepsilon)$

0 $\quad$ $t_\star(\varepsilon)$ $\quad$ t

Backward estimate $\quad$ *Forward estimate*

The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (r FDE)

(and in particular for the initial datum)

If v is a solves $(r\,\mathsf{FDE})$ for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} v_0\,dx \le A < \infty \tag{H_A}$$

then

$$(1-\varepsilon)\,\mathcal{B} \le v(t,\cdot) \le (1+\varepsilon)\,\mathcal{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and A

Global Harnack Principle

The *Global Harnack Principle* holds if for some $t > 0$ large enough

$$\mathcal{B}_{M_1}(t - \tau_1, x) \leq u(t, x) \leq \mathcal{B}_{M_2}(t + \tau_2, x) \tag{GHP}$$

[Vázquez, 2003], [Bonforte, Vázquez, 2006]: (GHP) holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$
[Vázquez, 2003], [Bonforte, Simonov, 2020]: (GHP) holds if

$$\mathsf{A}[u_0] := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d \setminus B_R(0)} |u_0|\, dx < \infty$$

Theorem

[Bonforte, Simonov, 2020] If $M + \mathsf{A}[u_0] < \infty$, then

$$\lim_{t\to\infty} \left\| \frac{u(t) - B(t)}{B(t)} \right\|_\infty = 0$$

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $T \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$ *and* $\mathcal{F}[u_0] \leq G$, *then*

$$\sup_{x\in\mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq T$$

The threshold time

Proposition

Let $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ *and* $G > 0$

$$T = \mathsf{c}_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a} = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$, $\alpha = d\,(m - m_c)$ *and* $\vartheta = \nu/(d+\nu)$

$$\mathsf{c}_\star = \mathsf{c}_\star(m,d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max\left\{\varepsilon\, \kappa_1(\varepsilon, m),\, \varepsilon^{\mathsf{a}} \kappa_2(\varepsilon, m),\, \varepsilon\, \kappa_3(\varepsilon, m)\right\}$$

$$\kappa_1(\varepsilon, m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m}\,\kappa_\star}{1 - (1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon, m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon, m) := \frac{8\,\alpha^{-1}}{1 - (1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathcal{I}[v] \geq (4 + \zeta)\, \mathcal{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v] = G$, $\int_{\mathbb{R}^d} v\, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\, v\, dx = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0,\, 2\,\varepsilon_\star)\;, \quad \varepsilon_\star := \frac{1}{2} \min\left\{\varepsilon_{m,d},\, \chi\,\eta\right\} \quad \text{with} \quad t_\star = t_\star(\varepsilon) = \frac{1}{2} \log R(T)$$

$$(1-\varepsilon)\,\mathcal{B} \leq v(t,\cdot) \leq (1+\varepsilon)\,\mathcal{B} \quad \forall\, t \geq t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathcal{I}[v(t,.)] \geq (4 + \zeta)\, \mathcal{F}[v(t,.)] \quad \forall\, t \in [0, t_\star] \quad \text{where} \quad \zeta = \frac{4\,\eta\, e^{-4\,t_\star}}{4 + \eta - \eta\, e^{-4\,t_\star}}$$

Two consequences

$$\zeta=\mathsf{Z}(A,\mathcal{F}[u_0])\,,\quad \mathsf{Z}(A,G):=\frac{\zeta_\star}{1+A^{(1-m)\,\frac{2}{\alpha}}+G}\,,\quad \zeta_\star:=\frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. If v is a solution of $(r\,\mathrm{FDE})$ with nonnegative initial datum $v_0\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0]=G$, $\int_{\mathbb{R}^d}v_0\,dx=\mathcal{M}$, $\int_{\mathbb{R}^d}x\,v_0\,dx=0$ and v_0 satisfies (H_A), then

$$\mathcal{F}[v(t,.)]\le\mathcal{F}[v_0]\,e^{-\,(4+\zeta)\,t}\quad\forall\,t\ge0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathcal{I}[v]-4\,\mathcal{F}[v]\ge\zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v]-4\,\mathcal{F}[v]\ge\frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f]:=\left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1}}{\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2}\right)^{\frac{2\,p}{d-p\,(d-4)}}\,,\quad \kappa[f]:=\frac{\mathcal{M}^{\frac{1}{2\,p}}}{\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}}$$

$$\mathsf{A}[f]:=\frac{\mathcal{M}}{\lambda[f]^{\frac{d-p\,(d-4)}{p-1}}\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{2\,p}}\,\sup_{r>0}r^{\frac{d-p\,(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2\,p}\,dx$$

$$\mathsf{E}[f]:=\frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2\,p}}}\,f^{p+1}-\mathsf{g}^{p+1}-\frac{1+p}{2\,p}\,\mathsf{g}^{1-p}\left(\frac{\kappa[f]^{2\,p}}{\lambda[f]^2}\,f^{2\,p}-\mathsf{g}^{2\,p}\right)\right)dx$$

$$\mathfrak{S}[f]:=\frac{\mathcal{M}^{\frac{p-1}{2\,p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}\,(\mathsf{A}[f],\,\mathsf{E}[f])$$

Theorem

Let $d\ge1$, $p\in(1,p^*)$

If $f\in\mathcal{W}_p(\mathbb{R}^d):=\left\{f\in\mathrm{L}^{2p}(\mathbb{R}^d)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d)\,,\;|x|\,f^{\,p}\in\mathrm{L}^2(\mathbb{R}^d)\right\}$,

$$\left(\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\right)^{2\,p\,\gamma}-\left(\mathcal{C}_{\mathrm{GN}}\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}\right)^{2\,p\,\gamma}\ge\mathfrak{S}[f]\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{2\,p\,\gamma}\,\mathsf{E}[f]$$

With $\mathcal{K}_{\mathrm{GNS}} = C(p,d)\,\mathcal{C}_{\mathrm{GNS}}^{2\,p\,\gamma}$, $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + 4\,\frac{d-p\,(d-2)}{p+1}\,\|f\|^{p+1}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} - \mathcal{K}_{\mathrm{GNS}}\,\|f\|^{2\,p\,\gamma}_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}$$

Theorem

Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathcal{C} = \mathcal{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}\big[f^{2p}\big] < \infty$,

$$\delta[f] \ge \mathcal{C}[f]\,\inf_{\varphi\in\mathfrak{M}}\int_{\mathbb{R}^d}\left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx$$

▷ The dependence of $\mathcal{C}[f]$ on $\mathsf{A}\big[f^{2p}\big]$ and $\mathcal{F}\big[f^{2p}\big]$ is explicit and does not degenerate if $f \in \mathfrak{M}$

▷ Can we remove the condition $\mathsf{A}\big[f^{2p}\big] < \infty$?

Stability in Sobolev's inequality (critical case)

- ▷ A constructive stability result
- ▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \Big\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\; |x|\,f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \Big\}$$

Deficit of the Sobolev inequality: $\delta[f] := \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$

Theorem

Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \le A$$

we have

$$\delta[f] \ge \frac{\mathcal{C}_\star(A)}{4 + \mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathfrak{C}_\star\,\big(1 + A^{1/(2\,d)}\big)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on d*

Peculiarities of the critical case

▷ We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \Big(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\Big)$$

the *Bianchi-Egnell type result*

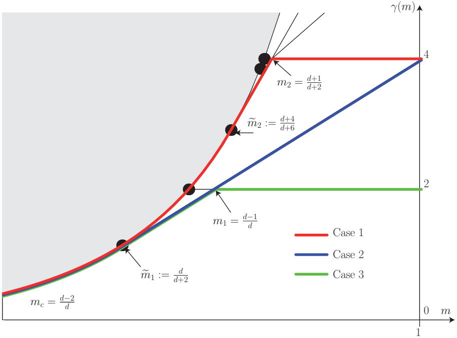
$$\delta[f] \geq \frac{\mathfrak{C}_\star\,Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathcal{J}[f|g]$$

with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

▷ Notion of time delay [JD, Toscani, 2014, 2015]

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathcal{B}_\lambda(x) = \lambda^{-d/2}\,\mathcal{B}\left(x/\sqrt{\lambda}\right)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves (r FDE) or to further rescale v according to



$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0)=0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

Stability in Caffarelli-Kohn-Nirenberg inequalities ?

Caffarelli-Kohn-Nirenberg inequalities

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

▷ *An optimal function among radial functions:*

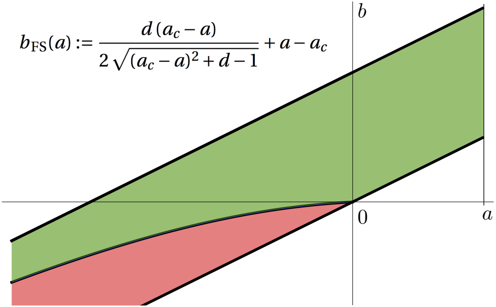
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Theorem

Let $d \ge 2$ and $p < 2^*$. $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ (symmetry) if and only if either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \ge b_{\mathrm{FS}}(a)$

[JD, Esteban, Loss, 2016]

Symmetry *versus* symmetry breaking



Symmetry and symmetry breaking regions

More Caffarelli-Kohn-Nirenberg inequalities

On $\mathbb{R}^d$ with $d \geq 1$, let us consider the *Caffarelli-Kohn-Nirenberg interpolation inequalities*

$$\|f\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \leq \mathcal{C}_{\beta,\gamma,p} \, \|\nabla f\|^{\theta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)} \, \|f\|^{1-\theta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}$$

$$\gamma-2 < \beta < \frac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2}\,,$$

with $\theta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$ and

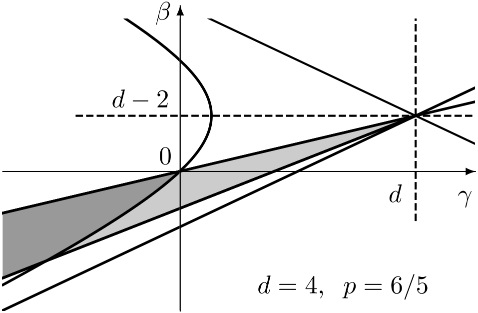
$\|f\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$ Symmetry means that equality is achieved by the *Aubin-Talenti type functions*

$$\mathrm{g}(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-\frac{1}{p-1}}$$

Theorem

[JD, Esteban, Loss, Muratori, 2017] Symmetry holds if and only if

$$\gamma < d\,, \quad \textit{and} \quad \gamma - 2 < \beta < \frac{d-2}{d}\,\gamma \quad \textit{and} \quad \beta \leq \beta_{\mathrm{FS}}(\gamma)$$



(γ, β) admissible region

An improved decay rate along the flow

In self-similar variables, with $m = (p+1)/(2\,p)$

$$|x|^{-\gamma}\,\frac{\partial v}{\partial t} + \nabla \cdot \left(|x|^{-\beta}\,v\,\nabla v^{m-1}\right) = \sigma\,\nabla \cdot \left(x\,|x|^{-\gamma}\,v\right)$$

$$\mathcal{F}[v] = \frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(v^{\frac{p+1}{2\,p}} - \mathrm{g}^{p+1} - \frac{p+1}{2\,p}\mathrm{g}^{1-p}\left(v - \mathrm{g}^{2p}\right)\right)|x|^{-\gamma}\,dx$$

Theorem

In the symmetry region, if $v \ge 0$ is a solution with a initial datum v_0 s.t.

$$A[v_0] := \sup_{R>0} R^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)}\int_{|x|>R} v_0(x)\,|x|^{-\gamma}\,dx < \infty$$

then there are some $\zeta > 0$ and some $T > 0$ such that, with $\alpha = 1 + \frac{\beta-\gamma}{2}$

$$\mathcal{F}[v(t,.)] \le \mathcal{F}[v_0]\,e^{-\left(4\,\alpha^2+\zeta\right)t} \quad \forall\, t \ge 2\,T$$

[Bonforte, JD, Nazaret, Simonov, 2022]