Introduction to statistical mechanics

General framework (Static setting)

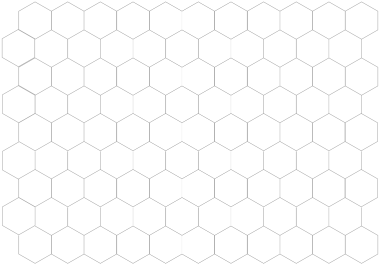
Dauphine, April 5, 2025

Statistical mechanics

Understand the large scale behavior of a physics system whose interactions are described on the microscopic level

Start from a model, whose general framework is the following.

- Structure of the physics system is represented by a graph $G = (V, E)$, finite.



Statistical mechanics

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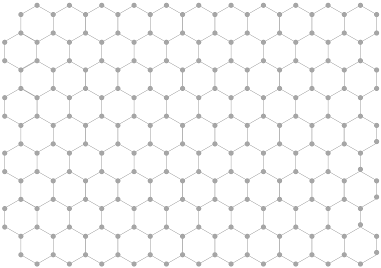
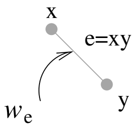


Statistical mechanics

- Set of configurations on the graph G: $\mathcal{C}(\mathsf{G})$,
  - vertex configurations,
  - edge configurations.
- Parameters:
  - intensity of interactions between microscopic components
  - external temperature.

$\Rightarrow$ Positive weight function $w = (w_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$ on the edges.



Statistical mechanics

- To a configuration C, one assigns an energy $\mathcal{E}_w(\mathsf{C})$.
- Boltzmann measure on configurations:

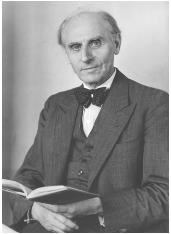
$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)},$$

where $Z(\mathsf{G}, w) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})}$ is the partition function.

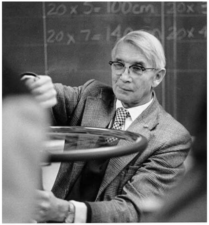
Understand the behavior of configurations when the graph is large (infinite).

The Ising model

Model of ferromagnetism - mixture of two materials



Wilhelm Lenz (1888-1957)



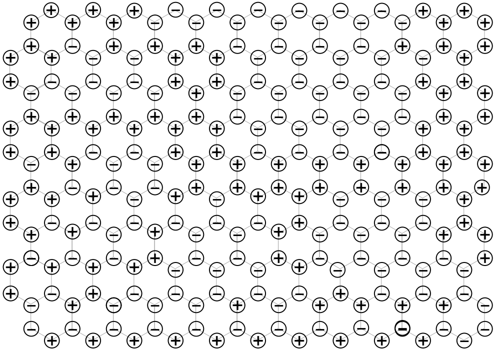
Ernst Ising (1900-1998)

- Graph $G = (V, E)$.
- A spin configuration σ assigns a spin $\sigma_x \in \{-1, 1\}$ to a each vertex x of the graph G.

$$\Rightarrow \mathcal{C}(G) = \{-1, 1\}^V = \text{set of spin configurations.}$$

The Ising model

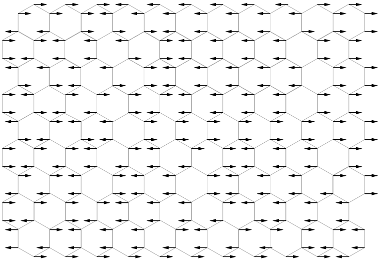
- A spin configuration



The Ising model

- A spin configuration / two interpretations

Magnetic moments:
$+1/\rightarrow$, $-1/\leftarrow$



The Ising model

- A spin configuration / two interpretations

Magnetic moments:

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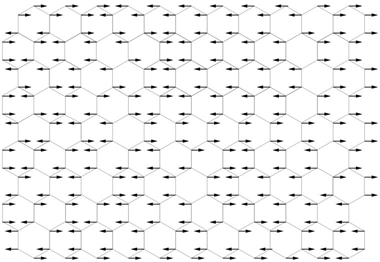
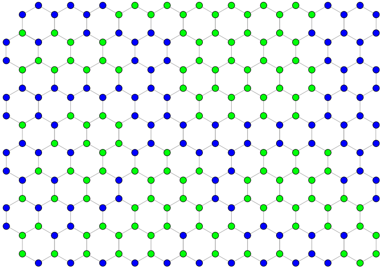


Mixture of two materials:

+1/●, -1/●.



The Ising model

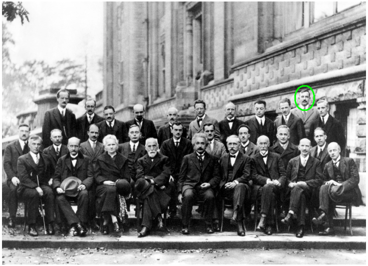
- Positive weight function: coupling constants $J = (J_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$.
- Energy of a spin configuration: $\mathcal{E}_J(\sigma) = -\sum_{\mathsf{e}=\mathsf{xy}\in\mathsf{E}} J_{\mathsf{xy}}\sigma_{\mathsf{x}}\sigma_{\mathsf{y}}$.
- Ising Boltzmann measure:

$$\forall\, \sigma \in \{-1,1\}^{\mathsf{V}}, \quad \mathbb{P}_{\text{Ising}}(\sigma) = \frac{e^{-\mathcal{E}_J(\sigma)}}{Z_{\text{Ising}}(\mathsf{G}, J)}.$$

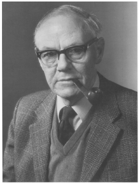
  - Two neighboring spins $\sigma_{\mathsf{x}}, \sigma_{\mathsf{y}}$ tend to align.
  - The higher the coupling J_{xy}, the higher this tendency.

The dimer model

Adsorption of di-atomic molecules on the surface of a cristal



Sir Ralph H. Fowler (1889-1944)
Congrès Solvay 1927.



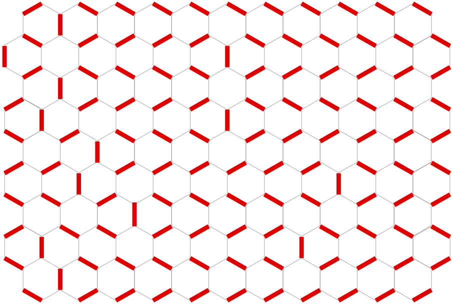
George S. Rushbrooke (1915-1995)

- Graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.
- A dimer configuration or perfect matching: subset of edges such that each vertex touches exactly one edge of this subset.

$$\Rightarrow \mathcal{C}(\mathsf{G}) = \mathcal{M}(\mathsf{G}) = \text{set of dimer configurations.}$$

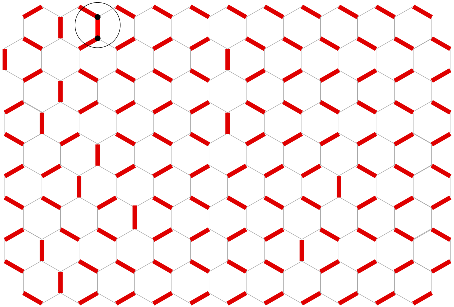
The dimer model

- A dimer configuration.



The dimer model

- A dimer configuration.



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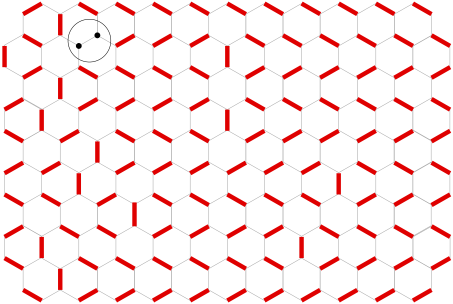

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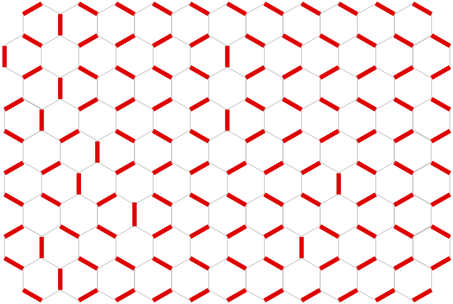


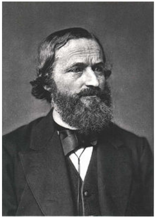
- Positive weight function: $\nu = (\nu_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$.
- Energy of a configuration M: $\mathcal{E}_\nu(\mathsf{M}) = -\sum_{\mathsf{e} \in \mathsf{M}} \log \nu_{\mathsf{e}}$.
- Dimer Botzmann measure:

$$\forall\, \mathsf{M} \in \mathcal{M}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e} \in \mathsf{M}} \nu_{\mathsf{e}}}{Z_{\mathrm{dimer}}(\mathsf{G}, \nu)}.$$

  - Edges with higher weights are more likely to occur.

Spanning trees

Related to electrical networks



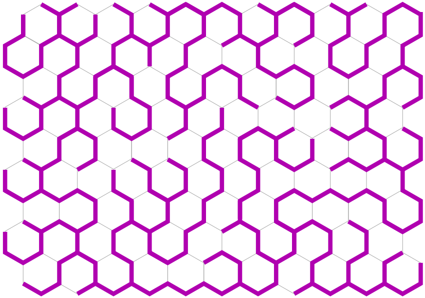
Gustav Kirchhoff (1824-1887)

- Graph $G = (V, E)$.
- A spanning tree: subset of edges covering all vertices of the graph, connected, with no cycle.

$$\Rightarrow \mathcal{C}(G) = \mathcal{T}(G) = \text{set of spanning trees.}$$

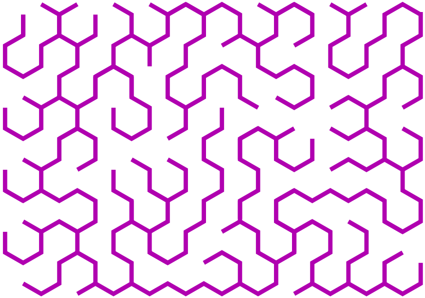
Spanning trees

- A spanning tree



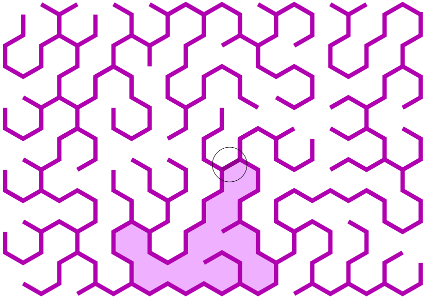
Spanning trees

- A spanning tree



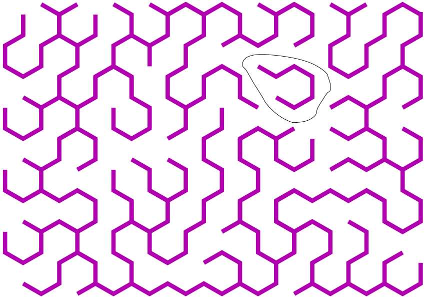
Spanning trees

- A spanning tree



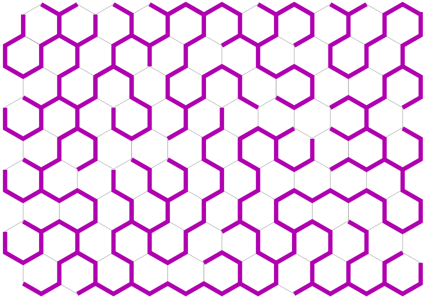
Spanning trees

- A spanning tree



Spanning trees

- A spanning tree



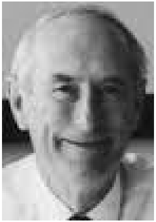
- Positive weight function: $\rho = (\rho_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$.
- Energy of a tree T: $\mathcal{E}_\rho(\mathsf{T}) = -\sum_{\mathsf{e}\in\mathsf{T}} \log \rho_{\mathsf{e}}$.
- Tree Boltzmann measure:

$$\forall\, \mathsf{T} \in \mathcal{T}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{tree}}(\mathsf{T}) = \frac{\prod_{\mathsf{e}\in\mathsf{T}} \rho_{\mathsf{e}}}{Z_{\mathrm{tree}}(\mathsf{G},\rho)}.$$

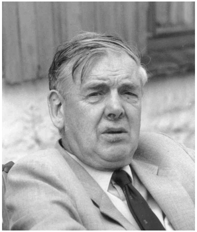
  - Edges with higher weights are more likely to occur.

Percolation

Flow of a liquid through a porous material



Simon Broadbent (1928-2002)



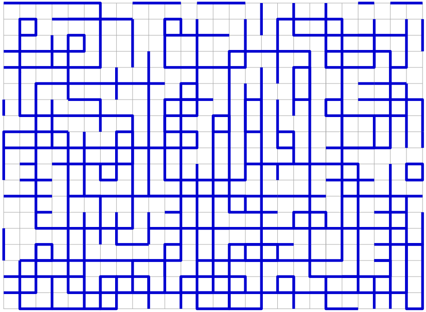
John Hammersley (1920-2004)

- Graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.
- Configuration of opened and closed edges: $\forall\, \mathsf{e} \in \mathsf{E},\ \omega_{\mathsf{e}} \in \{0,1\}$.

$$\Rightarrow \qquad \mathcal{C}(\mathsf{G}) = \{0,1\}^{\mathsf{E}}.$$

Percolation

- A percolation configuration



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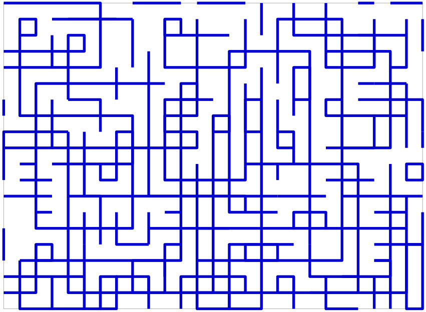


- Let $p \in [0,1]$. Each edge is opened/closed with probability p / $1-p$, independently.
- Percolation measure:

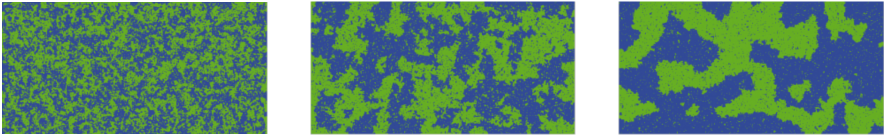
$$\forall\, \omega \in \{0,1\}^{\mathsf{E}}, \quad \mathbb{P}_{\text{perco}}(\omega) = p^{\sum_{e \in \mathsf{E}} \omega_e}(1-p)^{|\mathsf{E}| - \sum_{e \in \mathsf{E}} \omega_e}.$$

  - The higher p is, the more open edges there are
  - For which values of p do we percolate ?

Macroscopic behavior

Let the edge-length tend to 0
Look at a "typical" configuration.

- Ising model (Illustrations of R. Cerf)

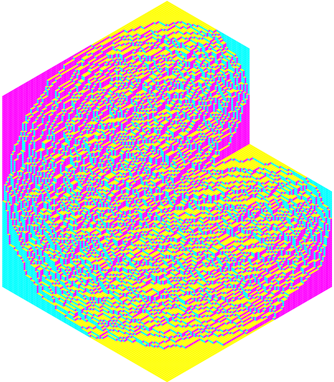


J small J critical J large

- On $\mathbb{Z}^2$: $J_c \equiv \frac{1}{2}\log(1+\sqrt{2})$ [Kramers et Wannier]
- Phase transition: studied through magnetization.

Macroscopic behavior

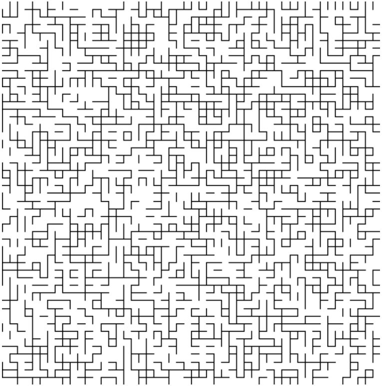
- Dimer model (Illustration of R. Kenyon)



- One sees two phases on the same figure.
- Phase transition studied through decay of correlations.

Macroscopic behavior

- Percolation (Illustration of Erzbischof)



$p = 0.51$

- On $\mathbb{Z}^2$: $p_c = 0.5$ [Kesten]
- Phase transition: (non) existence of an infinite connected component.

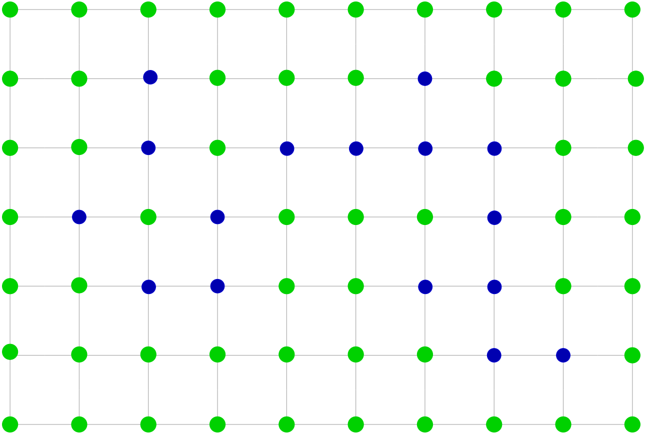
Macroscopic behavior

- Identification of the phase transition.
- Understand the sub/super critical models.
- Understand the critical model (at the phase transition):
  - Universality and conformal invariance.
  - Conjectures: Nienhuis, Cardy, Duplantier ...

    Proofs: Lawler, Schramm, Werner (Fields 2006), D. Chelkak, S. Smirnov (Fields 2010), H. Duminil-Copin (Fields 2022), ...

2. Relations between models: Ising - dimers [Fisher'66]

- Ising model on G.



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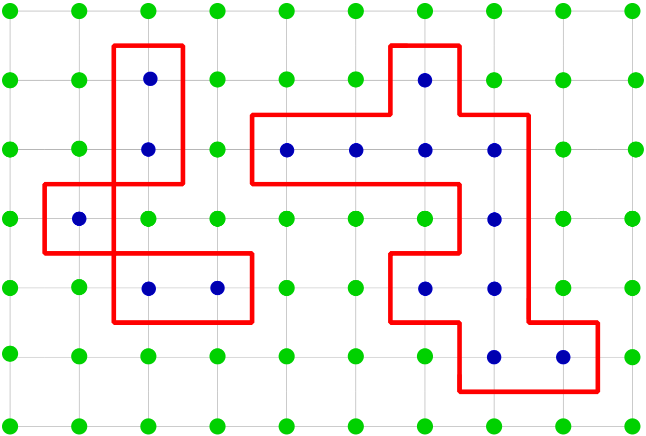
- Low temperature expansion [Kramers-Wannier].



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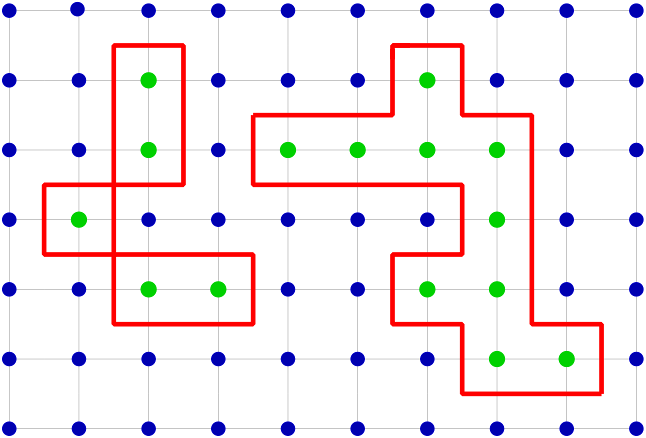
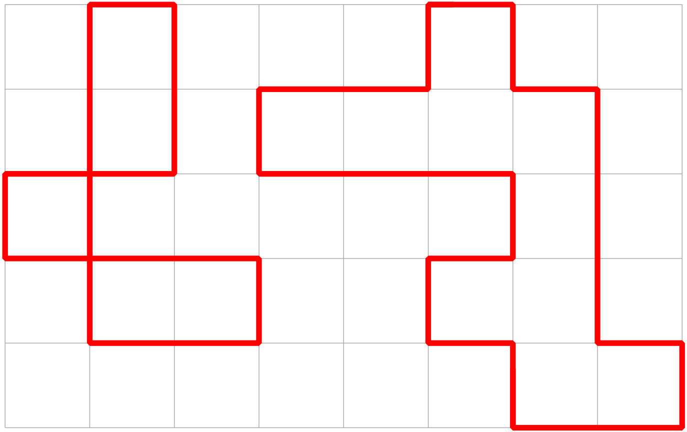
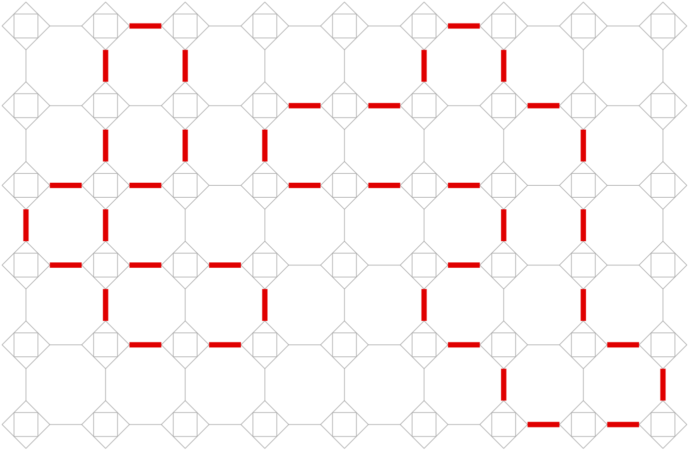


2. Relations between models: Ising - dimers [Fisher'66]

- Polygon contour configurations on G^*.

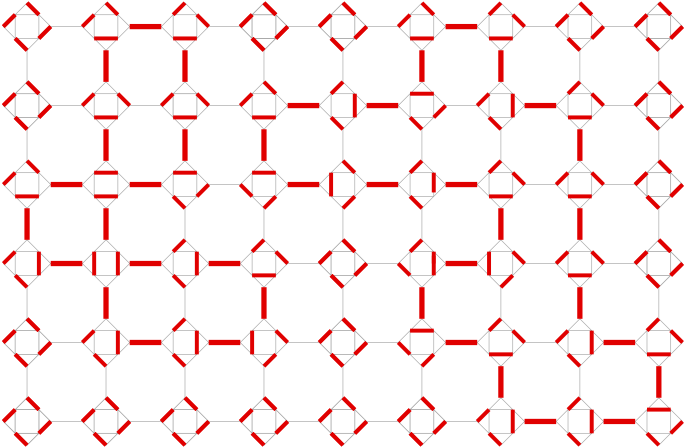


- Fisher's correspondence: exactly keep the polygon contour edges.



2. Relations between models: Ising - dimers [Fisher'66]

- Fill the decorations: $2^{|V^*|}$ possibilities.



Exactly solvable models

- One of the tools to study the macroscopic behavior is the partition function:

$$Z(\mathsf{G},w) = \sum_{\mathsf{C}\in\mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})},$$

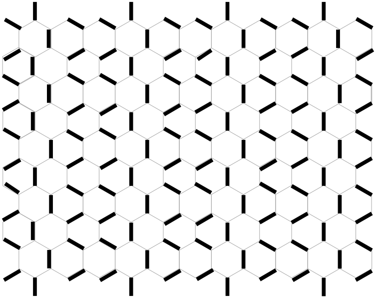
  the normalizing constant in the Boltzmann measure.

$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G},w)}.$$

- The model is exactly solvable if there exists an exact, explicit formula for the partition function
- Three exactly solvable models:
  - Ising-2d: Onsager (1944) - Fisher (1966).
  - Dimers-2d: Kasteleyn, Temperley-Fisher (1961).
  - Spanning trees: Kirchhoff (1848).

Preliminaries

- Let M_1, M_2 be two dimer configurations of G, and $M_1 \cup M_2$ be their superposition.



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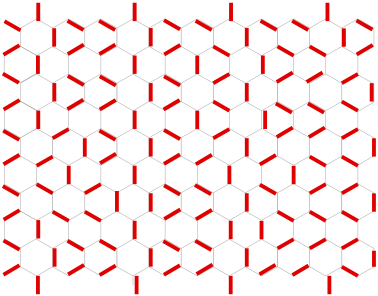

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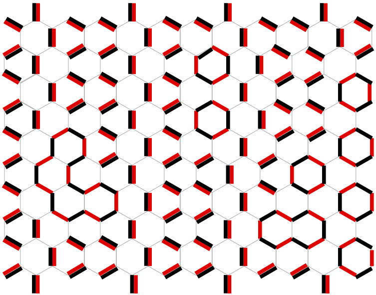

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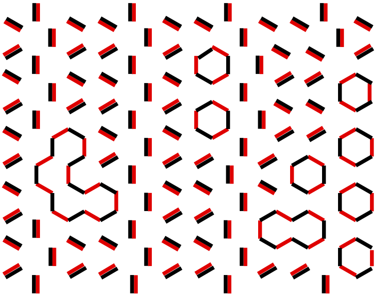

- $\mathsf{M}_1 \cup \mathsf{M}_2$ is a disjoint union of alternating cycles, where an alternating cycles has edges alternating between M_1 and M_2. Alternating cycles of length 2 are called doubled edges.