Free energy estimates for the two-dimensional Keller-Segel model

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The Keller-Segel model

The Keller-Segel(-Patlak) system for chemotaxis describes the collective motion of cells (bacteria or amoebae) [Othmer-Stevens, Horstman].
The complete Keller-Segel model is a system of two parabolic equations. Simplified two-dimensional version :

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot (n \nabla c) & x \in \mathbb{R}^2\,,\ t > 0 \\[2ex] -\Delta c = n & x \in \mathbb{R}^2\,,\ t > 0 \\[2ex] n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases} \qquad (1)$$

$n(x,t)$: the cell density
$c(x,t)$: concentration of chemo-attractant
$\chi > 0$: *sensitivity* of the bacteria to the chemo-attractant

I. Main results and *a priori* estimates

Dimension 2 is critical

The total mass of the system

$$M := \int_{\mathbb{R}^2} n_0 \, dx$$

is conserved

There are related models in gravitation which are defined in $\mathbb{R}^3$

The L^1-norm is critical in the sense that there exists a critical mass above which all solution blow-up in finite time and below which they globally exist. The critical space is $L^{d/2}(\mathbb{R}^d)$ for $d \geq 2$, see [Corrias-Perthame-Zaag]. In dimension $d = 2$, the Green kernel associated to the Poisson equation is a logarithm, namely

$$c = -\frac{1}{2\pi} \log|\cdot| * n$$

First main result

Theorem 1. *Assume that* $n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$ *and* $n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)$. *If* $M < 8\pi/\chi$, *then the Keller-Segel system (1) has a global weak non-negative solution* n *with initial data* n_0 *such that*

$$(1+|x|^2+|\log n|)\, n \in L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2)) \quad \int_0^t \int_{\mathbb{R}^2} n\, |\nabla \log n - \chi \nabla c|^2\, dx\, dt < \infty$$

and
$$\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx + 4M\left(1 - \frac{\chi M}{8\pi}\right) t$$

for any $t > 0$. *Moreover* $n \in L^\infty_{\rm loc}((\varepsilon, \infty), L^p(\mathbb{R}^2))$ *for any* $p \in (1,\infty)$ *and any* $\varepsilon > 0$, *and the following inequality holds for any* $t > 0$ *:*

$$F[n(\cdot,t)] + \int_0^t \int_{\mathbb{R}^2} n\, |\nabla (\log n) - \chi \nabla c|^2\, dx\, ds \le F[n_0]$$

Here $F[n] := \int_{\mathbb{R}^2} n \log n\, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} n\, c\, dx$

Notion of solution

The equation holds in the distributions sense. Indeed, writing

$$\Delta n - \chi \nabla\cdot(n \nabla c) = \nabla\cdot[n(\nabla \log n - \chi \nabla c)]$$

we can see that the flux is well defined in $L^1(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$ since

$$\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla \log n - \chi \nabla c|\;dx\,dt$$
$$\leq \left(\iint_{[0,T]\times\mathbb{R}^2} n\;dx\,dt\right)^{1/2} \left(\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla \log n - \chi \nabla c|^2\;dx\,dt\right)^{1/2} < \infty$$

Second main result : Large time behavior

Use asymptotically self-similar profiles given in the rescaled variables by the equation

$$u_\infty = M \frac{e^{\chi\, v_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{\chi\, v_\infty - |x|^2/2}\, dx} = -\Delta v_\infty \quad \text{with} \quad v_\infty = -\frac{1}{2\pi} \log|\cdot| * u_\infty \quad (2)$$

In the original variables :

$$\begin{aligned} n_\infty(x,t) &:= \frac{1}{1+2t}\, u_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right) \\ c_\infty(x,t) &:= v_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right) \end{aligned}$$

Theorem 2. *Under the same assumptions as in Theorem 1, there exists a stationary solution* (u_∞, v_∞) *in the self-similar variables such that*

$$\lim_{t\to\infty} \|n(\cdot,t) - n_\infty(\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,t) - \nabla c_\infty(\cdot,t)\|_{L^2(\mathbb{R}^2)} = 0$$

Assumptions

We assume that the initial data satisfies the following asssumptions :

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$$

$$n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)$$

The total mass is conserved

$$M := \int_{\mathbb{R}^2} n_0(x)\;dx = \int_{\mathbb{R}^2} n(x,t)\;dx$$

Goal : give a complete existence theory [J.D.-Perthame], [Blanchet-J.D.-Perthame] in the subcritical case, *i.e.* in the case

$$M < 8\pi/\chi$$

Alternatives

There are only two cases :

1. Solutions to (1) blow-up in finite time when $M > 8\pi/\chi$
2. There exists a global in time solution of (1) when $M < 8\pi/\chi$

The case $M = 8\pi/\chi$ is delicate and for radial solutions, some results have been obtained recently [Biler-Karch-Laurençot-Nadzieja]

Our existence theory completes the partial picture established in [Jäger-Luckhaus].

Convention

The solution of the Poisson equation $-\Delta c = n$ is given up to an harmonic function. From the beginning, we have in mind that the concentration of the chemo-attractant is defined by

$$c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, n(y,t)\, dy$$

$$\nabla c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\, n(y,t)\, dy$$

Blow-up for super-critical masses

Case $M > 8\pi/\chi$ (Case 1) : use moments estimates

Lemma 3. *Consider a non-negative distributional solution to (1) on an interval* $[0,T]$ *that satisfies the previous assumptions,* $\int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx < \infty$ *and such that* $(x,t) \mapsto \int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\, n(y,t)\, dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$ *and* $(x,t) \mapsto (1+|x|)\nabla c(x,t) \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$. *Then it also satisfies*

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx = 4M\left(1 - \frac{\chi\, M}{8\pi}\right)$$

Formal proof.

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx &= \int_{\mathbb{R}^2} |x|^2\, \Delta n(x,t)\, dx \\ &\quad + \frac{\chi}{2\,\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} 2x\cdot\frac{x-y}{|x-y|^2}\; n(x,t)\, n(y,t)\, dx\, dy\end{aligned}$$

Justification

Consider a smooth function φ_ε with compact support such that $\lim_{\varepsilon\to 0} \varphi_\varepsilon(|x|) = |x|^2$

$$\frac{d}{dt}\int_{\mathbb{R}^2} \varphi_\varepsilon\, n\, dx = \int_{\mathbb{R}^2} \Delta\varphi_\varepsilon\, n\, dx - \frac{\chi}{4\pi}\int_{\mathbb{R}^2} \underbrace{\frac{(\nabla\varphi_\varepsilon(x)-\nabla\varphi_\varepsilon(y))\cdot(x-y)}{|x-y|^2}}_{\to 1} n(x,t)\, n(y,t)\, dx\, dy$$

Since $\frac{d}{dt}\int_{\mathbb{R}^2} \varphi_\varepsilon\, n\, dx \leq C_\varepsilon \int_{\mathbb{R}^2} n_0\, dx$ where C_ε is some positive constant, as $\varepsilon \to 0$, $\int_{\mathbb{R}^2} \varphi_\varepsilon\, n\, dx \leq c_1 + c_2\, t$

$$\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx < \infty \quad \forall\, t \in (0,T)$$

$\square$

Weaker notion of solutions

We shall say that n is a solution to (1) if for all test functions $\psi \in \mathcal{D}(\mathbb{R}^2)$

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi(x)\, n(x,t)\, dx = \int_{\mathbb{R}^2} \Delta\psi(x)\, n(x,t)\, dx$$
$$- \frac{\chi}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} [\nabla\psi(x) - \nabla\psi(y)] \cdot \frac{x-y}{|x-y|^2}\, n(x,t)\, n(y,t)\, dx\, dy$$

Compared to standard distribution solutions, this is an improved concept that can handle some measure valued solutions because the term

$$[\nabla\psi(x) - \nabla\psi(y)] \cdot \frac{x-y}{|x-y|^2}$$

is continuous

However, this notion of solutions does not cover the case of all measure valued $n(\cdot,t)$

Finite time blow-up

Corollary 4. *Consider a non-negative distributional solution $n \in L^\infty(0,T^*;L^1(\mathbb{R}^2))$ to (1) and assume that $[0,T^*)$, $T^* \le \infty$, is the maximal interval of existence. Let*

$$I_0 := \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx < \infty \quad \textit{and} \quad \int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\, n(y,t)\, dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$$

If $\chi\, M > 8\pi$, then

$$T^* \le \frac{2\pi\, I_0}{M(\chi\, M - 8\pi)}$$

If $\chi\, M > 8\pi$ and $I_0 = \infty$: blow-up in finite time ?

Blow-up statements in bounded domains are available

Radial case : there exists a $L^1(\mathbb{R}^2 \times \mathbb{R}^+)$ radial function $\tilde{n}$ such that

$$n(x,t) \to \frac{8\pi}{\chi}\,\delta + \tilde{n}(x,t) \quad \text{as } t \nearrow T^*$$

Comments

1. $\chi M = 8\pi$ [Biler-Karch-Laurençot-Nadzieja] : blow-up only for $T^* = \infty$

2. If the problem is set in dimension $d \geq 3$, the critical norm is $L^p(\mathbb{R}^d)$ with $p = d/2$ [Corrias-Perthame-Zaag]

3. In dimension $d = 2$, the value of the mass M is therefore natural to discriminate between super- and sub-critical regimes. However, the limit of the L^p-norm is rather $\int_{\mathbb{R}^2} n \log n \, dx$ than $\int_{\mathbb{R}^2} n \, dx$, which is preserved by the evolution. This explains why it is natural to introduce the entropy, or better, as we shall see below, the *free energy*

The proof of Jäger and Luckhaus

[Corrias-Perthame-Zaag] Compute $\frac{d}{dt}\int_{\mathbb{R}^2} n \log n \, dx$. Using an integration by parts and the equation for c, we obtain :

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} n \log n \, dx &= -4\int_{\mathbb{R}^2} \left|\nabla\sqrt{n}\right|^2 dx + \chi \int_{\mathbb{R}^2} \nabla n \cdot \nabla c \, dx \\ &= -4\int_{\mathbb{R}^2} \left|\nabla\sqrt{n}\right|^2 dx + \chi \int_{\mathbb{R}^2} n^2 \, dx\end{aligned}$$

The entropy is nonincreasing if $\chi M \leq 4 C_{\mathrm{GNS}}^{-2}$, where $C_{\mathrm{GNS}} = C_{\mathrm{GNS}}^{(4)}$ is the best constant for $p = 4$ in the Gagliardo-Nirenberg-Sobolev inequality :

$$\|u\|^2_{L^p(\mathbb{R}^2)} \leq C_{\mathrm{GNS}}^{(p)} \, \|\nabla u\|^{2-4/p}_{L^2(\mathbb{R}^2)} \, \|u\|^{4/p}_{L^2(\mathbb{R}^2)} \quad \forall\, u \in H^1(\mathbb{R}^2) \quad \forall\, p \in [2, \infty)$$

Numerically : $\chi M \leq 4 C_{\mathrm{GNS}}^{-2} \approx 1.862... \times (4\pi) < 8\pi$

A sharper approach : free energy

The *free energy* :

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} n \, c \, dx$$

Lemma 5. *Consider a non-negative* $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ *solution* n *of (1) such that* $n(1+|x|^2)$, $n \log n$ *are bounded in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\nabla\sqrt{n} \in L^1_{\mathrm{loc}}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ *and* $\nabla c \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. *Then*

$$\frac{d}{dt} F[n(\cdot,t)] = -\int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) - \chi \nabla c \right|^2 \, dx =: \mathcal{I}$$

$\mathcal{I}$ is the *free energy production term* or *generalized relative Fisher information.*

Proof.

$$\frac{d}{dt} F[n(\cdot,t)] = \int_{\mathbb{R}^2} \left[\left(1 + \log n - \chi \, c \right) \nabla \cdot \left(\frac{\nabla n}{n} - \chi \nabla c \right) \right] \, dx$$

Hardy-Littlewood-Sobolev inequality

$$F[n(\cdot,t)] = \int_{\mathbb{R}^2} n \log n \, dx + \frac{\chi}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)\, n(y,t)\, \log|x-y| \, dx\, dy$$

Lemma 6. [Carlen-Loss, Beckner] *Let f be a non-negative function in $L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f\, dx = M$, then*

$$\int_{\mathbb{R}^2} f \log f \, dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x) f(y) \log|x-y| \, dx\, dy \geq -\, C(M)$$

with $C(M) := M(1 + \log\pi - \log M)$

The above inequality is the key functional inequality

Consequences

$$(1-\theta)\int_{\mathbb{R}^2} n\log n\; dx+\theta\left[\int_{\mathbb{R}^2} n\log n\; dx+\frac{\chi}{4\pi\theta}\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\;\log|x-y|\;dx\,dy\right]$$

Lemma 7. *Consider a non-negative $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solution n of (1) such that $n(1+|x|^2)$, $n\log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\int_{\mathbb{R}^2}\frac{1+|x|}{|x-y|}\,n(y,t)\;dy\in L^\infty\big((0,T)\times\mathbb{R}^2\big)$, $\nabla\sqrt{n}\in L^1_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $\nabla c\in L^\infty_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$. If $\chi\,M\leq 8\pi$, then the following estimates hold :*

$$M\;\log M-M\;\log[\pi(1+t)]-K\leq\int_{\mathbb{R}^2} n\log n\; dx\leq\frac{8\pi\,F_0+\chi\,M\,C(M)}{8\pi-\chi\,M}$$

$$0\leq\int_0^t ds\int_{\mathbb{R}^2} n(x,s)\Big|\nabla\left(\log n(x,s)\right)\quad-\quad\chi\nabla c(x,s)\Big|^2 dx$$

$$\leq C_1+C_2\left[M\;\log\left(\frac{\pi(1+t)}{M}\right)+K\right]$$

Lower bound

Because of the bound on the second moment

$$\frac{1}{1+t}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx \leq K \quad \forall\, t>0\,,$$

$$\int_{\mathbb{R}^2} n(x,t)\log n(x,t) \;\geq\; \frac{1}{1+t}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx - K + \int_{\mathbb{R}^2} n(x,t)\log n(x,t)\, dx$$

$$= \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)} \log\left(\frac{n(x,t)}{\mu(x,t)}\right) \mu(x,t)\, dx - M\, \log[\pi(1+t)] - K$$

with $\mu(x,t) := \frac{1}{\pi(1+t)}\, e^{-\frac{|x|^2}{1+t}}$. By Jensen's inequality,

$$\int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)} \log\left(\frac{n(x,t)}{\mu(x,t)}\right) d\mu(x,t) \geq X\, \log X \text{ where } X = \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\, d\mu = M$$

$L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ bound of the entropy term

Lemma 8. *For any $u \in L^1_+(\mathbb{R}^2)$, if $\int_{\mathbb{R}^2} |x|^2\, u\, dx$ and $\int_{\mathbb{R}^2} u\, \log u\, dx$ are bounded from above, then $u\,\log u$ is uniformly bounded in $L^\infty(\mathbb{R}^+_{\rm loc}, L^1(\mathbb{R}^2))$ and*

$$\int_{\mathbb{R}^2} u\,|\log u|\,dx \le \int_{\mathbb{R}^2} u\left(\log u + |x|^2\right)dx + 2\log(2\pi)\int_{\mathbb{R}^2} u\,dx + \frac{2}{e}$$

Proof. Let $\bar u := u\,1\!\!1_{\{u\le 1\}}$ and $m = \int_{\mathbb{R}^2} \bar u\, dx \le M$. Then

$$\int_{\mathbb{R}^2} \bar u\,(\log \bar u + \frac{1}{2}|x|^2)\,dx = \int_{\mathbb{R}^2} U \log U\; d\mu - m\log(2\pi)$$

$U := \bar u/\mu$, $d\mu(x) = \mu(x)\,dx$, $\mu(x) = (2\pi)^{-1} e^{-|x|^2/2}$. Jensen's inequality :

$$\int_{\mathbb{R}^2} \bar u\,\log \bar u\,dx \ge m\log\left(\frac{m}{2\pi}\right) - \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\,\bar u\,dx \ge -\frac{1}{e} - M\log(2\pi) - \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\,\bar u\,dx$$

and conclude using

$\int_{\mathbb{R}^2} u\,|\log u|\,dx = \int_{\mathbb{R}^2} u\,\log u\,dx - 2\int_{\mathbb{R}^2} \bar u\,\log \bar u\,dx$

II. Proof of the existence result

Weak solutions up to critical mass

Proposition 9. *If $M < 8\pi/\chi$, the Keller-Segel system (1) has a global weak non-negative solution such that, for any $T > 0$,*

$$(1 + |x|^2 + |\log n|)\, n \in L^\infty(0, T; L^1(\mathbb{R}^2))$$

and

$$\iint_{[0,T]\times\mathbb{R}^2} n\, |\nabla \log n - \chi \nabla c|^2\, dx\, dt < \infty$$

For $R > \sqrt{e}$, $R \mapsto R^2/\log R$ is an increasing function, so that

$$0 \leq \iint_{|x-y|>R} \log|x-y|\, n(x,t)\, n(y,t)\; dx\, dy \leq \frac{2\log R}{R^2}\, M \int_{\mathbb{R}^2} |x|^2\, n(x,t)\; dx$$

Since $\iint_{1<|x-y|<R} \log|x-y|\, n(x,t)\, n(y,t)\; dx\, dy \leq M^2\, \log R$, we only need a uniform bound for $|x-y| < 1$

A regularized model

Let $\mathcal{K}^\varepsilon(z) := \mathcal{K}^1\left(\frac{z}{\varepsilon}\right)$ with

$$\begin{cases} \mathcal{K}^1(z) = -\dfrac{1}{2\pi}\log|z| & \text{if } |z| \geq 4 \\ \mathcal{K}^1(z) = 0 & \text{if } |z| \leq 1 \end{cases}$$

$$0 \leq -\nabla\mathcal{K}^1(z) \leq \frac{1}{2\pi\,|z|} \quad \mathcal{K}^1(z) \leq -\frac{1}{2\pi}\log|z| \quad \text{and} \quad -\Delta\mathcal{K}^1(z) \geq 0$$

Since $\mathcal{K}^\varepsilon(z) = \mathcal{K}^1(z/\varepsilon)$, we also have

$$0 \leq -\nabla\mathcal{K}^\varepsilon(z) \cdot \frac{z}{|z|} \leq \frac{1}{2\pi\,|z|} \quad \forall\, z \in \mathbb{R}^2$$

Proposition 10. *For any fixed positive* ε*, if* $n_0 \in L^2(\mathbb{R}^2)$*, then for any* $T > 0$ *there exists* $n^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^2)) \cap C(0,T;L^2(\mathbb{R}^2))$ *which solves*

$$\begin{cases} \dfrac{\partial n^\varepsilon}{\partial t} = \Delta n^\varepsilon - \chi \nabla \cdot (n^\varepsilon \nabla c^\varepsilon) \\ \\ c^\varepsilon = \mathcal{K}^\varepsilon * n^\varepsilon \end{cases}$$

1. Regularize the initial data : $n_0 \in L^2(\mathbb{R}^2)$

2. Use the *Aubin-Lions compactness method* with the spaces $H := L^2(\mathbb{R}^2)$, $V := \{v \in H^1(\mathbb{R}^2) \ : \ \sqrt{|x|}\, v \in L^2(\mathbb{R}^2)\}$, $L^2(0,T;V)$, $L^2(0,T;H)$ and $\{v \in L^2(0,T;V) \ : \ \partial v/\partial t \in L^2(0,T;V')\}$

3. Fixed-point method

Uniform a priori estimates

Lemma 11. *Consider a solution n^ε of the regularized equation. If $\chi M < 8\pi$ then, uniformly as $\varepsilon \to 0$, with bounds depending only upon $\int_{\mathbb{R}^2}(1+|x|^2)\, n_0\, dx$ and $\int_{\mathbb{R}^2} n_0 \log n_0\, dx$, we have :*

(i) *The function $(t,x) \mapsto |x|^2 n^\varepsilon(x,t)$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^1(\mathbb{R}^2))$.*

(ii) *The functions $t \mapsto \int_{\mathbb{R}^2} n^\varepsilon(x,t) \log n^\varepsilon(x,t)\, dx$ and $t \mapsto \int_{\mathbb{R}^2} n^\varepsilon(x,t)\, c^\varepsilon(x,t)\, dx$ are bounded.*

(iii) *The function $(t,x) \mapsto n^\varepsilon(x,t) \log(n^\varepsilon(x,t))$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^1(\mathbb{R}^2))$.*

(iv) *The function $(t,x) \mapsto \nabla\sqrt{n^\varepsilon}(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

(v) *The function $(t,x) \mapsto n^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

(vi) *The function $(t,x) \mapsto n^\varepsilon(x,t)\, \Delta c^\varepsilon(x,t)$ is bounded in $L^1(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

(vii) *The function $(t,x) \mapsto \sqrt{n^\varepsilon}(x,t)\, \nabla c^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

Proof of (iv)

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\varepsilon \log n^\varepsilon \, dx \leq -4\int_{\mathbb{R}^2}\left|\nabla\sqrt{n^\varepsilon}\right|^2 dx + \chi\int_{\mathbb{R}^2} n^\varepsilon\cdot(-\Delta c^\varepsilon)\,dx$$

$$\int_{\mathbb{R}^2} n^\varepsilon\cdot(-\Delta c^\varepsilon)\,dx = \int_{\mathbb{R}^2} n^\varepsilon\cdot(-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon))\,dx = (\mathrm{I})+(\mathrm{II})+(\mathrm{III})$$

with

$$(\mathrm{I}) := \int_{n^\varepsilon<K} n^\varepsilon\cdot(-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon)),\ (\mathrm{II}) := \int_{n^\varepsilon\geq K} n^\varepsilon\cdot(-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon)) - (\mathrm{III}),\ (\mathrm{III}) = \int_{n^\varepsilon\geq K}|n^\varepsilon|^2$$

Let $\frac{1}{\varepsilon^2}\phi_1\left(\frac{\cdot}{\varepsilon}\right) := -\Delta\mathcal{K}^\varepsilon$: $\frac{1}{\varepsilon^2}\phi_1\left(\frac{\cdot}{\varepsilon}\right) = -\Delta\mathcal{K}^\varepsilon \rightharpoonup \delta$ in $\mathcal{D}'$

This heuristically explains why (II) should be small

III. Regularity and free energy

Weak regularity results

Theorem 12. [Goudon2004] *Let $n^\varepsilon : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ be such that for almost all $t \in (0,T)$, $n^\varepsilon(t)$ belongs to a weakly compact set in $L^1(\mathbb{R}^N)$ for almost any $t \in (0,T)$. If $\partial_t n^\varepsilon = \sum_{|\alpha| \leq k} \partial_x^\alpha g_\varepsilon^{(\alpha)}$ where, for any compact set $K \subset \mathbb{R}^n$,*

$$\limsup_{\substack{|E| \to 0 \\ E \subset \mathbb{R} \text{ is measurable}}} \left(\sup_{\varepsilon > 0} \int \int_{E \times K} |g_\varepsilon^{(\alpha)}| \, dt \, dx \right) = 0$$

then $(n^\varepsilon)_{\varepsilon > 0}$ is relatively compact in $C^0([0,T]; L^1_{\rm weak}(\mathbb{R}^N))$.

Corollary 13. *Let n^ε be a solution of the regularized problem with initial data $n_0^\varepsilon = \min\{n_0, \varepsilon^{-1}\}$ such that $n_0\,(1 + |x|^2 + |\log n_0|) \in L^1(\mathbb{R}^2)$. If n is a solution of (1) with initial data n_0, such that, for a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} \varepsilon_k = 0$, $n^{\varepsilon_k} \rightharpoonup n$ in $L^1((0,T) \times \mathbb{R}^2)$, then n belongs to $C^0(0,T; L^1_{\rm weak}(\mathbb{R}^2))$.*

L^p uniform estimates

Proposition 14. *Assume that $M < 8\pi/\chi$ holds. If n_0 is bounded in $L^p(\mathbb{R}^2)$ for some $p > 1$, then any solution n of (1) is bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^p(\mathbb{R}^2))$.*

$$\begin{aligned}
\frac{1}{2(p-1)}\frac{d}{dt}\int_{\mathbb{R}^2}|n(x,t)|^p\,dx &= -\frac{2}{p}\int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx + \chi\int_{\mathbb{R}^2}\nabla(n^{p/2})\cdot n^{p/2}\cdot\nabla c\,dx \\
&= -\frac{2}{p}\int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx + \chi\int_{\mathbb{R}^2}n^p\,(-\Delta c)\,dx \\
&= -\frac{2}{p}\int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx + \chi\int_{\mathbb{R}^2}n^{p+1}\,dx
\end{aligned}$$

Gagliardo-Nirenberg-Sobolev inequality with $n = v^{2/p}$:

$$\int_{\mathbb{R}^2}|v|^{2(1+1/p)}\,dx \le K_p\int_{\mathbb{R}^2}|\nabla v|^2\,dx\int_{\mathbb{R}^2}|v|^{2/p}\,dx$$

$$\frac{1}{2(p-1)}\frac{d}{dt}\int_{\mathbb{R}^2} n^p\,dx \le \int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2\,dx \left(-\frac{2}{p} + K_p\,\chi\,M\right)$$

which proves the decay of $\int_{\mathbb{R}^2} n^p\,dx$ if $M < \frac{2}{p\,K_p\,\chi}$

Otherwise, use the entropy estimate to get a bound : Let $K > 1$

$$\int_{\mathbb{R}^2} n^p\,dx = \int_{n\le K} n^p\,dx + \int_{n>K} n^p\,dx \le K^{p-1}\,M + \int_{n>K} n^p\,dx$$

Let $M(K) := \int_{n>K} n\,dx$:

$$M(K) \le \frac{1}{\log K}\int_{n>K} n\,\log n\,dx \le \frac{1}{\log K}\int_{\mathbb{R}^2} |n\,\log n|\,dx$$

Redo the computation for $\int_{\mathbb{R}^2}(n-K)_+^p\,dx$ [Jäger-Luckhaus]

The free energy inequality in a regular setting

Using the *a priori* estimates of the previous section for $p = 2 + \varepsilon$, we can prove that the free energy inequality holds.

Lemma 15. *Let n_0 be in a bounded set in $L^1_+(\mathbb{R}^2, (1+|x|^2)dx) \cap L^{2+\varepsilon}(\mathbb{R}^2, dx)$, for some $\varepsilon > 0$, eventually small. Then the solution n of (1) found before, with initial data n_0, is in a compact set in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$ and moreover the free energy production estimate holds :*

$$F[n] + \int_0^t \left(\int_{\mathbb{R}^2} n \left| \nabla (\log n) - \chi \nabla c \right|^2 \, dx \right) ds \leq F[n_0]$$

1. n is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$
2. ∇n is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$
3. Compactness in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$

Taking the limit in the Fisher information term

Up to the extraction of subsequences

$$\iint_{[0,T]\times\mathbb{R}^2} |\nabla n|^2\,dx\,dt \leq \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |\nabla n_k|^2\,dx\,dt$$

$$\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt \leq \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} n_k\,|\nabla c_k|^2\,dx\,dt$$

$$\iint_{[0,T]\times\mathbb{R}^2} n^2\,dx\,dt = \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |n_k|^2\,dx\,dt$$

Fisher information term :

$$\iint_{[[0,T]\times\mathbb{R}^2} n\,|\nabla(\log n) - \chi\nabla c|^2\,dx\,dt$$

$$= 4\iint_{[[0,T]\times\mathbb{R}^2} |\nabla\sqrt{n}|^2\,dx\,dt + \chi^2\iint_{[[0,T]\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt - 2\chi\iint_{[[0,T]\times\mathbb{R}^2} n^2\,dx\,dt \qquad \square$$

Hypercontractivity

Theorem 16. *Consider a solution n of (1) such that $\chi\, M < 8\pi$. Then for any $p \in (1,\infty)$, there exists a continuous function h_p on $(0,\infty)$ such that for almost any $t>0$, $\|n(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le h_p(t)$.*

Notice that unless n_0 is bounded in $L^p(\mathbb{R}^2)$, $\lim_{t\to 0_+} h_p(t) = +\infty$. Such a result is called an *hypercontractivity* result, since to an initial data which is originally in $L^1(\mathbb{R}^2)$ but not in $L^p(\mathbb{R}^2)$, we associate a solution which at almost any time $t>0$ is in $L^p(\mathbb{R}^2)$ with p arbitrarily large.

Proof. Fix $t>0$ and $p \in (1,\infty)$ and consider $q(s) := 1 + (p-1)\,\frac{s}{t}$. Define : $M(K) := \sup_{s\in(0,t)} \int_{n>K} n(\cdot,s)\,dx$

$$\int_{n>K} n(\cdot,s)\,dx \le \frac{1}{\log K}\int_{\mathbb{R}^2} |n(\cdot,s)\,\log n(\cdot,s)|\,dx$$

and

$$F(s) := \left[\int_{\mathbb{R}^2} (n-K)_+^{q(s)}(x,s)\,dx\right]^{1/q(s)}$$

$$F' \, F^{q-1} = \frac{q'}{q^2} \int_{\mathbb{R}^2} (n-K)_+^q \log\left(\frac{(n-K)_+^q}{F^q}\right) + \int_{\mathbb{R}^2} n_t \, (n-K)_+^{q-1}$$

$$\int_{\mathbb{R}^2} (n-K)_+^{q-1} \, n_t \, dx = -4 \, \frac{q-1}{q^2} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx + \chi \, \frac{q-1}{q} \int_{\mathbb{R}^2} v^{2(1+\frac{1}{q})} \, dx$$

with $v := (n-K)_+^{q/2}$

Logarithmic Sobolev inequality

$$\int_{\mathbb{R}^2} v^2 \log\left(\frac{v^2}{\int_{\mathbb{R}^2} v^2 \, dx}\right) dx \le 2\,\sigma \int_{\mathbb{R}^2} |\nabla v|^2 \, dx - (2 + \log(2\,\pi\,\sigma)) \int_{\mathbb{R}^2} v^2 \, dx$$

Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^2} |v|^{2(1+1/q)} \, dx \le \mathcal{K}(q) \, \|\nabla v\|^2_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |v|^{2/q} \, dx \quad \forall \, q \in [2, \infty) \qquad \square$$

The free energy inequality for weak solutions

Corollary 17. *Let $(n^k)_{k\in\mathbb{N}}$ be a sequence of solutions of (1) with regularized initial data n_0^k . For any $t_0>0, T\in\mathbb{R}^+$ such that $0<t_0<T$, $(n^k)_{k\in\mathbb{N}}$ is relatively compact in $L^2((t_0,T)\times\mathbb{R}^2)$, and if n is the limit of $(n^k)_{k\in\mathbb{N}}$, then n is a solution of (1) such that the free energy inequality holds.*

Proof.

$$F[n^k(\cdot,t)]+\int_{t_0}^t\left(\int_{\mathbb{R}^2}n^k\left|\nabla\left(\log n^k\right)-\chi\nabla c^k\right|^2\,dx\right)\,ds\le F[n^k(\cdot,t_0)]$$

Passing to the limit as $k\to\infty$, we get

$$F[n(\cdot,t)]+\int_{t_0}^t\left(\int_{\mathbb{R}^2}n\left|\nabla\left(\log n\right)-\chi\nabla c\right|^2\,dx\right)\,ds\le F[n(\cdot,t_0)]$$

Let $t_0\to 0_+$ and conclude $\square$

IV. Large time behaviour

Self-similar variables

$$n(x,t) = \frac{1}{R^2(t)}\,u\left(\frac{x}{R(t)},\tau(t)\right) \quad \text{and} \quad c(x,t) = v\left(\frac{x}{R(t)},\tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla\cdot(u(x+\chi\nabla v)) & x \in \mathbb{R}^2\,,\ t>0 \\ v = -\dfrac{1}{2\pi}\,\log|\cdot| * u & x \in \mathbb{R}^2\,,\ t>0 \\ u(\cdot,t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Free energy : $F^R[u] := \int_{\mathbb{R}^2} u \log u\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} u\,v\,dx + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\,u\,dx$

$$\frac{d}{dt}F^R[u(\cdot,t)] \leq -\int_{\mathbb{R}^2} u\,\left|\nabla \log u - \chi\nabla v + x\right|^2\,dx$$

Self-similar solutions : Free energy

Lemma 18. *The functional F^R is bounded from below on the set*

$$\left\{u \in L^1_+(\mathbb{R}^2) \ : \ |x|^2\, u \in L^1(\mathbb{R}^2) \ \int_{\mathbb{R}^2} u\, \log u\, dx < \infty\right\}$$

if and only if $\chi\, \|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$.

Proof. If $\chi\, \|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$, the bound is a consequence of the Hardy-Littlewood-Sobolev inequality

Scaling property. For a given u, let $u_\lambda(x) = \lambda^{-2} u(\lambda^{-1} x)$: $\|u_\lambda\|_{L^1(\mathbb{R}^2)} =: M$ does not depend on $\lambda > 0$ and

$$F^R[u_\lambda] = F^R[u] - 2M\left(1 - \frac{\chi\, M}{8\pi}\right) \log\lambda + \frac{\lambda - 1}{2} \int_{\mathbb{R}^2} |x|^2\, u\, dx$$

□

Strong convergence

Lemma 19. *Let $\chi M < 8\pi$. As $t \to \infty$, $(s,x) \mapsto u(x,t+s)$ converges in $L^\infty(0,T;L^1(\mathbb{R}^2))$ for any positive T to a stationary solution self-similar equation and*

$$\lim_{t\to\infty} \int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx = 2M\left(1 - \frac{\chi M}{8\pi}\right)$$

Proof. We use the free energy production term :

$$F^R[u_0] - \liminf_{t\to\infty} F^R[u(\cdot,t)] = \lim_{t\to\infty} \int_0^t \left(\int_{\mathbb{R}^2} u\, \left|\nabla \log u - \chi \nabla v + x\right|^2\, dx \right) ds$$

and compute $\int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx$:

$$\int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2\, n_0\, dx\, e^{-2t} + 2M\left(1 - \frac{\chi M}{8\pi}\right)\, (1 - e^{-2t}) \qquad \square$$

Stationary solutions

Notice that under the constraint $\|u_\infty\|_{L^1(\mathbb{R}^2)} = M$, u_∞ is a critical point of the free energy.

Lemma 20. *Let $u \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$ with $M := \int_{\mathbb{R}^2} u\,dx$, such that $\int_{\mathbb{R}^2} u \log u\,dx < \infty$, and define $v(x) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,u(y)\,dy$. Then there exists a positive constant C such that, for any $x \in \mathbb{R}^2$ with $|x| > 1$,*

$$\left| v(x) + \frac{M}{2\pi} \log|x| \right| \leq C$$

Lemma 21. [Naito-Suzuki] *Assume that V is a non-negative non-trivial radial function on $\mathbb{R}^2$ such that $\lim_{|x|\to\infty} |x|^\alpha V(x) < \infty$ for some $\alpha \geq 0$. If u is a solution of*

$$\Delta u + V(x)\,e^u = 0 \quad x \in \mathbb{R}^2$$

such that $u_+ \in L^\infty(\mathbb{R}^2)$, then u is radially symmetric decreasing w.r.t. the origin

Because of the asymptotic logarithmic behavior of v_∞, the result of Gidas, Ni and Nirenberg does not directly apply. The boundedness from above is essential, otherwise non-radial solutions can be found, even with no singularity. Consider for instance the perturbation $\delta(x) = \frac{1}{2}\,\theta\,(x_1^2 - x_2^2)$ for any $x = (x_1, x_2)$, for some fixed $\theta \in (0,1)$, and define the potential $\phi(x) = \frac{1}{2}\,|x|^2 - \delta(x)$. By a fixed-point method we can find a solution of

$$w(x) = -\frac{1}{2\pi}\,\log|\cdot| * M\,\frac{e^{\chi w - \phi(x)}}{\int_{\mathbb{R}^2} e^{\chi w(y) - \phi(y)}\,dy}$$

since, as $|x| \to \infty$, $\phi(x) \sim \frac{1}{2}\left[(1-\theta)x_1^2 + (1+\theta)x_2^2\right] \to +\infty$. This solution is such that $w(x) \sim -\frac{M}{2\pi}\,\log|x|$. Hence $v(x) := w(x) + \delta(x)/\chi$ is a non-radial solution of the self-similar equation, which behaves like $\delta(x)/\chi$ as $|x| \to \infty$ with $|x_1| \neq |x_2|$.

Lemma 22. *If $\chi\, M > 8\pi$, the rescaled equation has no stationary solution (u_∞, v_∞) such that $\|u_\infty\|_{L^1(\mathbb{R}^2)} = M$ and $\int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx < \infty$. If $\chi\, M < 8\pi$, the self-similar equation has at least one radial stationary solution. This solution is C^∞ and u_∞ is dominated as $|x| \to \infty$ by $e^{-(1-\varepsilon)|x|^2/2}$ for any $\varepsilon \in (0,1)$.*

Non-existence for $\chi\, M > 8\pi$:

$$0 = \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx = 4M \left(1 - \frac{\chi\, M}{8\pi}\right) - 2 \int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx$$

□

Uniqueness : [Biler-Karch-Laurençot-Nadzieja]

Intermediate asymptotics

Lemma 23.

$$\lim_{t\to\infty} F^R[u(\cdot,\cdot+t)] = F^R[u_\infty]$$

Proof. We know that $u(\cdot,\cdot+t)$ converges to u_∞ in $L^2((0,1)\times\mathbb{R}^2)$ and that $\int_{\mathbb{R}^2} u(\cdot,\cdot+t)\, v(\cdot,\cdot+t)\, dx$ converges to $\int_{\mathbb{R}^2} u_\infty\, v_\infty\, dx$. Concerning the entropy, it is sufficient to prove that $u(\cdot,\cdot+t)\,\log u(\cdot,\cdot+t)$ weakly converges in $L^1((0,1)\times\mathbb{R}^2)$ to $u_\infty\,\log u_\infty$. Concentration is prohibited by the convergence in $L^2((0,1)\times\mathbb{R}^2)$. Vanishing or dichotomy cannot occur either : Take indeed $R>0$, large, and compute $\int_{|x|>R} u\,|\log u| = (\mathrm{I})+(\mathrm{II})$, with $m := \int_{|x|>R,\ u<1} u\, dx$ and

$$(\mathrm{I}) = \int_{|x|>R,\ u\geq 1} u\,\log u\, dx \leq \frac{1}{2}\int_{|x|>R,\ u\geq 1} |u|^2\, dx$$

$$(\mathrm{II}) = -\int_{|x|>R,\ u<1} u\,\log u\, dx \leq \frac{1}{2}\int_{|x|>R,\ u<1} |x|^2\, u\, dx - m\log\left(\frac{m}{2\pi}\right)$$

Conclusion

The result we have shown above is actually slightly better : all terms converge to the corresponding values for the limiting stationary solution

$$F^R[u] - F^R[u_\infty] = \int_{\mathbb{R}^2} u \log\left(\frac{u}{u_\infty}\right)\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} |\nabla v - \nabla v_\infty|^2\,dx$$

Csiszár-Kullback inequality : for any nonnegative functions f, $g \in L^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f\,dx = \int_{\mathbb{R}^2} g\,dx = M$,

$$\|f-g\|^2_{L^1(\mathbb{R}^2)} \leq \frac{1}{4\,M}\int_{\mathbb{R}^2} f\,\log\left(\frac{f}{g}\right)\,dx$$

Corollary 24.

$$\lim_{t\to\infty}\|u(\cdot,\cdot+t)-u_\infty\|_{L^1(\mathbb{R}^2)}=0 \quad \textit{and} \quad \lim_{t\to\infty}\|\nabla v(\cdot,\cdot+t)-\nabla v_\infty\|_{L^2(\mathbb{R}^2)}=0$$