

# From $\varphi$ -entropies to hypocoercivity without confinement

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# Outline

- **From  $\varphi$ -entropies to  $H^1$  hypocoercivity**
    - ▷  $\varphi$ -entropies and diffusions
    - ▷  $\varphi$ -hypocoercivity ( $H^1$  framework)
  - **An  $L^2$  abstract hypocoercivity result**
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  - **$L^2$  framework: mode-by-mode hypocoercivity**
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    - ▷ Decay rates in the Euclidean space without confinement
- ▷ In collaboration with X. Li  
and E. Bouin, S. Mischler, C. Mouhot, C. Schmeiser

# From $\varphi$ -entropies to $H^1$ hypocoercivity

## ▷ Some references of related works

(Chafaï 2004), (Bolley, Gentil 2010)

(Baudoin 2017)

(Monmarché), (Evans, 2017)

(Arnold, Erb, 2014), (Arnold, Stürzer), (Achleitner, Arnold, Stürzer, 2016), (Achleitner, Arnold, Carlen, 2017), (Arnold, Einav, Wöhrer, 2017)

## Definition of the $\varphi$ -entropies

$$\mathcal{E}[w] := \int_{\mathbb{R}^d} \varphi(w) d\gamma$$

$\varphi$  is a nonnegative convex continuous function on  $\mathbb{R}^+$  such that  $\varphi(1) = 0$  and  $1/\varphi''$  is concave on  $(0, +\infty)$ :

$$\varphi'' \geq 0, \quad \varphi \geq \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \leq 0$$

Classical examples

$$\varphi_p(w) := \frac{1}{p-1} (w^p - 1 - p(w-1)) \quad p \in (1, 2]$$

$$\varphi_1(w) := w \log w - (w-1)$$

The invariant measure

$$d\gamma = e^{-\psi} dx$$

where  $\psi$  is a *potential* such that  $e^{-\psi}$  is in  $L^1(\mathbb{R}^d, dx)$

$d\gamma$  is a probability measure

# Diffusions

*Ornstein-Uhlenbeck equation or backward Kolmogorov equation*

$$\frac{\partial w}{\partial t} = \mathbf{L} w := \Delta w - \nabla \psi \cdot \nabla w$$

$$\bullet - \int_{\mathbb{R}^d} (\mathbf{L} w_1) w_2 d\gamma = \int_{\mathbb{R}^d} \nabla w_1 \cdot \nabla w_2 d\gamma \quad \forall w_1, w_2 \in H^1(\mathbb{R}^d, d\gamma)$$

$$\bullet 1 = \int_{\mathbb{R}^d} w_0 d\gamma = \int_{\mathbb{R}^d} w(t, \cdot) d\gamma \text{ and } \lim_{t \rightarrow +\infty} w(t, \cdot) = 1$$

$$\bullet \frac{d}{dt} \mathcal{E}[w] = - \int_{\mathbb{R}^d} \varphi''(w) |\nabla_x w|^2 d\gamma =: -\mathcal{J}[w] \quad (\text{Fisher information})$$

If for some  $\Lambda > 0$

$$\mathcal{J}[w] \geq \Lambda \mathcal{E}[w] \quad \forall w \in H^1(\mathbb{R}^d, d\gamma)$$

(entropy - entropy production inequality), then

$$\mathcal{E}[w(t, \cdot)] \leq \mathcal{E}[w_0] e^{-\Lambda t} \quad \forall t \geq 0$$

*Fokker-Planck equation* :  $u = w \gamma$  converges to  $u_* = \gamma$

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_x \cdot (u \nabla_x \psi)$$

# Generalized Csiszár-Kullback-Pinsker inequality

(Pinsker), (Csiszár 1967), (Kullback 1967), (Cáceres, Carrillo, JD, 2002)

## Proposition

Let  $p \in [1, 2]$ ,  $w \in L^1 \cap L^p(\mathbb{R}^d, d\gamma)$  be a nonnegative function, and assume that  $\varphi \in C^2(0, +\infty)$  is a nonnegative strictly convex function such that  $\varphi(1) = \varphi'(1) = 0$ . If  $A := \inf_{s \in (0, \infty)} s^{2-p} \varphi''(s) > 0$ , then

$$\mathcal{E}[w] \geq 2^{-\frac{2}{p}} A \min \left\{ 1, \|w\|_{L^p(\mathbb{R}^d, d\gamma)}^{p-2} \right\} \|w - 1\|_{L^p(\mathbb{R}^d, d\gamma)}^2$$

# Convexity, tensorization and sub-additivity

$$\int_{\mathbb{R}^{d_i}} \varphi''(w) |\nabla w|^2 d\gamma_i =: \mathcal{J}_{\gamma_i}[w] \geq \Lambda_i \mathcal{E}_{\gamma_i}[w] \quad \forall w \in H^1(\mathbb{R}^{d_i}, d\gamma_i)$$

## Theorem

If  $d\gamma_1$  and  $d\gamma_2$  are two probability measures on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ , then

$$\begin{aligned} \mathcal{J}_{\gamma_1 \otimes \gamma_2}[w] &= \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi''(w) |\nabla w|^2 d\gamma_1 d\gamma_2 \\ &\geq \min\{\Lambda_1, \Lambda_2\} \mathcal{E}_{\gamma_1 \otimes \gamma_2}[w] \quad \forall w \in H^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, d\gamma) \end{aligned}$$

$$\mathcal{J}_{\gamma_1 \otimes \gamma_2}[w] = \int_{\mathbb{R}^{d_2}} \mathcal{J}_{\gamma_1}[w] d\gamma_2 + \int_{\mathbb{R}^{d_1}} \mathcal{J}_{\gamma_2}[w] d\gamma_1$$

$$\mathcal{E}_{\gamma_1 \otimes \gamma_2}[w] \leq \int_{\mathbb{R}^{d_2}} \mathcal{E}_{\gamma_1}[w] d\gamma_2 + \int_{\mathbb{R}^{d_1}} \mathcal{E}_{\gamma_2}[w] d\gamma_1 \quad \forall w \in L^1(d\gamma_1 \otimes \gamma_2)$$

## Perturbation (Holley-Stroock type) results

With  $\bar{w} := \int_{\mathbb{R}^d} w \, d\gamma$ , assume that

$$\Lambda \left[ \int_{\mathbb{R}^d} \varphi(w) \, d\gamma - \varphi(\bar{w}) \right] \leq \int_{\mathbb{R}^d} \varphi''(w) |\nabla w|^2 \, d\gamma \quad \forall w \in H^1(d\gamma)$$

and, for some constants  $a, b \in \mathbb{R}$ ,

$$e^{-b} \, d\gamma \leq d\mu \leq e^{-a} \, d\gamma$$

### Lemma

If  $\varphi$  is a  $C^2$  function such that  $\varphi'' > 0$  and  $\tilde{w} := \int_{\mathbb{R}^d} w \, d\mu / \int_{\mathbb{R}^d} d\mu$ , then

$$e^{a-b} \Lambda \int_{\mathbb{R}^d} [\varphi(w) - \varphi(\tilde{w}) - \varphi'(\tilde{w})(w - \tilde{w})] \, d\mu \leq \int_{\mathbb{R}^d} \varphi''(w) |\nabla w|^2 \, d\mu$$



# Entropy – entropy production inequalities, linear flows

On a smooth convex bounded domain  $\Omega$ , consider

$$\frac{\partial w}{\partial t} = \mathbf{L} w := \Delta w - \nabla \psi \cdot \nabla w, \quad \nabla w \cdot \nu = 0 \quad \text{on } \partial\Omega$$

$$\frac{d}{dt} \int_{\Omega} \frac{w^p - 1}{p-1} d\gamma = -\frac{4}{p} \int_{\Omega} |\nabla z|^2 d\gamma \quad \text{and} \quad z = w^{p/2}$$

$$\frac{d}{dt} \int_{\Omega} |\nabla z|^2 d\gamma \leq -2\Lambda(p) \int_{\Omega} |\nabla z|^2 d\gamma$$

where  $\Lambda(p) > 0$  is the best constant in the inequality

$$\frac{2}{p} (p-1) \int_{\Omega} |\nabla X|^2 d\gamma + \int_{\Omega} \text{Hess } \psi : X \otimes X d\gamma \geq \Lambda(p) \int_{\Omega} |X|^2 d\gamma$$

## Proposition

$$\int_{\Omega} \frac{w^p - 1}{p-1} d\gamma \leq \frac{4}{p\Lambda} \int_{\Omega} |\nabla w^{p/2}|^2 d\gamma \quad \text{for any } w \text{ s.t. } \int_{\Omega} w d\gamma = 1$$

(Bakry, Emery 1985)

# An interpolation inequality

## Corollary

Assume that  $q \in [1, 2)$ . With  $\Lambda = \Lambda(2/q)$ , we have

$$\frac{\|f\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \|f\|_{L^q(\mathbb{R}^d, d\gamma)}^2}{2 - q} \leq \frac{1}{\Lambda} \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma \quad \forall f \in H^1(\mathbb{R}^d, d\gamma)$$

## Improved entropy – entropy production inequalities

In the special case  $\psi(x) = |x|^2/2$ , with  $z = w^{p/2}$ , we obtain that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma + \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma \leq -\frac{2}{p} \kappa_p \int_{\mathbb{R}^d} \frac{|\nabla z|^4}{z^2} d\gamma$$

with  $\kappa_p = (p-1)(2-p)/p$

Cauchy-Schwarz:  $\left(\int_{\mathbb{R}^d} |\nabla z|^2 d\gamma\right)^2 \leq \int_{\mathbb{R}^d} \frac{|\nabla z|^4}{z^2} d\gamma \int_{\mathbb{R}^d} z^2 d\gamma$

$$\frac{d}{dt} \mathcal{J}[w] + 2\mathcal{J}[w] \leq -\kappa_p \frac{\mathcal{J}[w]^2}{1 + (p-1)\mathcal{E}[w]}$$

### Proposition

Assume that  $q \in (1, 2)$  and  $d\gamma = (2\pi)^{-d/2} e^{-|x|^2/2} dx$ . There exists a strictly convex function  $F$  such that  $F(0) = 0$  and  $F'(0) = 1$  and

$$F\left(\|f\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - 1\right) \leq \|\nabla f\|_{L^2(\mathbb{R}^d, d\gamma)}^2 \quad \text{if } \|f\|_{L^q(\mathbb{R}^d, d\gamma)} = 1$$

# $\varphi$ -hypocoercivity ( $H^1$ framework)

- ▷ adapt the strategy of  $\varphi$ -entropies to kinetic equations
- ▷ Villani's strategy: derive  $H^1$  estimates (using a twisted Fisher information) and then use standard interpolation inequalities to establish entropy decay rates

The twisted Fisher information *is not* the derivative of the  $\varphi$ -entropy

The *kinetic Fokker-Planck equation*, or *Vlasov-Fokker-Planck equation*:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v f) \quad (1)$$

with  $\psi(x) = |x|^2/2$  and  $\|f\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$  has a unique nonnegative stationary solution

$$f_\star(x, v) = (2\pi)^{-d} e^{-\frac{1}{2}(|x|^2 + |v|^2)}$$

and the function  $g = f/f_\star$  solves the *kinetic Ornstein-Uhlenbeck equation*

$$\frac{\partial g}{\partial t} + \mathbb{T}g = \mathbb{L}g$$

with transport operator  $\mathbb{T}$  and Ornstein-Uhlenbeck operator  $\mathbb{L}$  given by

$$\mathbb{T}g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathbb{L}g := \Delta_v g - v \cdot \nabla_v g$$

# Sharp rates for the kinetic Fokker-Planck equation

Let  $\psi(x) = |x|^2/2$ ,  $d\mu := f_\star dx dv$ ,  $\mathcal{E}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_p(g) d\mu$

## Proposition

Let  $p \in [1, 2]$  and consider a nonnegative solution  $g \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$  of the kinetic Fokker-Planck equation. There is a constant  $\mathcal{C} > 0$  such that

$$\mathcal{E}[g(t, \cdot, \cdot)] \leq \mathcal{C} e^{-t} \quad \forall t \geq 0$$

and the rate  $e^{-t}$  is sharp as  $t \rightarrow +\infty$

(Villani), (Arnold, Erb): a twisted Fisher information functional

$$\mathcal{J}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2 d\mu$$

(Arnold, Erb) relies on  $\lambda = 1/2$  and  $\frac{d}{dt} \mathcal{J}_{1/2}[h(t, \cdot)] \leq -\mathcal{J}_{1/2}[h(t, \cdot)]$

## Improved rates (in the large entropy regime)

Rewrite the decay of the *Fisher information* functional as

$$-\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} X^\perp \cdot \mathfrak{M}_0 X \, d\mu = \int_{\mathbb{R}^d} X^\perp \cdot \mathfrak{M}_1 X \, d\mu + \int_{\mathbb{R}^d} Y^\perp \cdot \mathfrak{M}_2 Y \, d\mu$$

where  $X = (\nabla_v h, \nabla_x h)$ ,  $Y = (H_{vv}, H_{xv}, M_{vv}, M_{xv})$

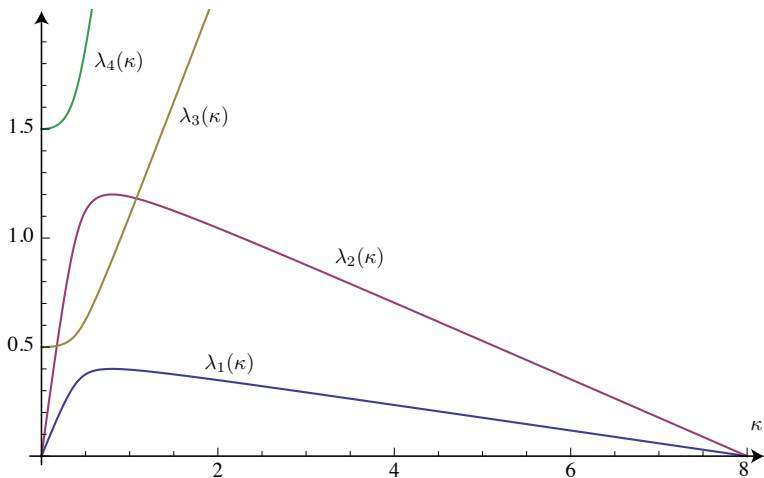
$$\mathfrak{M}_0 = \begin{pmatrix} 1 & \lambda \\ \lambda & \nu \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}, \quad \mathfrak{M}_1 = \begin{pmatrix} 1 - \lambda & \frac{1+\lambda-\nu}{2} \\ \frac{1+\lambda-\nu}{2} & \lambda \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}$$

$$\mathfrak{M}_2 = \begin{pmatrix} 1 & \lambda & -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} \\ \lambda & \nu & -\frac{\kappa\lambda}{2} & -\frac{\kappa\nu}{2} \\ -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} & 2\kappa & 2\kappa\lambda \\ -\frac{\kappa\lambda}{2} & -\frac{\kappa\nu}{2} & 2\kappa\lambda & 2\kappa\nu \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d \times \mathbb{R}^d}$$

With constant coefficients

$$\lambda_\star(\lambda, \nu) = \max \left\{ \min_X \frac{X^\perp \cdot \mathfrak{M}_1 X}{X^\perp \cdot \mathfrak{M}_0 X} : (\lambda, \nu) \in \mathbb{R}^2 \text{ s.t. } \mathfrak{M}_2 \geq 0 \right\}$$

For  $(\lambda, \nu) = (1/2)$ ,  $\lambda_\star = 1/2$  and the eigenvalues of  $\mathfrak{M}_2(\frac{1}{2}, 1)$  are given as a function of  $\kappa = 8(2-p)/p \in [0, 8]$  are all nonnegative





We know that

$$Y^\perp \cdot \mathfrak{M}_2 Y \geq \lambda_1(p, \lambda) |Y|^2$$

for some  $\lambda_1(p, \lambda) > 0$  and  $|Y|^2 \geq \|\mathbf{M}_{vv}\|^2$  so that, by Cauchy-Schwarz,

$$\left( \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu \right)^2 \leq \int_{\mathbb{R}^d} h^2 d\mu \int_{\mathbb{R}^d} \|\mathbf{M}_{vv}\|^2 d\mu \leq c_0 \int_{\mathbb{R}^d} \|\mathbf{M}_{vv}\|^2 d\mu$$

## Theorem

Let  $p \in (1, 2)$  and  $h$  be a solution of the kinetic Ornstein-Uhlenbeck equation. Then there exists a function  $\lambda : \mathbb{R}^+ \rightarrow [1/2, 1)$  such that  $\lambda(0) = \lim_{t \rightarrow +\infty} \lambda(t) = 1/2$  and a function  $\rho > 1/2$  s.t.

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq -2\rho(t) \mathcal{J}_{\lambda(t)}[h(t, \cdot)]$$

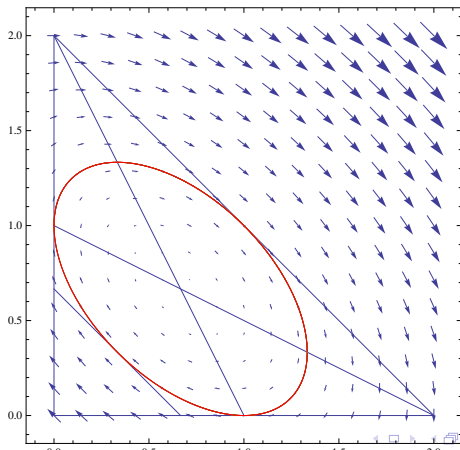
As a consequence, for any  $t \geq 0$  we have the global estimate

$$\mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq \mathcal{J}_{1/2}[h_0] \exp\left(-2 \int_0^t \rho(s) ds\right)$$

Let us define  $a := e^t \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu$ ,  $b := e^t \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu$ ,  
 $c := e^t \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu$  and  $j := a + b + c$

$$\frac{da}{dt} \leq a - 2(j - c), \quad \frac{dc}{dt} \leq 2(j - a) - c \quad \text{and} \quad \frac{dj}{dt} \leq 0$$

with the constraints  $a \geq 0$ ,  $c \geq 0$  and  $b^2 \leq ac$



# An abstract hypocoercivity result

- ▷ Abstract statement
- ▷ A toy model

# An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathbb{T}F = \mathbb{L}F \quad (2)$$

In the framework of kinetic equations,  $\mathbb{T}$  and  $\mathbb{L}$  are respectively the transport and the collision operators

We assume that  $\mathbb{T}$  and  $\mathbb{L}$  are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$

$$\mathbb{A} := (1 + (\mathbb{T}\Pi)^* \mathbb{T}\Pi)^{-1} (\mathbb{T}\Pi)^*$$

\* denotes the adjoint with respect to  $\langle \cdot, \cdot \rangle$

$\Pi$  is the orthogonal projection onto the null space of  $\mathbb{L}$

# The assumptions

$\lambda_m$ ,  $\lambda_M$ , and  $C_M$  are positive constants such that, for any  $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathbf{L}F, F \rangle \geq \lambda_m \|(1 - \Pi)F\|^2 \quad (\text{H1})$$

▷ *macroscopic coercivity:*

$$\|\mathbf{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2 \quad (\text{H2})$$

▷ *parabolic macroscopic dynamics:*

$$\Pi\mathbf{T}\Pi F = 0 \quad (\text{H3})$$

▷ *bounded auxiliary operators:*

$$\|\mathbf{A}\mathbf{T}(1 - \Pi)F\| + \|\mathbf{A}\mathbf{L}F\| \leq C_M \|(1 - \Pi)F\| \quad (\text{H4})$$

The estimate

$$\frac{1}{2} \frac{d}{dt} \|F\|^2 = \langle \mathbf{L}F, F \rangle \leq -\lambda_m \|(1 - \Pi)F\|^2$$

is not enough to conclude that  $\|F(t, \cdot)\|^2$  decays exponentially

## Equivalence and entropy decay

For some  $\delta > 0$  to be determined later, the  $L^2$  entropy / Lyapunov functional is defined by

$$\mathbf{H}[F] := \frac{1}{2} \|F\|^2 + \delta \operatorname{Re}\langle \mathbf{A}F, F \rangle$$

as in (J.D.-Mouhot-Schmeiser) so that  $\langle \mathbf{A}\Pi F, F \rangle \sim \|\Pi F\|^2$  and

$$\begin{aligned} -\frac{d}{dt}\mathbf{H}[F] &= : \mathbf{D}[F] \\ &= -\langle \mathbf{L}F, F \rangle + \delta \langle \mathbf{A}\Pi F, F \rangle \\ &\quad - \delta \operatorname{Re}\langle \mathbf{T}AF, F \rangle + \delta \operatorname{Re}\langle \mathbf{A}\mathbf{T}(1 - \Pi)F, F \rangle - \delta \operatorname{Re}\langle \mathbf{A}\mathbf{L}F, F \rangle \end{aligned}$$

▷ *entropy decay rate*: for any  $\delta > 0$  small enough and  $\lambda = \lambda(\delta)$

$$\lambda \mathbf{H}[F] \leq \mathbf{D}[F]$$

▷ *norm equivalence* of  $\mathbf{H}[F]$  and  $\|F\|^2$

$$\frac{2 - \delta}{4} \|F\|^2 \leq \mathbf{H}[F] \leq \frac{2 + \delta}{4} \|F\|^2$$

# Exponential decay of the entropy

$$\lambda = \frac{\lambda_M}{3(1+\lambda_M)} \min \left\{ 1, \lambda_m, \frac{\lambda_m \lambda_M}{(1+\lambda_M) C_M^2} \right\}, \quad \delta = \frac{1}{2} \min \left\{ 1, \lambda_m, \frac{\lambda_m \lambda_M}{(1+\lambda_M) C_M^2} \right\}$$

$$h_1(\delta, \lambda) := (\delta C_M)^2 - 4 \left( \lambda_m - \delta - \frac{2 + \delta}{4} \lambda \right) \left( \frac{\delta \lambda_M}{1 + \lambda_M} - \frac{2 + \delta}{4} \lambda \right)$$

## Theorem

Let  $\mathbf{L}$  and  $\mathbf{T}$  be closed linear operators (respectively Hermitian and anti-Hermitian) on  $\mathcal{H}$ . Under (H1)–(H4), for any  $t \geq 0$

$$\mathbf{H}[F(t, \cdot)] \leq \mathbf{H}[F_0] e^{-\lambda_* t}$$

where  $\lambda_*$  is characterized by

$$\lambda_* := \sup \left\{ \lambda > 0 : \exists \delta > 0 \text{ s.t. } h_1(\delta, \lambda) = 0, \lambda_m - \delta - \frac{1}{4} (2 + \delta) \lambda > 0 \right\}$$

## Sketch of the proof

• Since  $A\Pi = (1 + (\Pi)^*\Pi)^{-1} (\Pi)^*\Pi$ , from (H1) and (H2)

$$-\langle LF, F \rangle + \delta \langle A\Pi F, F \rangle \geq \lambda_m \|(1 - \Pi)F\|^2 + \frac{\delta \lambda_M}{1 + \lambda_M} \|\Pi F\|^2$$

• By (H4), we know that

$$|\operatorname{Re}\langle A(1 - \Pi)F, F \rangle + \operatorname{Re}\langle ALF, F \rangle| \leq C_M \|\Pi F\| \|(1 - \Pi)F\|$$

• The equation  $G = AF$  is equivalent to  $(\Pi)^*F = G + (\Pi)^*\Pi G$

$$\langle TAF, F \rangle = \langle G, (\Pi)^*F \rangle = \|G\|^2 + \|\Pi G\|^2 = \|AF\|^2 + \|TAF\|^2$$

By the Cauchy-Schwarz inequality, for any  $\mu > 0$

$$\langle G, (\Pi)^*F \rangle \leq \|TAF\| \|(1 - \Pi)F\| \leq \frac{1}{2\mu} \|TAF\|^2 + \frac{\mu}{2} \|(1 - \Pi)F\|^2$$

$$\|AF\| \leq \frac{1}{2} \|(1 - \Pi)F\|, \quad \|TAF\| \leq \|(1 - \Pi)F\|, \quad |\langle TAF, F \rangle| \leq \|(1 - \Pi)F\|^2$$

• With  $X := \|(1 - \Pi)F\|$  and  $Y := \|\Pi F\|$

$$D[F] - \lambda H[F] \geq (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M XY - \frac{2 + \delta}{4} \lambda (X^2 + Y^2)$$



# Hypocoercivity

## Corollary

For any  $\delta \in (0, 2)$ , if  $\lambda(\delta)$  is the largest positive root of  $h_1(\delta, \lambda) = 0$  for which  $\lambda_m - \delta - \frac{1}{4}(2 + \delta)\lambda > 0$ , then for any solution  $F$  of (2)

$$\|F(t)\|^2 \leq \frac{2 + \delta}{2 - \delta} e^{-\lambda(\delta)t} \|F(0)\|^2 \quad \forall t \geq 0$$

From the norm equivalence of  $\mathbf{H}[F]$  and  $\|F\|^2$

$$\frac{2 - \delta}{4} \|F\|^2 \leq \mathbf{H}[F] \leq \frac{2 + \delta}{4} \|F\|^2$$

We use  $\frac{2 - \delta}{4} \|F_0\|^2 \leq \mathbf{H}[F_0]$  so that  $\lambda_\star \geq \sup_{\delta \in (0, 2)} \lambda(\delta)$

## A toy problem

$$\frac{du}{dt} = (\mathbf{L} - \mathbf{T})u, \quad \mathbf{L} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad k^2 \geq \Lambda > 0$$

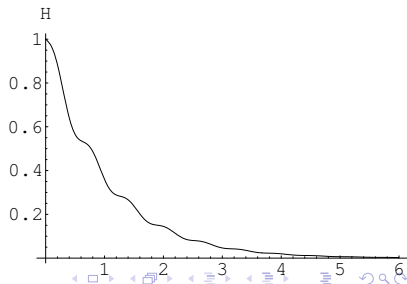
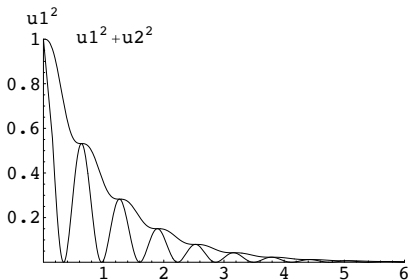
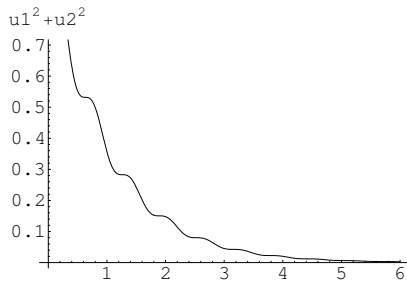
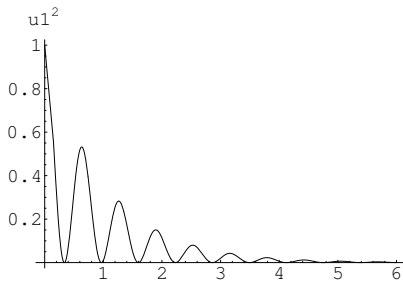
Non-monotone decay, a well known picture:

see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem:  $\frac{d}{dt}|u|^2 = -2u_2^2$
- macroscopic limit:  $\frac{du_1}{dt} = -k^2 u_1$
- generalized entropy:  $\mathbf{H}(u) = |u|^2 - \frac{\delta k}{1+k^2} u_1 u_2$

$$\begin{aligned} \frac{d\mathbf{H}}{dt} &= -\left(2 - \frac{\delta k^2}{1+k^2}\right) u_2^2 - \frac{\delta k^2}{1+k^2} u_1^2 + \frac{\delta k}{1+k^2} u_1 u_2 \\ &\leq -(2-\delta) u_2^2 - \frac{\delta \Lambda}{1+\Lambda} u_1^2 + \frac{\delta}{2} u_1 u_2 \end{aligned}$$

# Plots for the toy problem



# Mode-by-mode hypocoercivity

- ▷ Fokker-Planck equation and scattering collision operators
- ▷ A mode-by-mode hypocoercivity result
- ▷ Enlargement of the space by factorization
- ▷ Application to the torus and some numerical results

(Bouin, J.D., Mischler, Mouhot, Schmeiser)

# Fokker-Planck equation with general equilibria

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathbb{L}f, \quad f(0, x, v) = f_0(x, v) \quad (3)$$

for a distribution function  $f(t, x, v)$ , with *position* variable  $x \in \mathbb{R}^d$  or  $x \in \mathbb{T}^d$  the flat  $d$ -dimensional torus

*Fokker-Planck* collision operator with a general equilibrium  $M$

$$\mathbb{L}f = \nabla_v \cdot \left[ M \nabla_v (M^{-1} f) \right]$$

Notation and assumptions: an *admissible local equilibrium*  $M$  is positive, radially symmetric and

$$\int_{\mathbb{R}^d} M(v) dv = 1, \quad d\gamma = \gamma(v) dv := \frac{dv}{M(v)}$$

$\gamma$  is an *exponential weight* if

$$\lim_{|v| \rightarrow \infty} \frac{|v|^k}{\gamma(v)} = \lim_{|v| \rightarrow \infty} M(v) |v|^k = 0 \quad \forall k \in (d, \infty)$$

## Definitions

$$\Theta = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 M(v) dv = \int_{\mathbb{R}^d} (v \cdot e)^2 M(v) dv$$

for an arbitrary  $e \in \mathbb{S}^{d-1}$

$$\int_{\mathbb{R}^d} v \otimes v M(v) dv = \Theta \text{Id}$$

Then

$$\theta = \frac{1}{d} \|\nabla_v M\|_{L^2(d\gamma)}^2 = \frac{4}{d} \int_{\mathbb{R}^d} |\nabla_v \sqrt{M}|^2 dv < \infty$$

If  $M(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$ , then  $\Theta = 1$  and  $\theta = 1$

$$\bar{\sigma} := \frac{1}{2} \sqrt{\theta/\Theta}$$

*Microscopic coercivity property* (Poincaré inequality): for all  $u = M^{-1} F \in H^1(M dv)$

$$\int_{\mathbb{R}^d} |\nabla u|^2 M dv \geq \lambda_m \int_{\mathbb{R}^d} \left( u - \int_{\mathbb{R}^d} u M dv \right)^2 M dv$$

# Scattering collision operators

*Scattering* collision operator

$$\mathbb{L}f = \int_{\mathbb{R}^d} \sigma(\cdot, v') (f(v') M(\cdot) - f(\cdot) M(v')) dv'$$

Main assumption on the *scattering rate*  $\sigma$ : for some positive, finite  $\bar{\sigma}$

$$1 \leq \sigma(v, v') \leq \bar{\sigma} \quad \forall v, v' \in \mathbb{R}^d$$

Example: linear BGK operator

$$\mathbb{L}f = M\rho_f - f, \quad \rho_f(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$$

*Local mass conservation*

$$\int_{\mathbb{R}^d} \mathbb{L}f dv = 0$$

and we have

$$\int_{\mathbb{R}^d} |\mathbb{L}f|^2 d\gamma \leq 4\bar{\sigma}^2 \int_{\mathbb{R}^d} |M\rho_f - f|^2 d\gamma$$

The symmetry condition

$$\int_{\mathbb{R}^d} (\sigma(v, v') - \sigma(v', v)) M(v') dv' = 0 \quad \forall v \in \mathbb{R}^d$$

implies the *local mass conservation*  $\int_{\mathbb{R}^d} \mathbf{L}f dv = 0$

*Micro-reversibility*, i.e., the symmetry of  $\sigma$ , is not required

The null space of  $\mathbf{L}$  is spanned by the local equilibrium  $M$   
 $\mathbf{L}$  only acts on the velocity variable

*Microscopic coercivity property*: for some  $\lambda_m > 0$

$$\begin{aligned} \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(v, v') M(v) M(v') (u(v) - u(v'))^2 dv' dv \\ \geq \lambda_m \int_{\mathbb{R}^d} (u - \rho_u M)^2 M dv \end{aligned}$$

holds according to Proposition 2.2 of (Degond, Goudon, Poupaud, 2000) for all  $u = M^{-1} F \in L^2(M dv)$ . If  $\sigma \equiv 1$ , then  $\lambda_m = 1$



## Fourier modes

In order to perform a *mode-by-mode hypocoercivity* analysis, we introduce the Fourier representation with respect to  $x$ ,

$$f(t, x, v) = \int_{\mathbb{R}^d} \hat{f}(t, \xi, v) e^{-ix \cdot \xi} d\mu(\xi)$$

$d\mu(\xi) = (2\pi)^{-d} d\xi$  and  $d\xi$  is the Lebesgue measure if  $x \in \mathbb{R}^d$   
 $d\mu(\xi) = (2\pi)^{-d} \sum_{z \in \mathbb{Z}^d} \delta(\xi - z)$  is discrete for  $x \in \mathbb{T}^d$

Parseval's identity if  $\xi \in \mathbb{Z}^d$  and Plancherel's formula if  $x \in \mathbb{R}^d$  read

$$\|f(t, \cdot, v)\|_{L^2(dx)} = \|\hat{f}(t, \cdot, v)\|_{L^2(d\mu(\xi))}$$

The Cauchy problem is now decoupled in the  $\xi$ -direction

$$\partial_t \hat{f} + \mathbb{T} \hat{f} = \mathbb{L} \hat{f}, \quad \hat{f}(0, \xi, v) = \hat{f}_0(\xi, v)$$

$$\mathbb{T} \hat{f} = i(v \cdot \xi) \hat{f}$$

For any fixed  $\xi \in \mathbb{R}^d$ , let us apply the abstract result with

$$\mathcal{H} = L^2(d\gamma), \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2 d\gamma, \quad \Pi F = M \int_{\mathbb{R}^d} F dv = M \rho_F$$

and  $\mathbb{T}\hat{f} = i(v \cdot \xi)\hat{f}$ ,  $\mathbb{T}\Pi F = i(v \cdot \xi)\rho_F M$ ,

$$\|\mathbb{T}\Pi F\|^2 = |\rho_F|^2 \int_{\mathbb{R}^d} |v \cdot \xi|^2 M(v) dv = \Theta |\xi|^2 |\rho_F|^2 = \Theta |\xi|^2 \|\Pi F\|^2$$

(H2) *Macroscopic coercivity*  $\|\mathbb{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2 : \lambda_M = \Theta |\xi|^2$

(H3)  $\int_{\mathbb{R}^d} v M(v) dv = 0$

The operator  $A$  is given by

$$AF = \frac{-i \xi \cdot \int_{\mathbb{R}^d} v' F(v') dv'}{1 + \Theta |\xi|^2} M$$

## A mode-by-mode hypoocoercivity result

$$\begin{aligned} \|AF\| &= \|A(1 - \Pi)F\| \leq \frac{1}{1 + \Theta |\xi|^2} \int_{\mathbb{R}^d} \frac{|(1 - \Pi)F|}{\sqrt{M}} |v \cdot \xi| \sqrt{M} dv \\ &\leq \frac{1}{1 + \Theta |\xi|^2} \|(1 - \Pi)F\| \left( \int_{\mathbb{R}^d} (v \cdot \xi)^2 M dv \right)^{1/2} \\ &= \frac{\sqrt{\Theta} |\xi|}{1 + \Theta |\xi|^2} \|(1 - \Pi)F\| \end{aligned}$$

- Scattering operator  $\|LF\|^2 \leq 4\bar{\sigma}^2 \|(1 - \Pi)F\|^2$
- Fokker-Planck (FP) operator

$$\|ALF\| \leq \frac{2}{1 + \Theta |\xi|^2} \int_{\mathbb{R}^d} \frac{|(1 - \Pi)F|}{\sqrt{M}} |\xi \cdot \nabla_v \sqrt{M}| dv \leq \frac{\sqrt{\theta} |\xi|}{1 + \Theta |\xi|^2} \|(1 - \Pi)F\|$$

In both cases with  $\kappa = \sqrt{\theta}$  (FP) or  $\kappa = 2\bar{\sigma}\sqrt{\Theta}$  we obtain

$$\|ALF\| \leq \frac{\kappa |\xi|}{1 + \Theta |\xi|^2} \|(1 - \Pi)F\|$$

$$\mathbf{TAF}(v) = -\frac{(v \cdot \xi) M}{1 + \Theta |\xi|^2} \int_{\mathbb{R}^d} (v' \cdot \xi) (1 - \Pi) F(v') dv'$$

is estimated by

$$\|\mathbf{TAF}\| \leq \frac{\Theta |\xi|^2}{1 + \Theta |\xi|^2} \|(1 - \Pi)F\|$$

(H4) holds with  $C_M = \frac{\kappa |\xi| + \Theta |\xi|^2}{1 + \Theta |\xi|^2}$

Two elementary estimates

$$\frac{\Theta |\xi|^2}{1 + \Theta |\xi|^2} \geq \frac{\Theta}{\max\{1, \Theta\}} \frac{|\xi|^2}{1 + |\xi|^2}$$

$$\frac{\lambda_M}{(1 + \lambda_M) C_M^2} = \frac{\Theta (1 + \Theta |\xi|^2)}{(\kappa + \Theta |\xi|)^2} \geq \frac{\Theta}{\kappa^2 + \Theta}$$

# Mode-by-mode hypoocoercivity with exponential weights

## Theorem

Let us consider an admissible  $M$  and a collision operator  $\mathsf{L}$  satisfying Assumption (H), and take  $\xi \in \mathbb{R}^d$ . If  $\hat{f}$  is a solution such that  $\hat{f}_0(\xi, \cdot) \in L^2(d\gamma)$ , then for any  $t \geq 0$ , we have

$$\left\| \hat{f}(t, \xi, \cdot) \right\|_{L^2(d\gamma)}^2 \leq 3 e^{-\mu_\xi t} \left\| \hat{f}_0(\xi, \cdot) \right\|_{L^2(d\gamma)}^2$$

where

$$\mu_\xi := \frac{\Lambda |\xi|^2}{1 + |\xi|^2} \quad \text{and} \quad \Lambda = \frac{\Theta}{3 \max\{1, \Theta\}} \min \left\{ 1, \frac{\lambda_m \Theta}{\kappa^2 + \Theta} \right\}$$

with  $\kappa = 2\bar{\sigma} \sqrt{\Theta}$  for scattering operators  
and  $\kappa = \sqrt{\theta}$  for (FP) operators

## Enlargement of the space by factorization

A simple case (factorization of order 1) of the *factorization method* of (Gualdani, Mischler, Mouhot)

### Theorem

Let  $\mathcal{B}_1, \mathcal{B}_2$  be Banach spaces and let  $\mathcal{B}_2$  be continuously imbedded in  $\mathcal{B}_1$ , i.e.,  $\|\cdot\|_1 \leq c_1 \|\cdot\|_2$ . Let  $\mathfrak{B}$  and  $\mathfrak{A} + \mathfrak{B}$  be the generators of the strongly continuous semigroups  $e^{\mathfrak{B}t}$  and  $e^{(\mathfrak{A}+\mathfrak{B})t}$  on  $\mathcal{B}_1$ . If for all  $t \geq 0$ ,

$$\left\| e^{(\mathfrak{A}+\mathfrak{B})t} \right\|_{2 \rightarrow 2} \leq c_2 e^{-\lambda_2 t}, \quad \left\| e^{\mathfrak{B}t} \right\|_{1 \rightarrow 1} \leq c_3 e^{-\lambda_1 t}, \quad \|\mathfrak{A}\|_{1 \rightarrow 2} \leq c_4$$

where  $\|\cdot\|_{i \rightarrow j}$  denotes the operator norm for linear mappings from  $\mathcal{B}_i$  to  $\mathcal{B}_j$ . Then there exists a positive constant  $C = C(c_1, c_2, c_3, c_4)$  such that, for all  $t \geq 0$ ,

$$\left\| e^{(\mathfrak{A}+\mathfrak{B})t} \right\|_{1 \rightarrow 1} \leq \begin{cases} C (1 + |\lambda_1 - \lambda_2|^{-1}) e^{-\min\{\lambda_1, \lambda_2\}t} & \text{for } \lambda_1 \neq \lambda_2 \\ C (1 + t) e^{-\lambda_1 t} & \text{for } \lambda_1 = \lambda_2 \end{cases}$$

Integrating the identity  $\frac{d}{ds} (e^{(\mathfrak{A}+\mathfrak{B})s} e^{\mathfrak{B}(t-s)}) = e^{(\mathfrak{A}+\mathfrak{B})s} \mathfrak{A} e^{\mathfrak{B}(t-s)}$  with respect to  $s \in [0, t]$  gives

$$e^{(\mathfrak{A}+\mathfrak{B})t} = e^{\mathfrak{B}t} + \int_0^t e^{(\mathfrak{A}+\mathfrak{B})s} \mathfrak{A} e^{\mathfrak{B}(t-s)} ds$$

The proof is completed by the straightforward computation

$$\begin{aligned} \|e^{(\mathfrak{A}+\mathfrak{B})t}\|_{1 \rightarrow 1} &\leq c_3 e^{-\lambda_1 t} + c_1 \int_0^t \|e^{(\mathfrak{A}+\mathfrak{B})s} \mathfrak{A} e^{\mathfrak{B}(t-s)}\|_{1 \rightarrow 2} ds \\ &\leq c_3 e^{-\lambda_1 t} + c_1 c_2 c_3 c_4 e^{-\lambda_1 t} \int_0^t e^{(\lambda_1 - \lambda_2)s} ds \end{aligned}$$

## Weights with polynomial growth

Let us consider the measure

$$d\gamma_k := \gamma_k(v) dv \quad \text{where} \quad \gamma_k(v) = \pi^{d/2} \frac{\Gamma((k-d)/2)}{\Gamma(k/2)} (1 + |v|^2)^{k/2}$$

for an arbitrary  $k \in (d, +\infty)$

We choose  $\mathcal{B}_1 = L^2(d\gamma_k)$  and  $\mathcal{B}_2 = L^2(d\gamma)$

### Theorem

Let  $\Lambda = \frac{\Theta}{3 \max\{1, \Theta\}} \min \left\{ 1, \frac{\lambda_m \Theta}{\kappa^2 + \Theta} \right\}$  and  $k \in (d, \infty]$ . For any  $\xi \in \mathbb{R}^d$  if  $\hat{f}$  is a solution with initial datum  $\hat{f}_0(\xi, \cdot) \in L^2(d\gamma_k)$ , then there exists a constant  $C = C(k, d, \bar{\sigma})$  such that

$$\left\| \hat{f}(t, \xi, \cdot) \right\|_{L^2(d\gamma_k)}^2 \leq C e^{-\mu_\xi t} \left\| \hat{f}_0(\xi, \cdot) \right\|_{L^2(d\gamma_k)}^2 \quad \forall t \geq 0$$



• Fokker-Planck:  $\mathfrak{A}F = N \chi_R F$  and  $\mathfrak{B}F = -i(v \cdot \xi) F + \mathbb{L}F - \mathfrak{A}F$   
 $N$  and  $R$  are two positive constants,  $\chi$  is a smooth cut-off function  
 and  $\chi_R := \chi(\cdot/R)$

For any  $R$  and  $N$  large enough, according to Lemma 3.8 of (Mischler, Mouhot, 2016)

$$\int_{\mathbb{R}^d} (\mathbb{L} - \mathfrak{A})(F)F \, d\gamma_k \leq -\lambda_1 \int_{\mathbb{R}^d} F^2 \, d\gamma_k$$

for some  $\lambda_1 > 0$  if  $k > d$ , and  $\lambda_2 = \mu_\xi/2 \leq 1/4$

• Scattering operator:

$$\begin{aligned} \mathfrak{A}F(v) &= M(v) \int_{\mathbb{R}^d} \sigma(v, v') F(v') \, dv' \\ \mathfrak{B}F(v) &= - \left[ i(v \cdot \xi) + \int_{\mathbb{R}^d} \sigma(v, v') M(v') \, dv' \right] F(v) \end{aligned}$$

Boundedness:  $\|\mathfrak{A}F\|_{L^2(d\gamma)} \leq \bar{\sigma} \left( \int_{\mathbb{R}^d} \gamma_k^{-1} \, dv \right)^{1/2} \|F\|_{L^2(d\gamma_k)}$   
 $\lambda_1 = 1$  and  $\lambda_2 = \mu_\xi/2 \leq 1/4$

# Exponential convergence to equilibrium in $\mathbb{T}^d$

The unique global equilibrium in the case  $x \in \mathbb{T}^d$  is given by

$$f_\infty(x, v) = \rho_\infty M(v) \quad \text{with} \quad \rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0 dx dv$$

## Theorem

*Assume that  $k \in (d, \infty]$  and  $\gamma$  has an exponential growth if  $k = \infty$ .*

*We consider an admissible  $M$ , a collision operator  $\mathbb{L}$  satisfying Assumption (H), and  $\Lambda$  given by (11)*

*There exists a positive constant  $C_k$  such that the solution  $f$  of (3) on  $\mathbb{T}^d \times \mathbb{R}^d$  with initial datum  $f_0 \in L^2(dx d\gamma_k)$  satisfies*

$$\|f(t, \cdot, \cdot) - f_\infty\|_{L^2(dx d\gamma_k)} \leq C_k \|f_0 - f_\infty\|_{L^2(dx d\gamma_k)} e^{-\frac{1}{4} \Lambda t} \quad \forall t \geq 0$$

If we represent the flat torus  $\mathbb{T}^d$  by the box  $[0, 2\pi)^d$  with periodic boundary conditions, the Fourier variable satisfies  $\xi \in \mathbb{Z}^d$ . For  $\xi = 0$ , the microscopic coercivity implies

$$\left\| \hat{f}(t, 0, \cdot) - \hat{f}_\infty(0, \cdot) \right\|_{L^2(d\gamma)} \leq \left\| \hat{f}_0(0, \cdot) - \hat{f}_\infty(0, \cdot) \right\|_{L^2(d\gamma)} e^{-t}$$

Otherwise  $\mu_\xi \geq \Lambda/2$  for any  $\xi \neq 0$

Parseval's identity applies, with measure  $d\gamma(v)$  and  $C_\infty = \sqrt{3}$   
 The result with weight  $\gamma_k$  follows from the factorization result for some  $C_k > 0$

# Computation of the constants

▷ A more numerical point of view

Two simple examples:  $L$  denotes either the *Fokker-Planck operator*

$$L_1 f := \Delta_v f + \nabla_v \cdot (v f)$$

or the *linear BGK operator*

$$L_2 f := \Pi f - f$$

$\Pi f = \rho_f M$  is the projection operator on the normalized Gaussian function

$$M(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$$

and  $\rho_f := \int_{\mathbb{R}^d} f dv$  is the spatial density

Where do we have space for improvements ?

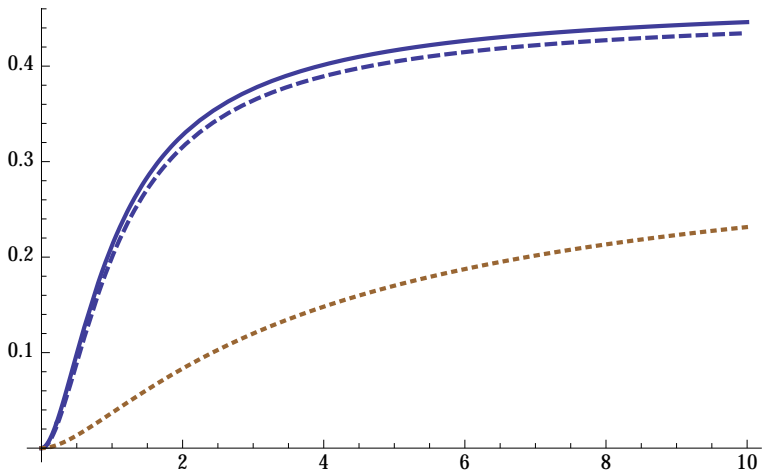
• With  $X := \|(1 - \Pi)F\|$  and  $Y := \|\Pi F\|$ , we wrote

$$\begin{aligned} & D[F] - \lambda H[F] \\ & \geq (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{\lambda}{2} (X^2 + Y^2 + \delta X Y) \\ & \geq (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{2 + \delta}{4} \lambda (X^2 + Y^2) \end{aligned}$$

• We can directly study the positivity condition for the quadratic form

$$(\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{\lambda}{2} (X^2 + Y^2 + \delta X Y)$$

$$\lambda_m = 1, \lambda_M = |\xi|^2 \text{ and } C_M = |\xi| (1 + |\xi|) / (1 + |\xi|^2)$$

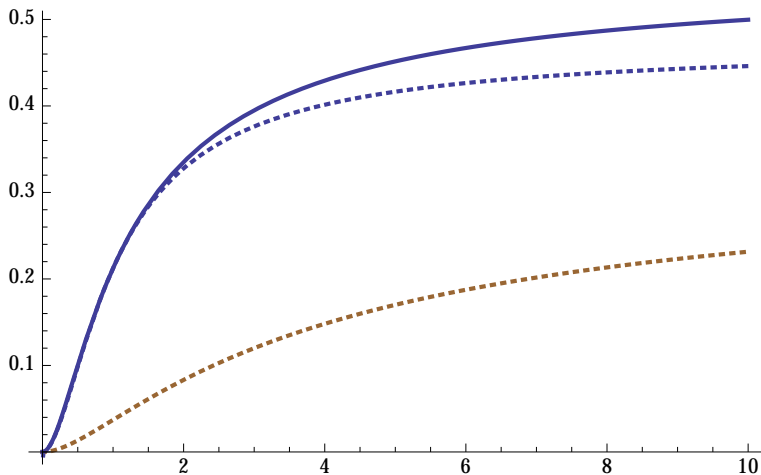


With  $\lambda_m = 1$ ,  $\lambda_M = |\xi|^2$  and  $C_M = |\xi|(1 + |\xi|)/(1 + |\xi|^2)$ , we optimize  $\lambda$  under the condition that the quadratic form

$$(\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{\lambda}{2} (X^2 + Y^2 + \delta X Y)$$

is positive, thus getting a  $\lambda(\xi)$

• By taking also  $\delta = \delta(\xi)$  where  $\xi$  is seen as a parameter, we get a better estimate of  $\lambda(\xi)$





By taking  $\delta = \delta(\xi)$ , for each value of  $\xi$  we build a different Lyapunov function, namely

$$H_\xi[F] := \frac{1}{2} \|F\|^2 + \delta(\xi) \operatorname{Re}\langle AF, F \rangle$$

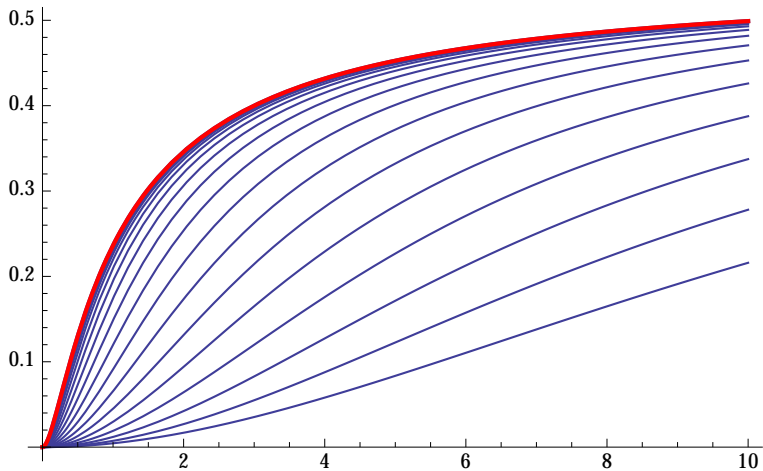
where the operator  $A$  is given by

$$AF = \frac{-i \xi \cdot \int_{\mathbb{R}^d} v' F(v') dv'}{1 + |\xi|^2} M$$

• We can consider

$$A_\varepsilon F = \frac{-i \xi \cdot \int_{\mathbb{R}^d} v' F(v') dv'}{\varepsilon + |\xi|^2} M$$

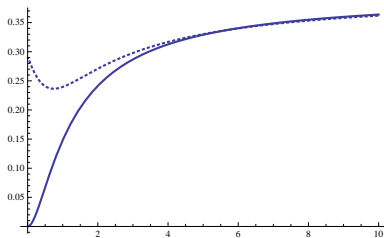
and look for the optimal value of  $\varepsilon$ ...

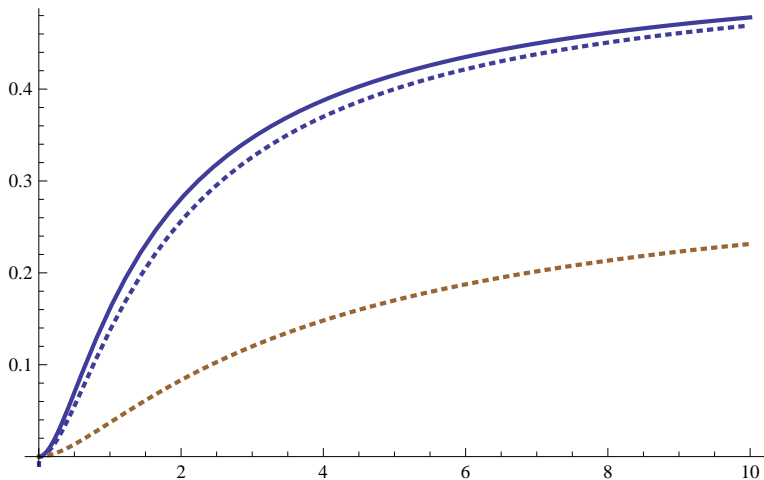


The dependence of  $\lambda$  in  $\varepsilon$  is monotone, and the limit as  $\varepsilon \rightarrow 0_+$  gives the optimal estimate of  $\lambda$ . The operator

$$A_0 F = \frac{-i \xi \cdot \int_{\mathbb{R}^d} v' F(v') dv'}{|\xi|^2} M$$

is not bounded anymore, but estimates still make sense and  $\lim_{\xi \rightarrow 0} \delta(\xi) = 0$  (see below)





## Theorem (Hypocoercivity on $\mathbb{T}^d$ with exponential weight)

Assume that  $L = L_1$  or  $L = L_2$ . If  $f$  is a solution, then

$$\|f(t, \cdot, \cdot) - f_\infty\|_{L^2(dx d\gamma)}^2 \leq \mathcal{C}_\star \|f_0\|_{L^2(dx d\gamma)}^2 e^{-\lambda_\star t} \quad \forall t \geq 0$$

with  $f_\infty(x, v) = M(v) \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) dx dv$

$\mathcal{C}_\star \approx 1.75863$  and  $\lambda_\star = \frac{2}{13}(5 - 2\sqrt{3}) \approx 0.236292$ .

(work in progress)

## Some comments on recent works

A more algebraic approach based on the spectral analysis of symmetric and non-symmetric operators

- On BGK models  
(Achleitner, Arnold, Carlen)
- On Fokker-Planck models  
(Arnold, Erb)  
(Arnold, Stürzer)  
(Arnold, Einav, Wöhler)

# Decay rates in the whole space

## Algebraic decay rates in $\mathbb{R}^d$

On the whole Euclidean space, we can define the entropy

$$\mathbf{H}[f] := \frac{1}{2} \|f\|_{L^2(dx d\gamma_k)}^2 + \delta \langle \mathbf{A}f, f \rangle_{dx d\gamma_k}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|_{L^2(dx)}^2 \leq \mathfrak{C}_{\text{Nash}} \|u\|_{L^1(dx)}^{\frac{4}{d+2}} \|\nabla u\|_{L^2(dx)}^{\frac{2d}{d+2}}$$

proves that

$$\mathbf{H}[f] \leq C \left( \mathbf{H}[f_0] + \|f_0\|_{L^1(dx dv)}^2 \right) (1+t)^{-\frac{d}{2}}$$

### Theorem

Assume that  $\gamma_k$  has an exponential growth ( $k = \infty$ ) or a polynomial growth of order  $k > d$

There exists a constant  $C > 0$  such that, for any  $t \geq 0$

$$\|f(t, \cdot, \cdot)\|_{L^2(dx d\gamma_k)}^2 \leq C \left( \|f_0\|_{L^2(dx d\gamma_k)}^2 + \|f_0\|_{L^2(d\gamma_k; L^1(dx))}^2 \right) (1+t)^{-\frac{d}{2}}$$



A direct proof... Recall that  $\mu_\xi = \frac{\Lambda |\xi|^2}{1+|\xi|^2}$

By the Plancherel formula

$$\|f(t, \cdot, \cdot)\|_{L^2(dx d\gamma_k)}^2 \leq C \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-\mu_\xi t} |\hat{f}_0|^2 d\xi \right) d\gamma_k$$

• if  $|\xi| < 1$ , then  $\mu_\xi \geq \frac{\Lambda}{2} |\xi|^2$

$$\begin{aligned} \int_{|\xi| \leq 1} e^{-\mu_\xi t} |\hat{f}_0|^2 d\xi &\leq C \|f_0(\cdot, v)\|_{L^1(dx)}^2 \int_{\mathbb{R}^d} e^{-\frac{\Lambda}{2} |\xi|^2 t} d\xi \\ &\leq C \|f_0(\cdot, v)\|_{L^1(dx)}^2 t^{-\frac{d}{2}} \end{aligned}$$

• if  $|\xi| \geq 1$ , then  $\mu_\xi \geq \Lambda/2$  when  $|\xi| \geq 1$

$$\int_{|\xi| > 1} e^{-\mu_\xi t} |\hat{f}_0|^2 d\xi \leq C e^{-\frac{\Lambda}{2} t} \|f_0(\cdot, v)\|_{L^2(dx)}^2$$

# Improved decay rate for zero average solutions

## Theorem

Assume that  $f_0 \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$  with  $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) dx dv = 0$  and  $\mathcal{C}_0 := \|f_0\|_{L^2(d\gamma_{k+2}; L^1(dx))}^2 + \|f_0\|_{L^2(d\gamma_k; L^1(|x| dx))}^2 + \|f_0\|_{L^2(dx d\gamma_k)}^2 < \infty$

Then there exists a constant  $c_k > 0$  such that

$$\|f(t, \cdot, \cdot)\|_{L^2(dx d\gamma_k)}^2 \leq c_k \mathcal{C}_0 (1+t)^{-(1+\frac{d}{2})}$$

## Step 1: Decay of the average in space, factorization

• *x*-average in space

$$f_{\bullet}(t, v) := \int_{\mathbb{R}^d} f(t, x, v) dx$$

with  $\int_{\mathbb{R}^d} f_{\bullet}(t, v) dv = 0$  and observe that  $f_{\bullet}$  solves a *Fokker-Planck* equation

$$\partial_t f_{\bullet} = \mathsf{L} f_{\bullet}$$

From the *microscopic coercivity property*, we deduce that

$$\|f_{\bullet}(t, \cdot)\|_{L^2(d\gamma)}^2 \leq \|f_{\bullet}(0, \cdot)\|_{L^2(d\gamma)}^2 e^{-\lambda_m t}$$

• *Factorisation*

$$\|f_{\bullet}(t, \cdot)\|_{L^2(|v|^2 d\gamma_k)}^2 \leq C \|f_0\|_{L^2(|v|^2 d\gamma_k; L^1(dx))}^2 e^{-\lambda t}$$

## Step 2: Improved decay of $f$

Let us define  $g := f - f_\bullet \varphi$ , with  $\varphi(x) := (2\pi)^{-d/2} e^{-|x|^2/2}$

The Fourier transform  $\hat{g}$  solves

$$\partial_t \hat{g} + \mathbb{T} \hat{g} = \mathbb{L} \hat{g} - f_\bullet \mathbb{T} \hat{\varphi} \text{ with } \mathbb{T} \hat{\varphi} = i(v \cdot \xi) \hat{\varphi}$$

Duhamel's formula

$$\hat{g} = \underbrace{e^{i(\mathbb{L}-\mathbb{T})t} \hat{g}_0}_{C e^{-\frac{1}{2} \mu_\xi t} \|\hat{g}_0(\xi, \cdot)\|_{L^2(d\gamma_k)}} + \int_0^t \underbrace{e^{i(\mathbb{L}-\mathbb{T})(t-s)} (-f_\bullet(s, v) \mathbb{T} \hat{\varphi}(\xi))}_{C e^{-\frac{\mu_\xi}{2}(t-s)} \|f_\bullet(s, \cdot)\|_{L^2(|v|^2 d\gamma_k)} |\xi| |\hat{\varphi}(\xi)|} ds$$

•  $\hat{g}_0(\xi, v) = \underbrace{\hat{g}_0(0, v)}_{=0} + \int_0^{|\xi|} \frac{\xi}{|\xi|} \cdot \nabla_\xi \hat{g}_0(\eta \frac{\xi}{|\xi|}, v) d\eta$  yields

$$|\hat{g}_0(\xi, v)| \leq |\xi| \|\nabla_\xi \hat{g}_0(\cdot, v)\|_{L^\infty(dv)} \leq |\xi| \|g_0(\cdot, v)\|_{L^1(|x| dx)}$$

•  $\mu_\xi = \Lambda |\xi|^2 / (1 + |\xi|^2) \geq \Lambda/2$  if  $|\xi| > 1$  (contribution  $O(e^{-\frac{\Lambda}{2}t})$ ) and

$$\int_{|\xi| \leq 1} \int_{\mathbb{R}^d} |e^{i(\mathbb{L}-\mathbb{T})t} \hat{g}_0|^2 d\gamma_k d\xi \leq \underbrace{\int_{\mathbb{R}^d} |\xi|^2 e^{-\frac{\Lambda}{2} |\xi|^2 t} d\xi}_{=O(t^{-(d+2)})} \|g_0\|_{L^2(d\gamma_k; L^1(|x| dx))}^2$$

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