Stability in functional inequalities

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1. Introduction

Outline

1. Introduction

- The Sobolev inequality and the non-constructive stability result of Bianchi–Egnell using concentration-compactness methods
- Duality and stability in Hardy-Littlewood-Sobolev inequalities
- An example of entropy methods on the Euclidean space

- ***2. Euclidean space.*** Stability results for Gagliardo-Nirenberg inequalities on the Euclidean space and extension (weights)

- ***3. Sphere.*** Stability results on the sphere and on the Gaussian space seen as an infinite dimensional limit of spheres

- ***4. Sobolev and LSI.*** Constructive stability results for the Sobolev and the logarithmic Sobolev inequalities

Introduction

1. The history of the problem
2. Sobolev and HLS inequalities
   - Duality and Yamabe flow
   - Entropy methods, improvements
   - (log)-HLS: Carlen's duality
3. An introduction to entropy methods

Stability for the Sobolev inequality: the history

▷ In the Sobolev inequality ([Rodemich, 1969], [Aubin, 1976], [Talenti, 1976])

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

the optimal constant is $S_d = \frac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{1-2/d}$ with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\textit{if} \;\lim_{n\to\infty} \|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = S_d \,,\; \textit{then} \;\lim_{n\to\infty}\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb], 1985 a quantitative stability result ?

▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 \geq S_d \, \|f\|_{2^*}^2 + c_{\mathrm{BE}} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

- The strategy of Bianchi & Egnell involves two steps:

– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$

– a local-to-global extension based on concentration-compactness : the *far away regime*

- The constant c_{BE} is not explicit

Sobolev and Hardy-Littlewood-Sobolev inequalities

▷ Stability in a weaker norm, with explicit constants

▷ From duality to improved estimates based on Yamabe's flow

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \qquad \text{(S)}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx \quad \forall\, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \qquad \text{(HLS)}$$

are dual of each other. Here S_d is the Aubin-Talenti constant, $2^* = \frac{2d}{d-2}$, $(2^*)' = \frac{2d}{d+2}$ and by the Legendre transform

$$\sup_u \left(\int_{\mathbb{R}^d} u\,v\,dx - \tfrac{1}{2}\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right) = \tfrac{1}{2}\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

$$\sup_u \left(\int_{\mathbb{R}^d} u\,v\,dx - \tfrac{1}{2}\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right) = \tfrac{1}{2}\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx$$

Improved Sobolev inequality by duality

Theorem

[JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} < 1$ such that

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)$$

for any $w \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

Proof: the completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1}\, v|^2\, dx = \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\, v\, dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla(-\Delta)^{-1}\, v\, dx = \int_{\mathbb{R}^d} u\, v\, dx = \int_{\mathbb{R}^d} u^{2^*}\, dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \mathsf{S}_d\, \|u\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\, \nabla u - \nabla(-\Delta)^{-1}\, v \right|^2 dx$$

shows that (with $\mathcal{C} = 1$)

$$0 \le \mathsf{S}_d\, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\mathsf{S}_d\, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right) - \left(\mathsf{S}_d\, \|u^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^q\,(-\Delta)^{-1}\, u^q\, dx \right)$$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0 \,, \quad x \in \mathbb{R}^d \tag{FDE}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \le$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\,dx + \mathsf{S}_d\left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\,dx$$

where $v = v(t,\cdot)$ is a solution of (FDE). With the choice $m = \frac{d-2}{d+2}$, we find that $m + 1 = \frac{2d}{d+2}$

A simple observation

Proposition

[JD] Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If v is a solution of (FDE) with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, then

$$\frac{1}{2}\,\frac{d}{dt}\left(\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right)$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left(\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right) \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (S) if v is optimal for (HLS)

Improved Sobolev inequality

By integrating along the flow defined by (FDE), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (S), with $d \geq 5$ for integrability reasons

Theorem

[JD] Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)$$

for any $w \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

Proof: use the convexity properties of $t \mapsto \mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1} \, dx$ to get an estimate of the *extinction time* and combine with a differential inequality for $t \mapsto \mathsf{H}(t)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[del Pino, Saez], [Vázquez, Esteban, Rodriguez] For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Another improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

Theorem

[JD, Jankowiak] Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left(\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right)\right)$$

$$\forall\, u \in \mathsf{D}_\alpha\,,\quad v = u^{\frac{d+2}{d-2}}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ for any $x \ge 0$

Proof: $\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t))\ \forall\, t \in [0,T)$, $\kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}$ and consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,,\quad \mathsf{Y}(0) = 0\,,\quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

$\mathcal{C} = 1$ is not optimal

$\mathcal{C} = 1$ is the constant in the expansion of the square method

Theorem

[JD, Jankowiak] In the inequality

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \le \mathcal{C}_d \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)$$

we have

$$\frac{d}{d+4} \le \mathcal{C}_d < 1$$

based on a (painful) linearization

Extensions:

- Moser-Trudinger-Onofri inequality
- fractional Laplacian operator [Jankowiak, Nguyen]

Stability for (log)-HLS inequality: Carlen's duality

- Logarithmic Hardy-Littlewood-Sobolev inequality

$$\mathcal{H}[f] := \int_{\mathbb{R}^2} f \, \log f \, dx + 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, f(y)\, \log|x-y|\, dx\, dy + 1 + \log\pi \geq 0$$

with manifold of optimal functions $\mathcal{M}$ generated from $f_\star(x) := \pi^{-1}\left(1+|x|^2\right)^{-2}$ by translations and scalings

Theorem

[Carlen 2024] If $f \geq 0$ is such that $\|f\|_{\mathrm{L}^1(\mathbb{R}^2)} = 1$

$$\mathcal{H}[f] \geq \frac{1}{32} \inf_{g\in\mathcal{M}} \|f-g\|^2_{\mathrm{L}^1(\mathbb{R}^2)}$$

Based on [Gui, Moradifam, 2018] (Onofri inequality) and [Carlen, Figalli 2014] on Keller-Segel, [Blanchet, JD, Perthame, 2006]

- Hardy-Littlewood-Sobolev inequality:
[Carlen 2017], [Chen, Lu, Tang 2023]

An introduction to entropy methods

- Entropies and diffusions on $\mathbb{R}^d$ (linear case)
- ▷ φ-entropies and entropy-entropy production inequalities
- ▷ The Bakry-Emery or *carré du champ* method
- ▷ Improvements and stability

The Fokker-Planck equation (domain in $\mathbb{R}^d$)

The linear Fokker-Planck (FP) equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \psi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \psi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \psi \cdot \nabla v =: \mathcal{L} v$$

[Bakry, Emery, 1985], [Arnold, Markowich, Toscani, Unterreiter, 2001]
With mass normalized to 1, the unique stationary solution of (FP) is

$$u_s = e^{-\psi} \quad \Longleftrightarrow \quad v_s = 1$$

Definition of the φ-entropies

If $d\gamma = e^{-\psi}\,dx$ is the invariant probability measure, let

$$\mathcal{E}[v] := \int_{\mathbb{R}^d} \varphi(v)\,d\gamma$$

φ is a nonnegative convex continuous function on $\mathbb{R}^+$ such that $\varphi(1)=0$ and $1/\varphi''$ is concave on $(0,+\infty)$:

$$\varphi'' \ge 0\,, \quad \varphi \ge \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \le 0$$

Classical examples

$$\varphi_p(v) := \tfrac{1}{p-1}\left(v^p - 1 - p\,(v-1)\right) \quad p \in (1,2]$$

$$\varphi_1(v) := v\,\log v - (v-1)\,, \qquad \varphi_2(v) := |v-1|^2$$

The invariant measure

$$d\gamma = e^{-\psi}\,dx$$

where ψ is a *potential* such that $e^{-\psi}$ is in $\mathrm{L}^1(\mathbb{R}^d, dx)$
$d\gamma$ is a probability measure

Entropy – entropy production inequalities, linear flows

Case of a smooth convex bounded domain Ω

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \psi \cdot \nabla v \,, \quad \nabla v \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

$$\frac{d}{dt} \int_\Omega \frac{v^q - 1}{q-1}\, d\gamma = -\frac{4}{q} \int_\Omega |\nabla w|^2\, d\gamma \quad \text{and} \quad w = v^{q/2}$$

$$\frac{d}{dt} \int_\Omega |\nabla w|^2\, d\gamma \le -\,2\,\Lambda(q) \int_\Omega |\nabla w|^2\, d\gamma$$

where $\Lambda(q) > 0$ is the best constant in the inequality

$$\frac{2}{q}\,(q-1) \int_\Omega |\nabla X|^2\, d\gamma + \int_\Omega \mathrm{Hess}\,\psi : X \otimes X\, d\gamma \ge \Lambda(q) \int_\Omega |X|^2\, d\gamma$$

Proposition

$$\int_\Omega \frac{v^q - 1}{q-1}\, d\gamma \le \frac{4}{q\,\Lambda(q)} \int_\Omega \left|\nabla v^{q/2}\right|^2 d\gamma \quad \textit{for any } v \textit{ s.t.} \quad \int_\Omega v\, d\gamma = 1$$

[Bakry, Emery, 1984] [JD, Nazaret, Savaré, 2008]

The Bakry-Emery method (domain in $\mathbb{R}^d$)

With $d\gamma = u_s\,dx$ and v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$

- *q-entropy*

$$\mathcal{E}_q[v] := \frac{1}{q-1}\int_\Omega \big(v^q - 1 - q\,(v-1)\big)\,d\gamma$$

- *q-Fisher information* with $w = v^{q/2}$

$$\mathcal{I}_q[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$$

▷ The strategy

$$\frac{d}{dt}\mathcal{E}_q[v(t,\cdot)] = -\,\mathcal{I}_q[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\Big(\mathcal{I}_q[v] - 2\,\lambda\,\mathcal{E}_q[v]\Big) \le 0$$

▷ The decay rates

$$\mathcal{I}_q[v(t,\cdot)] \le \mathcal{I}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t} \quad \text{and} \quad \mathcal{E}_q[v(t,\cdot)] \le \mathcal{E}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$$

▷ The entropy-entropy production inequality

$$\mathcal{I}_q[v] \ge 2\,\lambda\,\mathcal{E}_q[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma)$$

Properties of the φ-entropies

- Generalized Csiszár-Kullback-Pinsker inequality: [Pinsker], [Csiszár 1967], [Kullback 1967], [Cáceres, Carrillo, JD, 2002]

$$\mathcal{E}[v] \geq \mathcal{C}_q \, \|v-1\|^2_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)} \,, \quad \mathcal{C}_q = \inf_{s\in(0,\infty)} \frac{s^{2-q}\,\varphi''(s)}{2^{2/q}} \, \min\left\{1, \|v\|^{q-2}_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}\right\}$$

- Tensorization and sub-additivity

$$\iint_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \varphi''(v)\,|\nabla v|^2\,d\gamma_1\,d\gamma_2 \geq \min\{\Lambda_1,\Lambda_2\}\,\mathcal{E}_{\gamma_1\otimes\gamma_2}[v]$$

- Holley-Stroock type perturbation results: if for some constants a, $b\in\mathbb{R}$, $e^{-b}\,d\gamma \leq d\mu \leq e^{-a}\,d\gamma$, then

$$e^{a-b}\,\Lambda \int_{\mathbb{R}^d} \left[\varphi(v)-\varphi(\tilde v)-\varphi'(\tilde v)(v-\tilde v)\right] d\mu \leq \int_{\mathbb{R}^d} \varphi''(v)\,|\nabla v|^2\,d\mu$$

Improved entropy – entropy production inequalities

In the special case $\psi(x)=|x|^2/2+\frac{d}{2}\,\log(2\,\pi)$, with $w=v^{q/2}$, we obtain that

$$\frac12\,\frac{d}{dt}\int_{\mathbb R^d}|\nabla w|^2\,d\gamma+\int_{\mathbb R^d}|\nabla w|^2\,d\gamma\le-\,\frac2q\,\kappa_q\int_{\mathbb R^d}\frac{|\nabla w|^4}{w^2}\,d\gamma$$

with $\kappa_q=(q-1)\,(2-q)/q$

Cauchy-Schwarz: $\left(\int_{\mathbb R^d}|\nabla w|^2\,d\gamma\right)^2\le\int_{\mathbb R^d}\frac{|\nabla w|^4}{w^2}\,d\gamma\int_{\mathbb R^d}w^2\,d\gamma$

$$\frac{d}{dt}\mathcal I[v]+2\,\mathcal I[v]\le-\,\kappa_q\,\frac{\mathcal I[v]^2}{1+(q-1)\,\mathcal E[v]}$$

Proposition

Assume that $q\in(1,2)$ and $d\gamma=(2\pi)^{-d/2}\,e^{-|x|^2/2}\,dx$. There exists a strictly convex function Ψ such that $\Psi(0)=0$ and $\Psi'(0)=1$ and

$$\Psi\left(\|f\|^2_{\mathrm L^2(\mathbb R^d,d\gamma)}-1\right)\le\|\nabla f\|^2_{\mathrm L^2(\mathbb R^d,d\gamma)}\quad\text{if}\quad\|f\|_{\mathrm L^q(\mathbb R^d,d\gamma)}=1$$

Two references

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Improved inequalities and stability results

Entropy – entropy production inequality

$$\boxed{\mathcal{I}[u] \ge \Lambda\,\mathcal{E}[u]}$$

▷ ***Improved entropy – entropy production inequality*** (weaker form)

$$\mathcal{I} \ge \Lambda\,\Psi(\mathcal{E})$$

for some Ψ such that $\Psi(0) = 0$, $\Psi'(0) = 1$ and $\Psi'' > 0$

$$\mathcal{I} - \Lambda\,\mathcal{E} \ge \Lambda\left(\Psi(\mathcal{E}) - \mathcal{E}\right) \ge 0$$

▷ ***Improved constant*** means ***stability***

Under some restrictions on the functions, there is some $\Lambda_\star > \Lambda$ such that

$$\mathcal{I} - \Lambda\,\mathcal{E} \ge (\Lambda_\star - \Lambda)\,\mathcal{E} \ge 0 \quad \text{or} \quad \mathcal{I} - \Lambda\,\mathcal{E} \ge \left(1 - \frac{\Lambda}{\Lambda_\star}\right)\mathcal{I} \ge 0$$

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