An entropy minimization approach to second-order variational mean-field games

Jean-David Benamou [*] Guillaume Carlier[†]
Simone Di Marino [‡] Luca Nenna [§]

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Abstract

We propose a new viewpoint on variational mean-field games with diffusion and quadratic Hamiltonian. We show the equivalence of such mean-field games with a relative entropy minimization at the level of probabilities on curves. We also address the time-discretization of such problems, establish Γ-convergence results as the time step vanishes and propose an efficient algorithm relying on this entropic interpretation as well as on the Sinkhorn scaling algorithm.

Keywords: Mean-Field Games, Fokker-Planck equation, entropy minimization, Schrödinger bridges, Sinkhorn algorithm.

MS Classification: 65K10, 49M05.

1 Introduction

It is well-known since the seminal work of Lasry and Lions [25, 26, 27] that the mean-field game system

$$\begin{cases} -\partial_t u - \frac{1}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = f[\rho_t], & (t,x) \in (0,T)\times\mathbb{R}^d \\ \partial_t \rho - \frac{1}{2}\Delta\rho - \operatorname{div}(\rho\nabla u) = 0, & (t,x)\in(0,T)\times\mathbb{R}^d \\ \rho_{|t=0} = \rho_0,\ u_{|t=T} = g[\rho_T], \end{cases} \tag{1.1}$$

[*]INRIA-Paris, MOKAPLAN, rue Simone Iff, 75012, Paris, FRANCE and CEREMADE Jean-David.Benamou@inria.fr.

[†]CEREMADE, UMR CNRS 7534, Université Paris IX Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, FRANCE and INRIA-Paris, MOKAPLAN carlier@ceremade.dauphine.fr

[‡]Indam, Unità SNS, Pisa, Italy simone.dimarino@altamatematica.it

[§]CNRS and CEREMADE nenna@ceremade.dauphine.fr

may be seen, at least formally (see [15] for a detailed analysis), as the system of optimality conditions for the minimization problem:

$$\inf_{(\rho,v)} \left\{ \mathcal{J}(\rho, v) \ : \partial_t \rho - \frac{1}{2}\Delta\rho + \operatorname{div}(\rho v) = 0, \rho_{|t=0} = \rho_0 \right\} \tag{1.2}$$

where

$$\mathcal{J}(\rho, v) := \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |v_t|^2 \mathrm{d}\rho_t(x)\mathrm{d}t + \int_0^T F(\rho_t)\mathrm{d}t + G(\rho_T). \tag{1.3}$$

We assume here that f and g are *potentials* i.e. admit F and G as primitives that is, $f[\rho]$ and $g[\rho]$ represent the directional derivative of the functionals F and G at ρ in some suitable sense.

Our aim is to relate precisely the *Eulerian* variational problem (1.2)-(1.3) with a *Lagrangian* relative entropy minimization problem and to develop a suitable efficient algorithm based on this entropic interpretation. This entropy minimization approach has its roots in the classical paper of Schrödinger [34]. The Schrödinger bridge problem has deep connections with large deviations and has been extensively analyzed and developed by Mikami [30] and Léonard [28], [29] who in particular proved convergence of Schrödinger bridges to optimal transport geodesics as the noise intensity vanishes. We also refer to Cattiaux and Léonard [17] for further connections between entropy minimization, large deviations and optimal control.

Entropy minimization has also proved to be an efficient computational strategy for optimal transport by Cuturi [20] who made the connection with the powerful and versatile Sinkhorn scaling algorithm (also see [9]). One advantage of reformulating (1.2)-(1.3) as an entropy minimization is therefore that it enables one to use specific numerical schemes based on the Sinkhorn algorithm. However, the entropic viewpoint is, to the best of our knowledge, restricted to the quadratic Hamiltonian case and there are now efficient solvers, developed by Achdou and his coauthors [2, 1, 3] based on the PDE system which go much beyond the quadratic case. Guéant in [24] designed a monotone scheme specially intended for the quadratic Hamiltonian case, relying on ingenious changes of variables and the Hopf-Cole transform, the approach we propose in this paper shares some common features with Guéant's method. For variational numerical methods based on more traditional convex optimization techniques, we refer for instance to [8], [5] and [14].

The starting point of our analysis is the equivalence between the Schrödinger bridge problem and the optimal control (with kinetic energy as cost) of the Fokker-Planck equation as emphasized by Chen, Georgiou and Pavon [18] and Gentil, Léonard and Ripani [22]. The recent work of Gigli and Tamanini [23] provides a very general and analytical approach to the dynamic formulation of entropic transport problems. The entropy minimization viewpoint has been recently fruitfully used in the context of incompressible flows by Arnaudon et al. [6] also see [11] for a numerical approach. Indeed, incompressibility (in the case of the flat torus instead of $\mathbb{R}^d$) can be seen as an instance of (1.2)-(1.3) in the somehow extreme case where $F(\rho)$ is 0 when ρ is uniform and $+\infty$ otherwise.

The paper is organized as follows. Section 2 is devoted to some preliminaries and introduces an entropy minimization problem we propose as a Lagrangian counterpart of (1.2). In section 3, the time discretization of both the Lagrangian and Eulerian problems are shown to be equivalent. We then establish a Γ-convergence result which enables to show the equivalence of the two formulations at the continuous time level as well in section 4. In section 5, we propose a numerical scheme, based on the Sinkhorn scaling algorithm to solve the entropy minimization problem and we present some numerical experiments.

2 Preliminaries

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of Borel probability measures with finite second moments, endowed with the Wasserstein distance $\mathcal{W}_2$, whose square is defined by

$$\mathcal{W}_2^2(\rho_0, \rho_1) := \inf_{\gamma \in \Pi(\rho_0,\rho_1)} \Big\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \mathrm{d}\gamma(x,y) \Big\},\ (\rho_0, \rho_1) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$$

$\Pi(\rho_0, \rho_1)$ being the set of transport plans between ρ_0 and ρ_1 i.e. the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ having ρ_0 and ρ_1 as marginals.

We denote by $\mathcal{L}^d$ the Lebesgue measure on $\mathbb{R}^d$ and whenever a measure ρ is absolutely continuous with respect to $\mathcal{L}^d$, we will denote by ρ both the measure itself and its density with respect to $\mathcal{L}^d$. Throughout the paper we will assume that F and G can be written as the sum of a convex local term and a term which is continuous for the Wasserstein distance $\mathcal{W}_2$ i.e.

$$F = F_1 + F_2,\ G = G_1 + G_2,$$

such that

- there are convex continuous integrands L_F and $L_G : \mathbb{R}^d \to \mathbb{R}_+$ such that $L_F(0) = L_G(0) = 0$ and

$$F_1(\rho) = \begin{cases} \int_{\mathbb{R}^d} L_F(\rho(x))\mathrm{d}x \text{ if } \rho \ll \mathcal{L}^d \\ +\infty \text{ otherwise} \end{cases},$$

$$G_1(\rho) = \begin{cases} \int_{\mathbb{R}^d} L_G(\rho(x))\mathrm{d}x \text{ if } \rho \ll \mathcal{L}^d \\ +\infty \text{ otherwise} \end{cases}$$

- F_2, $G_2 : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}_+$ are continuous for $\mathcal{W}_2$.

Given a Polish space X, a Borel probability measure q on X and r a σ-finite measure on X, the relative entropy of q with respect to r is given[1] by

$$\mathrm{H}(q|r) := \begin{cases} \int_X \log\left(\frac{dq}{dr}\right)\mathrm{d}q \text{ if } q \ll r \\ +\infty \text{ otherwise} \end{cases}$$

where $\frac{dq}{dr}$ stands for the Radon-Nikodym derivative of q with respect to r. If $X = \mathbb{R}^d$ and $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ we shall simply denote by $\mathrm{Ent}(\rho)$ the relative entropy of ρ with respect to $\mathcal{L}^d$. Let us also recall that Ent is controlled from below by the second moment (see (4.2) for a linear lower bound). We shall always assume that the initial condition ρ_0 satisfies

$$\rho_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \text{ and } \mathrm{Ent}(\rho_0) < +\infty \tag{2.1}$$

so that $-\infty < \mathrm{Ent}(\rho_0) < +\infty$.

As outlined in the introduction, our goal is to relate (1.2) with a relative entropy minimization over the path space $\Omega := C([0,T], \mathbb{R}^d)$. Let e_t denote the evaluation at time $t \in [0,T]$ i.e. $e_t(\omega) = \omega(t)$ for every $\omega \in \Omega$. We take as reference measure on Ω the reversible Wiener measure R defined by

$$R := \int_{\mathbb{R}^d} \delta_{x+B}\, \mathrm{d}x \tag{2.2}$$

where $B = (B_t)_{t\in[0,T]}$ is a standard d-dimensional Brownian motion starting at 0. Given $Q \in \mathcal{P}(\Omega)$ a Borel probability measure on Ω, set

$$Q_t := e_{t\#}Q,\ Q_{s,t} := (e_t, e_s)_{\#}Q,\ 0 \le s < t \le T, \tag{2.3}$$

[1]some caution must be taken when r is σ-finite but unbounded, in this case one can find a measurable and bounded from below potential $V\colon X \mapsto \mathbb{R}$ such that $e^{-V}r$ is a probability measure on X and then define $H(q|r)$ as $H(q|r) = H(q|e^{-V}r) - \int_X V\mathrm{d}q$ see Appendix 1 in [29] for details.

and more generally if $t_1, \cdots, t_n \in [0,T]^n$

$$Q_{t_1,\cdots,t_n} := (e_{t_1}, \cdots, e_{t_n})_\# Q.$$

We define similarly R_t, $R_{s,t}$ and $R_{t_1,\cdots,t_n}$ and observe that by construction $R_t := e_{t\#} R = \mathcal{L}^d$ for every $t \in [0,T]$ and for every $0 \le s < t \le T$, $R_{s,t} = (e_t, e_s)_\# R$ is given by

$$R_{s,t}(\mathrm{d}x, \mathrm{d}y) = P_{t-s}(x-y)\mathrm{d}x\mathrm{d}y$$

where P_t is the heat kernel:

$$P_t(z) := \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\Big(-\frac{|z|^2}{2t}\Big),\ t > 0,\ x \in \mathbb{R}^d. \tag{2.4}$$

Let us finally introduce a few notations. Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $h \in (0,T)$ let us define $\mathrm{FP}_h(\mu,\nu)$ as the infimum of the kinetic energy

$$\mathrm{FP}_h(\mu,\nu) := \inf_{(\rho,v)} \Big\{ \frac{1}{2} \int_0^h \int_{\mathbb{R}^d} |v_t|^2 \mathrm{d}\rho_t(x) \mathrm{d}t \Big\} \tag{2.5}$$

among pairs (ρ, v) solving the Fokker-Planck equation on $(0,h) \times \mathbb{R}^d$ with endpoints μ and ν

$$\partial_t \rho - \frac{1}{2}\Delta\rho + \mathrm{div}(\rho v) = 0,\ \rho_{|t=0} = \mu,\ \rho_{|t=h} = \nu.$$

Also define

$$S_h(\mu,\nu) := \inf\{H(Q|R)\ :\ Q \in \mathcal{P}(C([0,h],\mathbb{R}^d),\ Q_0 = \mu,\ Q_h = \nu\} \tag{2.6}$$

(where, slightly abusing notations in the formula above, R denotes the reversible Wiener measure on $C([0,h],\mathbb{R}^d)$). Following Léonard [29], and writing the disintegrations of Q and R with respect to their marginals $Q_{0,h}$, $R_{0,h}$:

$$Q = \int_{\mathbb{R}^d \times \mathbb{R}^d} Q(.|x_0, x_h) \mathrm{d}Q_{0,h}(x_0, x_h),\ R = \int_{\mathbb{R}^d \times \mathbb{R}^d} R(.|x_0, x_h) \mathrm{d}R_{0,h}(x_0, x_h)$$

(so that $R(.|x_0, x_h)$ is the probability Law of a Brownian bridge) since

$$H(Q|R) = H(Q_{0,h}|R_{0,h}) + \int_{\mathbb{R}^d \times \mathbb{R}^d} H(Q(.|x_0,x_h)|R(.|x_0,x_h)) \mathrm{d}Q_{0,h}(x_0,x_h)$$

and $H(Q(.|x_0,x_h)|R(.|x_0,x_h)) \ge 0$ with an equality when $Q(.|x_0,x_h) = R(.|x_0,x_h)$ (which means that R and the optimal Q share the same bridge) one can express S_h as the value of a static entropy minimization problem:

$$S_h(\mu,\nu) = \inf\{H(\gamma|R_{0,h})\ : \gamma \in \Pi(\mu,\nu)\} \tag{2.7}$$

and the solution of the dynamic problem (2.6) is obtained by

$$Q = \int_{\mathbb{R}^d \times \mathbb{R}^d} R(.|x_0, x_h) \mathrm{d}\gamma(x_0, x_h)$$

with γ optimal for the static problem (2.7).

The following result, proven in Chen, Georgiou, Pavon [18] and Gentil, Léonard and Ripani [22] (in a more general setting) connects FP_h to S_h and can be viewed as a noisy version of the Benamou-Brenier formula:

Theorem 2.1. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $h \in (0, T)$, $\mathrm{FP}_h(\mu, \nu)$ and $S_h(\mu, \nu)$ are related by*

$$S_h(\mu, \nu) = \mathrm{FP}_h(\mu, \nu) + \mathrm{Ent}(\mu). \tag{2.8}$$

Let us also mention an alternative formulation, involving the Fisher information (see [22, 18]):

$$\mathrm{FP}_h(\mu, \nu) = \inf_{(\rho, v)} \Big\{ \int_0^h \Big(\int_{\mathbb{R}^d} \frac{1}{2} |v_t|^2 \mathrm{d}\rho_t(x) + \frac{1}{8} I(\rho_t) \Big) \mathrm{d}t \Big\} + \frac{1}{2} (\mathrm{Ent}(\nu) - \mathrm{Ent}(\mu)) \tag{2.9}$$

where the infimum is taken among solutions of the continuity equation with fixed endpoints

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \ \rho_{|t=0} = \mu, \ \rho_{|t=h} = \nu$$

and I denotes the Fisher information

$$I(\rho) := \int_{\mathbb{R}^d} |\nabla \log(\rho)|^2 \rho = \int_{\mathbb{R}^d} \frac{|\nabla \rho|^2}{\rho} = 4 \|\nabla(\sqrt{\rho})\|^2_{L^2(\mathbb{R}^d)}. \tag{2.10}$$

Given a continuous curve of measures $\mu \in C([0, T], (\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)) : t \in [0, T] \mapsto \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ define

$$\mathcal{E}(\mu) := \inf_v \Big\{ \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |v_t|^2 \mathrm{d}\mu_t(x) \mathrm{d}t \ : \ \partial_t \mu - \frac{1}{2} \Delta \mu + \mathrm{div}(\mu v) = 0 \Big\}$$

as well as the cost

$$\mathcal{C}(\mu) := G(\mu_T) + \int_0^T F(\mu_t) \mathrm{d}t$$

so that the variational formulation (1.2)-(1.3) of the MFG system can be simply rewritten as

$$\inf\{\mathcal{E}(\mu) + \mathcal{C}(\mu) \ : \ \mu \in C([0, T], (\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)), \ \mu_{|t=0} = \rho_0\} \tag{2.11}$$

For $\mu = (\mu_t)_{t\in[0,T]} \in C([0,T], (\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$, define also the minimal entropic cost

$$\mathcal{S}(\mu) := \inf\{H(Q|R) \ : \ Q_t = \mu_t, \ \forall t \in [0,T]\} \tag{2.12}$$

which can be viewed as a variant of (2.6) with infinitely many marginal constraints. We shall prove, by a careful inspection of a suitable time discretization (section 3) and Γ-convergence arguments (see section 4), that when $\mathrm{Ent}(\mu_0) < +\infty$ the following relation holds

$$\mathcal{S}(\mu) = \mathcal{E}(\mu) + \mathrm{Ent}(\mu_0)$$

so that (2.11) can be reformulated as

$$\inf\Big\{H(Q|R) + \mathcal{C}((Q_t)_{t\in[0,T]}) \ : \ Q \in \mathcal{P}(\Omega), \ Q_0 = \rho_0\Big\}. \tag{2.13}$$

3 Time discretization

3.1 Discretization

Given a positive integer N and $\mu_0, \cdots, \mu_N \in \mathcal{P}_2(\mathbb{R}^d)$, define the respective discretization of $\mathcal{E}$, $\mathcal{C}$ and $\mathcal{S}$:

$$\mathcal{E}^N(\mu_0, \cdots, \mu_N) := \sum_{k=0}^{N-1} \mathrm{FP}_{\frac{T}{N}}(\mu_k, \mu_{k+1}), \tag{3.1}$$

$$\mathcal{C}^N(\mu_0, \cdots, \mu_N) := \frac{T}{N} \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N) \tag{3.2}$$

and

$$\mathcal{S}^N(\mu_0, \cdots, \mu_N) := \inf\Big\{H(Q|R) \ : \ Q \in \mathcal{P}(\Omega), \ Q_{\frac{kT}{N}} = \mu_k, \ k = 0, \cdots N\Big\} \tag{3.3}$$

and observe that, exactly as for (2.6), $\mathcal{S}^N$ can be written as the value of the static multi-marginal problem

$$\mathcal{S}^N(\mu_0, \cdots, \mu_N) := \inf_{\gamma\in\Pi(\mu_0,\cdots,\mu_N)} H(\gamma^N|R^N) \tag{3.4}$$

where $\Pi(\mu_0, \cdots, \mu_N)$ is the set of probability measures on $(\mathbb{R}^d)^{N+1}$ having $\mu_0, \cdots, \mu_N$ as marginals and

$$R^N := R_{0,\frac{T}{N},\cdots,T}.$$

One then discretizes (2.11) as

$$\inf\{\mathcal{E}^N(\mu_0,\cdots,\mu_N)+\mathcal{C}^N(\mu_0,\cdots,\mu_N),\ \mu_i\in\mathcal{P}_2(\mathbb{R}^d),\ \mu_0=\rho_0\}. \tag{3.5}$$

In an analogous way, the discrete counterpart of (2.13) becomes

$$\inf\{\mathcal{S}^N(\mu_0,\cdots,\mu_N)+\mathcal{C}^N(\mu_0,\cdots,\mu_N),\ \mu_i\in\mathcal{P}_2(\mathbb{R}^d),\ \mu_0=\rho_0\}. \tag{3.6}$$

3.2 On entropy minimization and Markovianity

Definition 3.1. *Given Polish spaces X_1, X_2, X_3, two measures $P_{12}\in\mathcal{M}(X_1\times X_2)$ and $P_{23}\in\mathcal{M}(X_2\times X_3)$ such that $\rho_2=(e_2)_\#P_{12}=(e_2)_\#P_{23}$, we define the Markov-concatenation of the two plans P_{12} and P_{23}, which we denote by $P_{12}\circ P_{23}$, as the measure*

$$P_{12}\circ P_{23}=\int_{X_2}\nu_1^x\otimes\delta_x\otimes\nu_3^x\,d\rho_2(x),$$

where $\nu_1^x\in\mathcal{P}(X_1),\nu_3^x\in\mathcal{P}(X_3)$ and the following disintegration formulas hold:

$$P_{12}=\int\nu_1^x\otimes\delta_x\,d\rho_2(x)\ \text{and}\ P_{23}=\int\delta_x\otimes\nu_3^x\,d\rho_2(x).$$

Lemma 3.2. *Given the product of Polish spaces $X=X_1\times X_2\times\cdots\times X_n$, a probability $P\in\mathcal{P}(X)$ and a measure $R\in\mathcal{M}(X)$. Let us denote $P_i=(e_i)_\#P$, $R_i=(e_i)_\#R$, $P_{i,i+1}=(e_i,e_{i+1})_\#P$ and $R_{i,i+1}=(e_i,e_{i+1})_\#R$. Let us suppose that R is Markovian, that is $R=R_{1,2}\circ R_{2,3}\circ\cdots\circ R_{n-1,n}$, then we have*

$$H(P|R)\geq\sum_{i=1}^{n-1}H(P_{i,i+1}|R_{i,i+1})-\sum_{i=2}^{n-1}H(P_i|R_i),$$

with equality if and only if $P=P_{1,2}\circ P_{2,3}\circ\ldots\circ P_{n-1,n}$ i.e. P is itself Markov.

Proof. We will prove the lemma by induction on n, starting with the first nontrivial case $n=3$. For $n=3$, we have

$$H(P_{12}|R_{12})=H(P_2|R_2)+\int_{X_2}H(P_{12}(\mathrm{d}x_1|x_2)|R_{12}(\mathrm{d}x_1|x_2))\mathrm{d}P_2(x_2)$$

and

$$H(P_{23}|R_{23})=H(P_2|R_2)+\int_{X_2}H(P_{23}(\mathrm{d}x_3|x_2)|R_{23}(\mathrm{d}x_3|x_2))\mathrm{d}P_2(x_2)$$

but

$$H(P|R) = H(P_2|R_2) + \int_{X_2} H(P(\mathrm{d}x_1\mathrm{d}x_3|x_2)|R(\mathrm{d}x_1\mathrm{d}x_3|x_2))\mathrm{d}P_2(x_2).$$

Let us recall that it follows from the strict convexity of $t \in \mathbb{R}_+ \mapsto t\log(t)$ that if γ is a probability measure on a product space with marginals γ_1 and γ_2 then $H(\gamma|\mu_1 \otimes \mu_2) \geq H(\gamma_1|\mu_1) + H(\gamma_2|\mu_2)$ with equality exactly when γ is a product measure $\gamma = \gamma_1 \otimes \gamma_2$. Then, observing that for P_2 almost every x_2, the conditional probability, $P(\mathrm{d}x_1\mathrm{d}x_3|x_2)$ has marginals $P_{12}(\mathrm{d}x_1|x_2)$ and $P_{23}(\mathrm{d}x_3|x_2)$ and that since R is Markov $R(\mathrm{d}x_1\mathrm{d}x_3|x_2) = R_{12}(\mathrm{d}x_1|x_2)R_{23}(\mathrm{d}x_3|x_2)$, we have

$$\begin{aligned} H(P(\mathrm{d}x_1\mathrm{d}x_3|x_2)|R(\mathrm{d}x_1\mathrm{d}x_3|x_2)) \geq\ & H(P_{12}(\mathrm{d}x_1|x_2)|R_{12}(\mathrm{d}x_1|x_2)) \\ & + H(P_{23}(\mathrm{d}x_3|x_2)|R_{23}(\mathrm{d}x_3|x_2)) \end{aligned}$$

with an equality if and only if $P(\mathrm{d}x_1\mathrm{d}x_3|x_2) = P_{12}(\mathrm{d}x_1|x_2)P_{23}(\mathrm{d}x_3|x_2)$ i.e. P is Markov. This proves the claim for $n = 3$. Assume the claim is satisfied for n and consider $P \in \mathcal{P}(X_1 \times \cdots \times X_{n+1})$ and R a Markov measure on $X_1 \times \cdots \times X_{n+1}$, we then have

$$\begin{aligned} H(P|R) =\ & H(P_{1,\cdots,n}|R_{1,\cdots,n}) \\ & + \int_{X_1\times\cdots\times X_n} H(P(\mathrm{d}x_{n+1}|x_1,\cdots x_n)|R(\mathrm{d}x_{n+1}|x_1,\cdots x_n))\mathrm{d}P_{1,\cdots,n}(x_1,\cdots,x_n). \end{aligned}$$

From the validity of the claim for $P_{1,\cdots,n}$ and $R_{1,\cdots,n}$, we have

$$H(P_{1,\cdots,n}|R_{1,\cdots,n}) \geq \sum_{i=1}^{n-1} H(P_{i,i+1}|R_{i,i+1}) - \sum_{i=2}^{n-1} H(P_i|R_i). \tag{3.7}$$

So we are left to show that

$$\begin{aligned} & \int_{X_1\times\cdots\times X_n} H(P(\mathrm{d}x_{n+1}|x_1,\cdots x_n)|R(\mathrm{d}x_{n+1}|x_1,\cdots x_n))\mathrm{d}P_{1,\cdots,n}(x_1,\cdots,x_n) \\ & \geq H(P_{n,n+1}|R_{n,n+1}) - H(P_n|R_n) \\ & = \int_{X_n} H(P_{n,n+1}(\mathrm{d}x_{n+1}|x_n)|R_{n,n+1}(\mathrm{d}x_{n+1}|x_n))\mathrm{d}P_n(x_n) \end{aligned}$$

Now we observe that since R is Markov $R(\mathrm{d}x_{n+1}|x_1,\cdots x_n) = R_{n,n+1}(\mathrm{d}x_{n+1}|x_n)$, we then write

$$\begin{aligned} & \int_{X_1\times\cdots\times X_n} H(P(\mathrm{d}x_{n+1}|x_1,\cdots x_n)|R_{n,n+1}(\mathrm{d}x_{n+1}|x_n)\mathrm{d}P_{1,\cdots,n}(x_1,\cdots,x_n) \\ & = \int_{X_n}\Big(\int_{X_1\times\cdots\times X_{n-1}} H(P(\mathrm{d}x_{n+1}|x_1,\cdots x_n)|R(\mathrm{d}x_{n+1}|x_n))\mathrm{d}P_{1,\cdots,n}(x_1,\cdots,x_{n-1}|x_n)\Big)\mathrm{d}P_n(x_n) \end{aligned}$$

using the convexity of $H(.|R(\mathrm{d}x_{n+1}|x_n))$ and the fact that

$$\int_{X_1\times\cdots\times X_{n-1}} P(\mathrm{d}x_{n+1}|x_1\cdots,x_n)\mathrm{d}P_{1,\cdots,n}(x_1,\cdots,x_{n-1}|x_n)=P_{n,n+1}(\mathrm{d}x_{n+1}|x_n)$$

we deduce

$$\int_{X_1\times\cdots\times X_{n-1}} H(P(\mathrm{d}x_{n+1}|x_1,\cdots x_n)|R(\mathrm{d}x_{n+1}|x_n))\mathrm{d}P_{1,\cdots,n}(x_1,\cdots,x_{n-1}|x_n)$$
$$\geq H(P_{n,n+1}(\mathrm{d}x_{n+1}|x_n)|R_{n,n+1}(\mathrm{d}x_{n+1}|x_n))$$

integrating with respect to $\mathrm{d}P_n(x_n)$ gives the desired inequality. Now in the equality case, there should be an equality in (3.7) so that $P_{1,\cdots,n}$ should be Markov, but there should also be an inequality in the convexity inequality for the relative entropy above which implies that $P(\mathrm{d}x_{n+1}|x_1\cdots,x_n)=P_{n,n+1}(\mathrm{d}x_{n+1}|x_n)$ and these two conditions imply that P is Markov. □

As a consequence of Lemma 3.2 and the Markovianity of the reversible Wiener measure we deduce

Corollary 3.3. *Let $(\mu_0,\cdots,\mu_N)\in(\mathcal{P}_2(\mathbb{R}^d))^{N+1}$, one has*

$$\begin{aligned}\mathcal{S}^N(\mu_0,\cdots,\mu_N)&=\sum_{i=0}^{N-1}S_{\frac{T}{N}}(\mu_i,\mu_{i+1})-\sum_{i=1}^{N-1}\mathrm{Ent}(\mu_i)\\&=\mathcal{E}^N(\mu_0,\cdots,\mu_N)+\mathrm{Ent}(\mu_0).\end{aligned}$$

Proof. The first formula follows from Lemma 3.2, the Markovianity of R together with the fact that $H(\mu_i|R_i)=\mathrm{Ent}(\mu_i)$, the second identity then directly follows from the first one and (2.8). □

4 Γ-convergence and equivalence

4.1 Γ-convergence

We state now some lemmas which will be useful for the proof of the Γ-convergence result, stated in Theorem 4.3.

Lemma 4.1. *Let $\{\rho_t\}_{0\leq t\leq T}$ be a curve of probability measures such that $\partial_t\rho_t+\nabla\cdot(v_t\rho_t)=\frac{1}{2}\Delta\rho_t$ with $\iint_0^T|v_t|^2\,d\rho_t=E<\infty$. Let us suppose also that $\int_{\mathbb{R}^d}|x|^2\,d\rho_0=c_0<\infty$ is finite. Then we have:*

(i) $\int |x|^2 \, d\rho_t \leq C(T, c_0, E)$ *for all* $0 \leq t \leq T$.

(ii) $\mathrm{Ent}(\rho_t) \in L^\infty(\tau, T)$ *for all* $\tau > 0$.

(iii) $I(\rho_t) \in L^1(\tau, T)$ *and* $\mathrm{Ent}(\rho_t) \in W^{1,1}(\tau, T)$ *for all* $\tau > 0$.

(iv) $\iint_\tau^T |v_t|^2 \, d\rho_t = \iint_\tau^T |w_t|^2 \, d\rho_t + \frac{1}{4} \int_\tau^T I(\rho_t) \, dt + (\mathrm{Ent}(\rho_T) - \mathrm{Ent}(\rho_\tau))$, *where* $w_t = v_t - \frac{1}{2} \nabla \log(\rho_t)$.

(v) $\mathcal{W}_2^2(\rho_\tau, \rho_T) \leq (T - \tau)(E + (\mathrm{Ent}(\rho_\tau) - \mathrm{Ent}(\rho_T)))$. *In particular we have that* $\rho_t \in H^1([\tau, T]; (\mathcal{P}(\mathbb{R}^n), \mathcal{W}_2))$.

Moreover if we add the assumption $\mathrm{Ent}(\rho_0) < \infty$ *we can take also* $\tau = 0$ *in (ii), (iii), (iv) and (v).*

Proof. Let us take $\delta > 0$ and let us consider $\rho_t^\delta = \eta_\delta * \rho_t$ a convolution with a C^∞ (and everywhere strictly positive) kernel $\eta_\delta = \eta(x/\delta)/\delta^d$. Denoting $m_t^\delta = \eta_\delta * (v_t \rho_t)$ we have that $\partial_t \rho_t^\delta + \nabla \cdot (m_t^\delta) = \frac{1}{2} \Delta \rho_t^\delta$. In particular setting $v_t^\delta = m_t^\delta / \rho_t^\delta$, we have, by convexity of the function $(v, s) \mapsto \frac{|v|^2}{s}$

$$\iint_0^T |v_t^\delta|^2 \, d\rho_t^\delta = \iint_0^T \frac{|(v_t \rho_t) * \eta_\delta|^2}{\rho_t * \eta_\delta} \, dx \leq \iint_0^T \frac{|v_t \rho_t|^2}{\rho_t} \, dx = E.$$

Moreover we can compute, letting $c_t^\delta = \int |x|^2 \, d\rho_t^\delta$:

$$\frac{d}{dt} c_t^\delta = 2 \int \langle x, v_t^\delta \rangle \, d\rho_t^\delta + 2d \leq c_t^\delta + \int |v_t^\delta|^2 \, d\rho_t^\delta + 2d$$

$$\frac{d}{dt}(c_t^\delta e^{-t}) \leq e^{-t} \left(\int |v_t^\delta|^2 \, d\rho_t^\delta + 2d \right) \leq \int |v_t^\delta|^2 \, d\rho_t^\delta + 2d.$$

So we get

$$c_t^\delta \leq e^t (c_0^\delta + E + 2dt) \leq e^T (c_0 + \alpha\delta + E + 2dT) \tag{4.1}$$

for some α depending on η; now we can let δ to 0 to obtain (i). Moreover, we can obtain a bound on the entropy. Indeed, $e^{-\pi|x|^2}$ is the density of a probability measure and so for any ρ bounded density of a probability measure with finite second moment we have:

$$0 \leq H(\rho | e^{-\pi|x|^2}) = \pi \int |x|^2 \, d\rho + \int \rho \log(\rho) \, dx$$

$$\int \rho \log(\rho) \, dx \geq -\pi \int |x|^2 \, d\rho. \tag{4.2}$$

Taking the function $f(t) = e^{-\frac{2}{d}\operatorname{Ent}(\rho_t^\delta)}$, we have:

$$
\begin{aligned}
\frac{d}{dt}f(t) &= -\frac{2f(t)}{d}\left(\int \nabla\rho_t^\delta \cdot v_t^\delta dx - \frac{1}{2}\int \frac{|\nabla\rho_t^\delta|^2}{\rho_t^\delta}\,dx\right)\\
&\geq -\frac{2f(t)}{d}\left(\int |v_t^\delta|^2\,d\rho_t^\delta + \frac{1}{4}\int \frac{|\nabla\rho_t^\delta|^2}{\rho_t^\delta}\,dx - \frac{1}{2}\int \frac{|\nabla\rho_t^\delta|^2}{\rho_t^\delta}\,dx\right)\\
&= -\frac{2f(t)}{d}\left(\int |v_t^\delta|^2\,d\rho_t^\delta - \int |\nabla\sqrt{\rho_t^\delta}|^2\,dx\right)\\
&\geq \pi e - \frac{2f(t)}{d}\int |v_t^\delta|^2\,d\rho_t^\delta,
\end{aligned}
$$

where in the last passage we used the Log-Sobolev inequality. So we can conclude $f(t) \geq e\pi t e^{-\frac{2}{d}\iint_0^t |v_s^\delta|^2\,d\rho_s^\delta}$, that is

$$
\operatorname{Ent}(\rho_t^\delta) \leq -\frac{d}{2}\log(\pi e t) + \iint_0^t |v_s^\delta|^2\,d\rho_s^\delta.
$$

This proves (ii). In order to establish (iii), we first notice that

$$
\operatorname{Ent}(\rho_r^\delta) - \operatorname{Ent}(\rho_s^\delta) = \iint_s^r v_t^\delta \cdot \nabla\rho_t^\delta\,dxdt - \frac{1}{2}\iint_s^r \frac{|\nabla\rho_t^\delta|^2}{\rho_t^\delta}\,dxdt. \tag{4.3}
$$

In particular taking $s = \tau$ and $r = T$ and using $v_t^\delta \cdot \nabla\rho_t^\delta \leq |v_t^\delta|^2\rho_t^\delta + \frac{1}{4}\frac{|\nabla\rho_t^\delta|^2}{\rho_t^\delta}$ we get, thanks to the L^∞ bound on the entropy:

$$
\frac{1}{4}\iint_\tau^T \frac{|\nabla\rho_t^\delta|^2}{\rho_t^\delta}\,dxdt \leq E + \operatorname{Ent}(\rho_\tau^\delta) - \operatorname{Ent}(\rho_T^\delta) \leq C(\tau, T, E, c_0), \tag{4.4}
$$

which proves that $I(\rho_t)$ is in $L^1(\tau, T)$. Equation (4.3) also yields

$$
|\operatorname{Ent}(\rho_t^\delta) - \operatorname{Ent}(\rho_s^\delta)| \leq \int_s^t \left(2I(\rho_t^\delta) + \frac{1}{4}\int_{\mathbb{R}^d} |v_t^\delta|^2\rho_t^\delta\right)dt, \tag{4.5}
$$

by convexity of I we have $0 \leq I(\rho_t^\delta) \leq I(\rho_t) \in L^1(\tau, T)$ and a similar convexity argument gives an integrable bound for the kinetic energy as well, hence $\operatorname{Ent}(\rho_t) \in W^{1,1}(\tau, T)$ for every $\tau > 0$. Statements (iv) and (v) easily follow. Finally, whenever $\operatorname{Ent}(\rho_0) < \infty$ from equation (4.4) we get immediately $I(\rho_t) \in L^1(0, T)$ and then from (4.5) we have $\operatorname{Ent}(\rho_t) \in W^{1,1}(0, T)$.

□

Lemma 4.2. *Let μ and ν be two fixed probability measures such that $E := 2\mathrm{FP}_T(\mu,\nu) < \infty$. Let $c_0 = \int |x|^2\, d\mu < \infty$ and let us fix $T/2 \le T' \le 2T$. Then we have*

(i) $\mathrm{Ent}(\nu) \le \mathrm{Ent}(\mu) + E$*;*

(ii) $\mathrm{Ent}(\nu) \ge -\pi e^T (c_0 + E + 2dT)$*;*

(iii) $2\mathrm{FP}_{T'}(\mu,\nu) \le 5E + (\mathrm{Ent}(\mu) - \mathrm{Ent}(\nu))$*.*

Proof. Inequality (*i*) directly follows from Lemma 4.1-(*iv*); (*ii*) is implied directly by (4.1)-(4.2) in the proof of lemma 4.1. As for the last estimate let us consider an almost minimiser (ρ_t, v_t) for the problem $\mathrm{FP}_T(\mu,\nu)$; in particular we have $\partial\rho_t + \nabla(v_t\rho_t) = \frac{1}{2}\Delta\rho_t$ and $\iint_0^T v_t^2\, d\rho_t \le E + \delta$. Let us take $(\tilde\rho_t, \tilde v_t) = (\rho_{Tt/T'}, v_{Tt/T'})$; this curve is such that $\tilde\rho_{T'} = \nu$ and $\tilde\rho_0 = \mu$ and moreover $\partial_t\tilde\rho_t + \frac{T}{T'}\nabla(\tilde v_t\tilde\rho_t) = \frac{T}{2T'}\Delta\tilde\rho_t$ and in particular we have

$$\partial_t\tilde\rho_t + \nabla\cdot\left(\Big(\frac{T}{T'}\tilde v_t + \frac{1}{2}\big(1 - T/T'\big)\frac{\nabla\tilde\rho_t}{\tilde\rho_t}\Big)\tilde\rho_t\right) = \frac{1}{2}\Delta\tilde\rho_t.$$

We can thus estimate

$$\begin{aligned}
2\mathrm{FP}_{T'}(\mu,\nu) &\le \iint_0^{T'} \Big|\frac{T}{T'}\tilde v_t + \frac{1}{2}\big(1 - T/T'\big)\frac{\nabla\tilde\rho_t}{\tilde\rho_t}\Big|^2 d\tilde\rho_t\, dt \\
&\le 2\iint_0^{T'} \frac{T^2}{T'^2}|\tilde v_t|^2\, d\tilde\rho_t + 2\Big(1 - \frac{T}{T'}\Big)^2 \iint_0^{T'} \frac{1}{4}\frac{|\nabla\tilde\rho_t|^2}{\tilde\rho_t}\, dx\, dt \\
&\le \frac{2T}{T'}(E+\delta) + \frac{2T'}{T}\Big(1 - \frac{T}{T'}\Big)^2 \int_0^T \frac{1}{4} I(\rho_t)\, dt \\
&\le \frac{2T}{T'}(E+\delta) + \frac{2T'}{T}\Big(1 - \frac{T}{T'}\Big)^2 (E + \delta + \mathrm{Ent}(\mu) - \mathrm{Ent}(\nu)) \\
&\le 5(E+\delta) + (\mathrm{Ent}(\mu) - \mathrm{Ent}(\nu)).
\end{aligned}$$

In the last two inequalities we used the estimates on $\int I(\rho_t)$ given by lemma 4.1-(*iv*) and then the fact that $T'/T \in [1/2, 2]$ which is easily seen to imply that $\frac{2T'}{T}(1 - \frac{T}{T'})^2 \le 1$. Letting $\delta \to 0$ we thus obtain (*iii*).

□

We now fix the initial condition ρ_0 such that

$$\rho_0 \in \mathcal{P}_2(\mathbb{R}^d),\ \mathrm{Ent}(\rho_0) < +\infty, \tag{4.6}$$

and define

$$\mathcal{A}_{\rho_0} := \{\mu \in C([0,T], (\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)) \ : \ \mu_0 = \rho_0\} \tag{4.7}$$

and its discretization

$$\mathcal{A}_{\rho_0}^N := \{\mu^N := (\mu_0^N, \cdots, \mu_N^N) \in \mathcal{P}_2(\mathbb{R}^d)^{N+1}, \ \mu_0 = \rho_0\}. \tag{4.8}$$

Given $\mu^N \in \mathcal{A}_{\rho_0}^N$, it will be convenient to extend it in a piecewise constant way as

$$\widetilde{\mu}_t^N := \mu_k^N, \ \ t \in \Big(\frac{(k-1)T}{N}, \frac{kT}{N}\Big], \ k = 1, \cdots, N, \ \widetilde{\mu}_0 := \mu_0^N = \rho_0. \tag{4.9}$$

By construction, we have

$$\mathcal{C}^N(\mu^N) = \int_0^{T-T/N} F(\widetilde{\mu}_s^N) ds + G(\widetilde{\mu}_T^N).$$

We will say that a sequence $\mu^N \in \mathcal{A}_{\rho_0}^N$ converges to a curve of measures $\mu \in \mathcal{A}_{\rho_0}$ whenever

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathcal{W}_2(\widetilde{\mu}_t^N, \mu_t) = 0. \tag{4.10}$$

For further use, let us point out that if $\mathcal{E}^N(\mu^N)$ is bounded, it follows from Lemmas 4.1-4.2 and a refined version of Ascoli-Arzelà's theorem (see [4] chapter 3) that (up to a subsequence), μ^N may be assumed to converge (in the sense of (4.10)) to some $\mu \in \mathcal{A}_{\rho_0}$. Thanks to Lemmas 4.1, we have uniform (in t and N) bounds on second moments and entropy of $\widetilde{\mu}_t^N$ hence we may also assume that $\widetilde{\mu}^N$ converges to μ weakly in $L^1((0,T)\times\mathbb{R}^d)$ and also that $\widetilde{\mu}_T^N$ converges weakly in $L^1(\mathbb{R}^d)$ to μ_T.

Theorem 4.3. *The sequence of functionals $\mathcal{E}^N + \mathcal{C}^N : \mathcal{A}_{\rho_0}^N \to \mathbb{R}_+ \cup \{+\infty\}$ Γ-converges to the functional $\mathcal{E} + \mathcal{C}$: $\mathcal{A}_{\rho_0} \to \mathbb{R}_+ \cup \{+\infty\}$ as $N \to \infty$.*

Proof. Let us start with the Γ-liminf inequality. Let $\mu^N \in \mathcal{A}_{\rho_0}^N$ converge (in the sense of (4.10)) to $\mu \in \mathcal{A}_{\rho_0}$ and assume that $\mathcal{E}^N(\mu^N)+\mathcal{C}^N(\mu^N)$ is bounded (hence so is $\mathcal{E}^N(\mu^N)$). As observed above, one may assume that $\widetilde{\mu}^N$ converges to μ weakly in $L^1(0,T)\times\mathbb{R}^d)$ and also that $\widetilde{\mu}_T^N$ converges weakly in $L^1(\times\mathbb{R}^d)$ to μ_T. In the terminal term $G(\widetilde{\mu}_T^N) = \int_{\mathbb{R}^d} L_G(\widetilde{\mu}_T^N(x))dx + G_2(\widetilde{\mu}_T^N)$, we have the sum of a convex nonnegative integral term, which is then weakly l.s.c. in $L^1(\mathbb{R}^d)$, and a term, $G_2(\widetilde{\mu}_T^N)$, which is continuous for $\mathcal{W}_2$, so that

$$\liminf_N G(\widetilde{\mu}_T^N) \geq G(\mu_T). \tag{4.11}$$

Fix $\varepsilon > 0$ and let $N > T\varepsilon^{-1}$, by Fatou's Lemma, the non-negativity of $F = F_1 + F_2$ and invoking the same weak L^1 lower-semi continuity argument as above (but with respect to both variables t and x), we get

$$\liminf_N \int_0^{T-T/N} \Big(\int_{\mathbb{R}^d} L_F(\widetilde{\mu}_s^N)dx + F_2(\widetilde{\mu}_s^N) \Big) ds \geq \int_0^{T-\varepsilon} F(\mu_s)ds$$

letting $\varepsilon \to 0$ and recalling (4.11), we deduce

$$\liminf_N \mathcal{C}^N(\mu^N) \geq \mathcal{C}(\mu). \tag{4.12}$$

For the energy term, we consider $m_t^N := \widetilde{\mu}_t^N v_t^N$, an almost minimizer for $\mathcal{E}^N(\mu^N)$ i.e. $\partial_t \widetilde{\mu}_t^N - \frac{1}{2}\Delta\widetilde{\mu}_t^N + \operatorname{div}(m_t^N) = 0$ and

$$\mathcal{E}^N(\mu^N) \geq \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \frac{|m_t^N|^2}{\widetilde{\mu}_t^N} - \frac{1}{N}. \tag{4.13}$$

By Young's inequality and the bound on $\mathcal{E}^N(\mu^N)$, one deduces a bound on the total variation (in t and x) of m^N from which (up to a further extraction) one may assume that for every $R > 0$, m^N converges weakly $*$ to some vector-valued measure m on $[0,T] \times B_R$, where m satisfies $\partial_t \mu_t - \frac{1}{2}\Delta\mu_t + \operatorname{div}(m_t) = 0$. By a well known convexity/lower semi-continuity property of the Benamou-Brenier functional (see [4]) we have

$$\liminf_N \mathcal{E}^N(\mu^N) \geq \frac{1}{2}\int_0^T \int_{B_R} \frac{|m|^2}{\mu}.$$

Letting $R \to \infty$ we deduce that $m = \mu v$ and $\int_0^T \int_{\mathbb{R}^d} |v_t|^2 d\mu_t dt < +\infty$ so that

$$\liminf_N \mathcal{E}^N(\mu^N) \geq \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |v_t|^2 d\mu_t dt \geq \mathcal{E}(\mu).$$

Combining it with (4.12) we obtain the desired Γ-liminf inequality.

For the Γ-limsup inequality we will do a smoothing construction in order to deal with the nonlinear term F. For $\delta \in [1/2, 3/2]$, let us consider the time reparametrization $\phi_{\delta,N}$ defined by $\phi_{\delta,N}(0) = 0$ and

$$\phi'_{\delta,N}(t) = \begin{cases} \delta & \text{if } 0 \leq t \leq T/N \\ 1 & \text{if } T/N \leq t \leq T - T/N \\ 2-\delta & \text{if } T - T/N \leq t \leq T \end{cases}.$$

Notice in particular, that, if $\frac{1}{2} \leq \delta \leq \frac{3}{2}$, then

$$\frac{1}{2}(t-s) \leq \phi_{\delta,N}(t) - \phi_{\delta,N}(s) \leq 2(t-s), \ |\phi_{\delta,N}(t) - t| \leq \frac{T}{2N}, \ \forall 0 \leq s \leq t \leq T. \tag{4.14}$$

Let $\mu \in \mathcal{A}_{\rho_0}$ be such that $\mathcal{E}(\mu) + \mathcal{C}(\mu) < +\infty$ (otherwise there is nothing to prove). We then define

$$\tilde{\rho}_t^N = \int_{1/2}^{3/2} \mu_{\phi_{\delta,N}(t)}\, d\delta.$$

In particular it is worthwhile to notice that

$$\tilde{\rho}_t^N = \frac{N}{T} \int_{t-\frac{T}{2N}}^{t+\frac{T}{2N}} \mu_s\, ds \qquad \text{if } T/N \leq t \leq T - T/N. \tag{4.15}$$

Our recovery sequence will be then $\mu_k^N = \tilde{\rho}_{kT/N}^N$, $k = 0, \cdots, N$. Notice that μ^N converges to μ thanks to (4.14) and the fact that the curve $t \in [0,T] \mapsto \mu_t$ is $C^{0,1/2}$ with respect to $\mathcal{W}_2$ as a consequence of Lemma 4.1. By construction, $\mu_N^N = \mu_T$, so the convergence of the terminal term G is trivial. The convergence of the discretization of F_2 then follows directly from its continuity in the Wasserstein metric and Lebesgue's dominated convergence theorem. For the integral term F_1, we use equation (4.15), the convexity and the non-negativity of F_1 to conclude that

$$\begin{aligned} \frac{T}{N}\sum_{k=1}^{N-1} F_1(\mu_k^N) &\leq \frac{T}{N}\sum_{k=1}^{N-1} \frac{N}{T}\int_{\frac{kT}{N}-\frac{T}{2N}}^{\frac{kT}{N}+\frac{T}{2N}} F_1(\mu_s)\, ds \\ &= \int_{\frac{T}{2N}}^{T-\frac{T}{2N}} F_1(\mu_s)\ ds \leq \int_0^T F_1(\mu_s)\ ds \end{aligned}$$

hence

$$\limsup_{N\to\infty} \mathcal{C}^N(\mu^N) \leq \mathcal{C}(\mu). \tag{4.16}$$

For the energy part, we first use the convexity of $\mathrm{FP}_{\frac{T}{N}}$ and the representation of $\tilde{\rho}_t^N$ to get that for $1 \leq k \leq N-2$

$$\mathrm{FP}_{\frac{T}{N}}(\mu_k^N, \mu_{k+1}^N) \leq \frac{N}{T}\int_{kT/N-T/2N}^{kT/N+T/2N} \mathrm{FP}_{\frac{T}{N}}(\mu_s, \mu_{s+T/N})\, ds.$$

Now we take v an optimal vector field for $\mathcal{E}(\mu)$ (existence can be obtained by classical arguments, indeed, taking $m = \mu v$ as new unknown, this is a convex minimization problem with a linear constraint). By construction, we then have $\mathrm{FP}_{\frac{T}{N}}(\mu_s, \mu_{s+T/N}) \leq \frac{1}{2}\int_s^{s+T/N}\int_{\mathbb{R}^d}|v_t|^2 d\mu_t dt$. So, summing up over k we get

$$\sum_{k=1}^{N-2}\mathrm{FP}_{\frac{T}{N}}(\mu_k^N, \mu_{k+1}^N) \leq \frac{N}{T}\sum_{k=1}^{N-2}\int_{kT/N-T/2N}^{kT/N+T/2N}\int_s^{s+T/N}\frac{1}{2}\int_{\mathbb{R}^d}|v_t|^2 d\mu_t dt ds$$
$$\leq \int_{T/2N}^{T-T/2N}\frac{1}{2}\int_{\mathbb{R}^d}|v_t|^2 d\mu_t dt \leq \mathcal{E}(\mu)$$

For the last pieces, we use again the convexity of $\mathrm{FP}_{\frac{T}{N}}$, the definition of $\mu_1^N = \int_{1/2}^{3/2}\mu_{st/N}ds$ and Lemma 4.2 (*iii*) to get

$$\begin{aligned}
\mathrm{FP}_{\frac{T}{N}}(\mu_0^N, \mu_1^N) &= \mathrm{FP}_{\frac{T}{N}}(\rho_0, \mu_1^N) \\
&\leq \int_{1/2}^{3/2}\mathrm{FP}_{\frac{T}{N}}(\rho_0, \mu_{sT/N})\, ds \\
&\leq 5\int_{1/2}^{3/2}\mathrm{FP}_{\frac{sT}{N}}(\rho_0, \mu_{sT/N})\, ds + \frac{1}{2}\int_{1/2}^{3/2}\mathrm{Ent}(\rho_0) - \mathrm{Ent}(\mu_{sT/n})\, ds \\
&\leq 5\int_0^{\frac{3T}{2N}}\int_{\mathbb{R}^d}|v_t|^2\, d\rho_t dt + \frac{1}{2}\int_{1/2}^{3/2}\mathrm{Ent}(\rho_0) - \mathrm{Ent}(\mu_{sT/N})\, ds
\end{aligned}$$

But now the first term is converging to zero as $N \to \infty$ and the second term as well thanks to the uniform continuity of $t \mapsto \mathrm{Ent}(\mu_t)$ (proven in Lemma 4.1 (*iii*)). In a similar way, we can also prove that $\mathrm{FP}_{\frac{T}{N}}(\mu_{N-1}^N, \mu_N^N) \to 0$ thus completing the proof that

$$\limsup_{N\to\infty}\mathcal{E}^N(\mu^N) \leq \mathcal{E}(\mu)$$

which, together with (4.16), gives the desired Γ-limsup inequality for the recovery sequence μ^N.

□

Remark 4.4. For the sake of simplicity, we have taken a continuous local part L_F but is clear from the proof of Theorem 4.3 that our results extend to the case where there is an additional pointwise closed and convex state constraint on the density, for instance a hard congestion constraint such as $\rho \leq M$ where M is a prescribed saturation threshold (a case we will actually consider in our numerical simulations in section 5). It also goes without saying that allowing F and G to depend on x does not create extra difficulties neither.

In particular, when $F = G = 0$, we have

Corollary 4.5. *The sequence of functionals $\mathcal{E}^N\colon \mathcal{A}^N_{\rho_0} \to \mathbb{R}_+ \cup \{+\infty\}$ Γ-converges to the functional $\mathcal{E}\colon \mathcal{A}_{\rho_0} \to \mathbb{R}_+ \cup \{+\infty\}$ as $N \to \infty$.*

4.2 Equivalence

Our aim now is to show that thanks to Theorem 4.3, we have a continuous in time counterpart of the formulas of Corollary 3.3. This will enable us to deduce that (2.11) can be equivalently rewritten as an entropy minimization problem. We first need a technical result:

Lemma 4.6. *Let ρ_0 satisfy (4.6), if $(Q^N)_N$ is a family in $\mathcal{P}(\Omega)$ such that $Q^N_0 = \rho_0$ and $H(Q^N|R)$ is bounded then Q^N is tight. Moreover, if there exists $\mu \in A_{\rho_0}$ such that $Q^N_{kT/N} = \mu_{kT/N}$ for every k and Q^N converges narrowly to Q then $Q_t = \mu_t$ for every $t \in [0, T]$.*

Proof. Let us first define the (equivalent to R) probability measure $\widetilde{R}$ on Ω by $\widetilde{R}(\mathrm{d}\omega) := e^{-\pi|\omega(0)|^2} R(\mathrm{d}\omega)$. We then observe that

$$H(Q^N|R) = H(Q^N|\widetilde{R}) - \pi \int_{\mathbb{R}^d} |x|^2 \mathrm{d}\rho_0.$$

Since Q^N is absolutely continuous with respect to $\widetilde{R}$, it can be identified to its Radon-Nikodym derivative $\widetilde{Q}^N$ with respect to $\widetilde{R}$, we thus deduce from the boundedness of $H(Q^N|R)$ that $\widetilde{Q}^N$ is uniformly integrable hence admits a subsequence which converges weakly in $L^1(\widetilde{R})$, the tightness claim directly follows. Let us now prove the second statement,

Since Ω is Polish and Q^N is tight, by Prokhorov Theorem, for every $\varepsilon > 0$ there exists a compact subset K_ε of Ω such that $Q^N(\Omega \setminus K_\varepsilon) \leq \varepsilon$ for every N. Now let $t \in [0, T]$ and $t_N = k_N T/N$ be such that $|t - t_N| \leq T/N$. For a given $\varphi \in C^0_b(\mathbb{R}^d)$ we then have

$$\int_{\mathbb{R}^d} \varphi \ \mathrm{d}(Q^N_t - \mu_t) = \int_{\mathbb{R}^d} \varphi \ \mathrm{d}((Q^N_t - Q^N_{t_N}) - (\mu_t - \mu_{t_N}))$$

the second term tends to 0 by continuity of the curve $t \mapsto \mu_t$ and we can decompose the first term as

$$\int_{K_\varepsilon} (\varphi(\omega(t)) - \varphi(\omega(t_N)))\mathrm{d}Q^N(\omega) + \int_{\Omega\setminus K_\varepsilon} (\varphi(\omega(t)) - \varphi(\omega(t_N)))\mathrm{d}Q^N(\omega)$$

By Ascoli-Arzéla's Theorem, K_ε is uniformly equicontinuous so that the first term tends to 0 as $N \to +\infty$ whereas the second term is less in absolute value than $2\varepsilon\|\varphi\|_\infty$. Since Q_t^N converges narrowly to Q_t this shows that $Q_t = \mu_t$. □

Theorem 4.7. *Let ρ_0 satisfy (4.6), then we have*

$$\mathcal{S}(\mu) = \mathcal{E}(\mu) + \mathrm{Ent}(\rho_0), \ \forall \mu \in \mathcal{A}_{\rho_0}. \tag{4.17}$$

Proof. Let $\mu \in \mathcal{A}_{\rho_0}$ be such that $\mathcal{S}(\mu) < +\infty$, by definition of $\mathcal{S}^N$ and $\mathcal{S}$ and using Corollary 3.3, we have

$$\mathcal{S}(\mu) \geq \mathcal{S}^N(\mu_0, \mu_{T/N}, \cdots, \mu_T) = \mathcal{E}^N(\mu_0, \mu_{T/N}, \cdots, \mu_T) + \mathrm{Ent}(\rho_0).$$

Taking the liminf and using Theorem 4.3, we get $\mathcal{S}(\mu) \geq \mathcal{E}(\mu) + \mathrm{Ent}(\rho_0)$. Now take $\mu \in \mathcal{A}_{\rho_0}$ such that $\mathcal{E}(\mu) < +\infty$, we then have

$$\mathcal{E}(\mu) \geq \mathcal{E}^N(\mu_0, \mu_{T/N}, \cdots, \mu_T) = \mathcal{S}^N(\mu_0, \mu_{T/N}, \cdots, \mu_T) - \mathrm{Ent}(\rho_0)$$

let then $Q^N \in \mathcal{P}(\Omega)$ be such that $Q_{kT/N} = \mu_{kT/N}$ for $k = 0, \cdots, N$ and

$$\mathcal{S}^N(\mu_0, \mu_{T/N}, \cdots, \mu_T) \geq H(Q^N|R) - \frac{1}{N}$$

passing to a subsequence if necessary, thanks to Lemma 4.6, we may assume that Q^N converges narrowly to some $Q \in \mathcal{P}(\Omega)$ such that $Q_t = \mu_t$ for every $t \in [0, T]$ so that, thanks to the weak lower semi-continuity of the entropy

$$\mathcal{E}(\mu) + \mathrm{Ent}(\rho_0) \geq \liminf_N H(Q^N|R) \geq H(Q|R) \geq \mathcal{S}(\mu).$$

□

5 Algorithm and Numerical Results

5.1 Multi-Marginal Sinkhorn

We now introduce a numerical scheme to solve the discretized in time problem (3.6). The scheme is based on a variant of the celebrated Sinkhorn algorithm [35], already used to solve many variational problem related to optimal transport (see for instance [20, 21, 11, 10, 9, 13, 19, 31, 32]). Note that the iterative method proposed in [24] for MFGs with quadratic hamiltonians can be reinterpreted in this framework.

We recall that (3.6) reads as:

$$\inf\{\mathcal{S}^N(\mu_0,\cdots,\mu_N)+\frac{T}{N}\sum_{k=1}^{N-1}F(\mu_k)+G(\mu_N),\ \mu_i\in\mathcal{P}_2(\mathbb{R}^d),\ \mu_0=\rho_0\}$$

where $\mathcal{S}^N$ is itself defined by (3.4) which is an entropy minimization with multi-marginal constraints. Denoting $\pi^k:(\mathbb{R}^d)^{N+1}\to(\mathbb{R}^d)$ the $k-$th canonical projection we can obviously rewrite (3.6) as an optimization problem over plans γ^N only:

$$\inf\left\{H(\gamma^N|R^N)+i_{\rho_0}(\pi^0_\#\gamma^N)+\frac{T}{N}\sum_{k=1}^{N-1}F(\pi^k_\#\gamma^N)+G(\pi^N_\#\gamma^N)\ :\ \gamma^N\in\mathcal{P}((\mathbb{R}^d)^{N+1})\right\}, \tag{5.1}$$

where

$$i_{\rho_0}(\rho)=\begin{cases}0 \text{ if } \rho=\rho_0\\ +\infty \text{ otherwise}\end{cases}$$

is the indicator function in the convex analysis sense and is used to enforce the initial condition.

We recall that R^N is defined as

$$R^N:=R_{0,\frac{T}{N},\cdots,T}$$

Since R is the reversible Wiener measure, R^N can be decomposed by using the heat kernel (2.4) as

$$R^N(\mathrm{d}x_0,\cdots,\mathrm{d}x_N):=\left(\prod_{k=1}^N P_{\frac{T}{N}}(x_k-x_{k-1})\right)\mathrm{d}x_0\cdots\mathrm{d}x_N. \tag{5.2}$$

We also need to discretize in space (for example we use M grid-points to discretize $\mathbb{R}^d$), then γ^N and R^N become tensors in $\mathbb{R}^{MN}$ (see remark 5.3 for a further decomposition of R^N). Under the assumption that F and G are convex, (5.1) is now a finite-dimensional strictly convex minimization[2]. In order to simplify the presentation, we will keep the continuous in space notation. Integrals must therefore be understood as finite sums and $x_0,..x_N$ as M vectors.

One can now generalise the algorithm formalized in [19, theorem 3.2] (the case $N=1$ or two marginals) as follows. We first state without proof a classical duality result:

[2]We will also allow F to contain a nonconvex but regular term given by a convolution which we will treat in a semi-implict way as explained in paragraph 5.3.3.

Proposition 5.1. *The dual problem of* (5.1) *is*

$$\sup_{(u_0,\cdots,u_N)} -\tilde{F}^\star(-u_0) - \frac{T}{N}\sum_{k=1}^{N-1} F^\star(-u_k) - G^\star(-u_N) - \int \left(\exp(\oplus_{k=0}^N u_k) - 1\right) R^N \tag{5.3}$$

where $\oplus_{k=0}^N u_k : (x_0,\cdots,x_N) \mapsto u_0(x_0) + \cdots + u_N(x_N)$*. Strong duality holds in the sense that the minimum in (5.1) coincides with the maximum in (5.3).*

Denoting by $u_k^\star$ and $\gamma^\star$ the optimal solutions to (5.3) and (5.1) respectively, it follows that the unique solution to (5.1) has the form

$$\gamma^\star(x_0,\cdots,x_N) := \left(\otimes_{k=0}^N e^{u_k^\star(x_k)}\right) R^N(x_0,\cdots,x_N). \tag{5.4}$$

The algorithm is obtained by relaxations of the maximizations on the dual problem (5.3). We get the iterative method computing a sequence of potentials (denoted with the superscripts $.^{(n)}$) :
given $N+1$ vectors $u_k^{(0)}$ with $k = 0,\cdots,N$, then the update at step n is defined as

$$\begin{cases} u_k^{(n)} := \mathrm{argmax}_u \, -\tilde{F}^\star(-u) - \int \exp(u) I_k^u \mathrm{d}x_1 \cdots \mathrm{d}x_N & \text{for } k = 0, \\ u_k^{(n)} := \mathrm{argmax}_u \, -\frac{T}{N} F^\star(-u) - \int \exp(u) I_k^u \mathrm{d}x_0 \cdots \mathrm{d}x_{k-1}\mathrm{d}x_{k+1}\cdots \mathrm{d}x_N & \text{for } k = 1,\cdots,N-1, \\ u_k^{(n)} := \mathrm{argmax}_u \, -G^\star(-u) - \int \exp(u) I_k^u \mathrm{d}x_0 \cdots \mathrm{d}x_{N-1} & \text{for } k = N, \end{cases} \tag{5.5}$$

where

$$I_k^u := \exp(\oplus_{i=0}^{k-1} u_i^{(n)}) \exp(\oplus_{i=k+1}^N u_i^{(n-1)}) R^N.$$

Remark 5.2 (Examples of Energies). For many interesting energies F and G, the relaxed maximizations can be computed pointwise in space and analytically. We list here a few which are tested in the numerical section.

Marginal constraint : In this case the functional F takes the form $F(\rho) = i_{\rho_0}(\rho)$.

Hard congestion : For the hard congestion one has

$$F(\rho) = \begin{cases} 0 \text{ if } \rho \le \overline{\rho}, \\ +\infty \text{ otherwise.} \end{cases}$$

Potentials and Obstacles In this case $F(\rho) = \int V(t,x)\,\rho(x)\,dx$ is linear in ρ.

Obstacles correspond to

$$V(t,x) = \begin{cases} +\infty \text{ if } x \in \Omega(t), \\ 0 \text{ otherwise} \end{cases}$$

where $t \mapsto \Omega(t)$ represent one or more (possibly moving with time) bounded domains.

Remark 5.3 (Implementation). The iterations of Sinkhorn might seem tedious at a first glance because of the integration against R^N, but due to the special form of R^N (5.2), these are just series of convolution with the kernel P_t. Moreover, the heat kernel can be further decomposed along the dimension as follows

$$P_t(z) = \prod_{j=1}^{d} p_t(z_j)$$

where $p_t(z_j)$ is the heat kernel in dimension one. This implies that instead of storing a matrix $P_t \in \mathbb{R}^{M \times M}$ (which is already better than the full tensor R^N) one can just store d small matrices belonging to $\mathbb{R}^{\sqrt[d]{M} \times \sqrt[d]{M}}$.

5.2 Adding a viscosity parameter

It is straightforward to extend the theory and the numerical method to a slightly more general model with a viscosity parameter ε. The Mean Field Game system (1.1) now takes the form :

$$\begin{cases} -\partial_t u - \frac{\varepsilon}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = f[\rho_t], & (t,x) \in (0,T) \times \mathbb{R}^d \\ \partial_t \rho - \frac{\varepsilon}{2}\Delta \rho - \operatorname{div}(\rho \nabla u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d \\ \rho_{|t=0} = \rho_0,\ u_{|t=T} = g[\rho_T]. \end{cases} \tag{5.6}$$

The Lagrangian formulation (2.13) we have proposed becomes

$$\inf \Big\{ H(Q|R_\varepsilon) + \mathcal{C}((Q_t)_{t \in [0,T]}) \ : \ Q \in \mathcal{P}(\Omega),\ Q_0 = \rho_0 \Big\}. \tag{5.7}$$

where R_ε is the reversible Wiener measure induced by a Brownian motion with variance ε. In particular, if we discretize the problem in time, we have that the reference measure R^N_ε can be still decomposed by using the heat kernel $P_{\varepsilon t}(z)$. Notice that we can still use the algorithm we have introduced in the previous section, but the performance, in terms of iterations to converge, will be affected by small values of ε. At least formally, when the viscosity is small, (5.7) is an approximation of the following Lagrangian formulation of first-order variational mean-field games (see [12] for more details about this formulation)

$$\inf \left\{ \mathcal{K}(Q) + \mathcal{C}((Q_t)_{t\in[0,T]}) \ : \ Q \in \mathcal{P}(\Omega),\ Q_0 = \rho_0 \right\}, \tag{5.8}$$

where

$$\mathcal{K}(Q) := \frac{1}{2} \int_\Omega \int_0^T |\dot{\omega}(t)|^2 \mathrm{d}t \mathrm{d}Q(\omega). \tag{5.9}$$

This also implies that we can use the Sinkhorn algorithm, with small ε, in order to approximate the solution to first-order MFGs.

Remark 5.4 ($\Gamma-$convergence)*.* We do not prove here $\Gamma-$convergence results as $\varepsilon \to 0$, for the time discretization, it is indeed, at least on the flat torus, a straightforward generalization of a similar result obtained in [11] for the so-called Bredinger problem.

Remark 5.5 (Numerical limitations)*.* Denoting by h and dt respectively the space and time discretization steps, the heat kernel (2.4) used in the Sinkhorn iterations scales like $e^{-\frac{h^2}{dt\,\varepsilon}}$. It induces a limit to the stability of the numerical method when letting ε go to 0 or increasing, respectively decreasing, h and dt. We refer to [33] for a review of several approaches to mitigate this problem in the 2-marginal case.

5.3 Numerical Results

5.3.1 Planning MFG

We illustrate the effect of varying ε with the planning MFG problem on the torus. This correspond to $F := 0$ and $G := i_{\rho_1}$ (see remark 5.2), the characteristic function which forces, as for the initial density, the final density to match exactly a prescribed density. This problem is equivalent to the dynamic formulation of optimal transport [7] which is a first-order planning MFG with the additional expected diffusion effect linked to the second-order term.

On figure 1, for $\varepsilon = 1$, one can see the effect of the diffusion and that mass will travel across the periodic boundary. On figures 2 ($\varepsilon = 0.1$) and 3 ($\varepsilon = 0.01$), the transport gets closer to the optimal transport/first-order MFG.

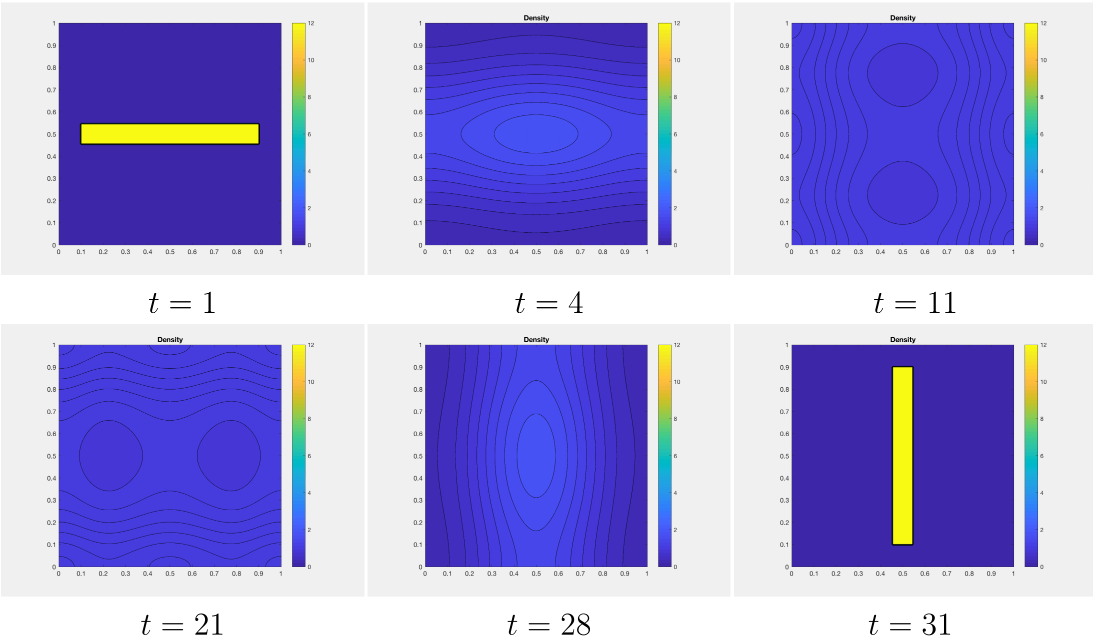


Figure 1: Planning MFG on the torus, Densities at different time steps. $\varepsilon = 1$ and 31 time steps.

5.3.2 Moving Obstacles

We now use the same planning MFG setting but add multiple obstacles moving with time (see remark 5.2). The boundaries of the obstacles are the white circles on the snapshots displayed in figures 4-6. The density is zero inside as agents pay an infinite cost to be there.

5.3.3 Non local interactions

In this section, we investigate a model for which F is not convex in ρ. More precisely, the overall running cost will be of the form $F = F_1 + F_2$, with F_1 a hard congestion term preventing concentration of mass:

$$F_1(\rho) = \begin{cases} 0 \text{ if } \rho \leq 1, \\ +\infty \text{ otherwise.} \end{cases}$$

As for the F_2 term, it is given by a non local interaction functional:

$$F_2(\rho) = -\frac{1}{2} \int \int K(x - y)\rho(y) \, \mathrm{d}y \, \rho(x) \, \mathrm{d}x.$$

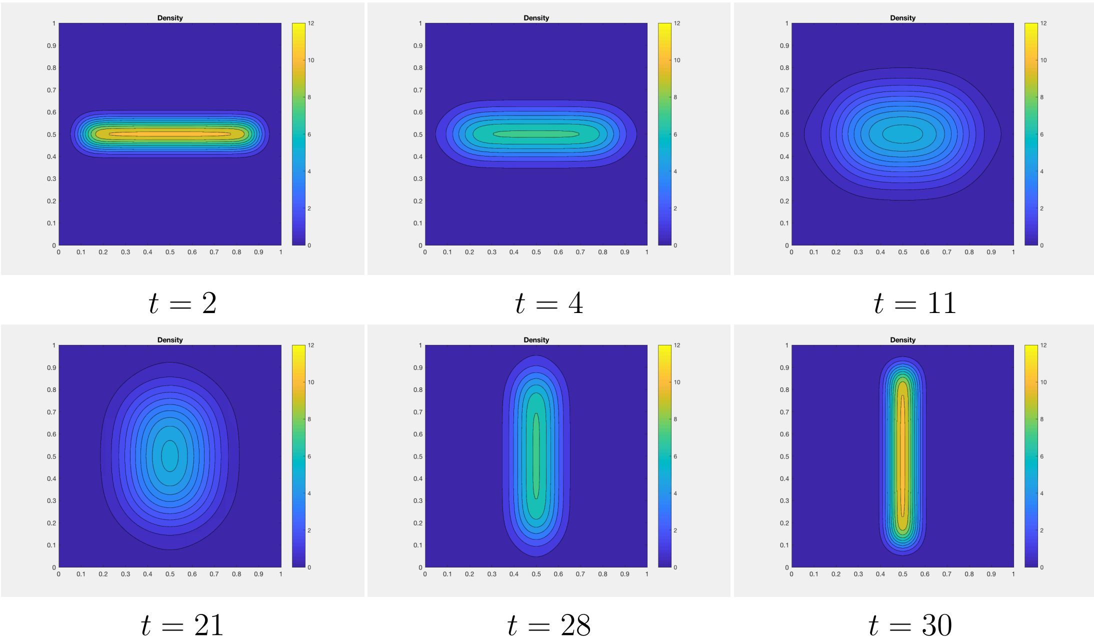


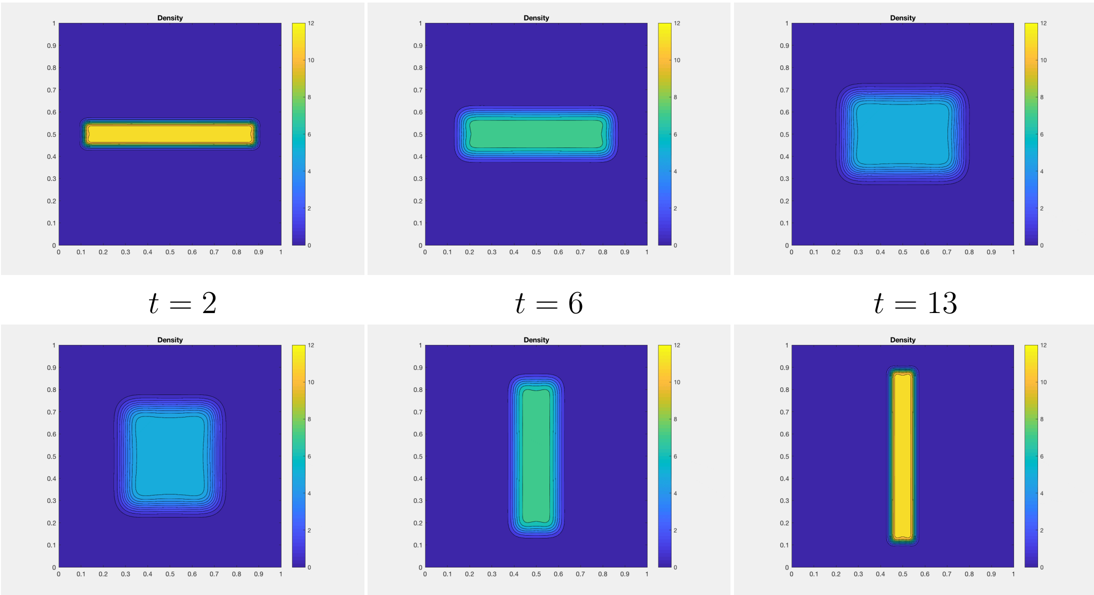
Figure 2: Planning MFG on the torus, fixed initial and final densities similar to figure 1, Densities at different time steps. $\varepsilon = 0.1$ and 31 time steps.

If the kernel is non symmetric ($K(x-y) \neq K(y-x)$) the mean field game is not variational, we only solve for a fixed point of the system.

The first kernel in (see first frame figures 7) in polar coordinates, is the tensor product of Gaussians in radius and direction. It is non symmetric and (because of the minus sign above) will favor streching the mass in the mean direction (here 45 deg.). The initial density is prescribed and the final density is free. The kinetic energy cost penalizes displacement and the game seems to give a smooth deformation to a stationary optimal shape. The periodic boundary conditions are not active as we use a low diffusion parameter ε.

The second proposed kernel in figure 8 is the symmetrized version of the first. We now get a perfectly symmetric evolution toward a stationary shape.

From an algorithmic point of view, we are out of the domain of application of Sinkhorn (5.5). In order to solve the problem, we adopt a semi-implicit approach (see [16]). At each Sinkhorn iteration $n+1$ we relax the non local functional as a potential linear cost using the density of the previous iteration: $f_2^{(n)}(x) = -\int K(x-y)\rho^{(n)}(y)\, dy$.



$t = 2$ $t = 6$ $t = 13$

$t = 17$ $t = 26$ $t = 30$

Figure 3: Planning MFG on the torus, fixed initial and final densities similar to figure 1, Densities at different time steps. $\varepsilon = 0.01$ and 31 time steps.

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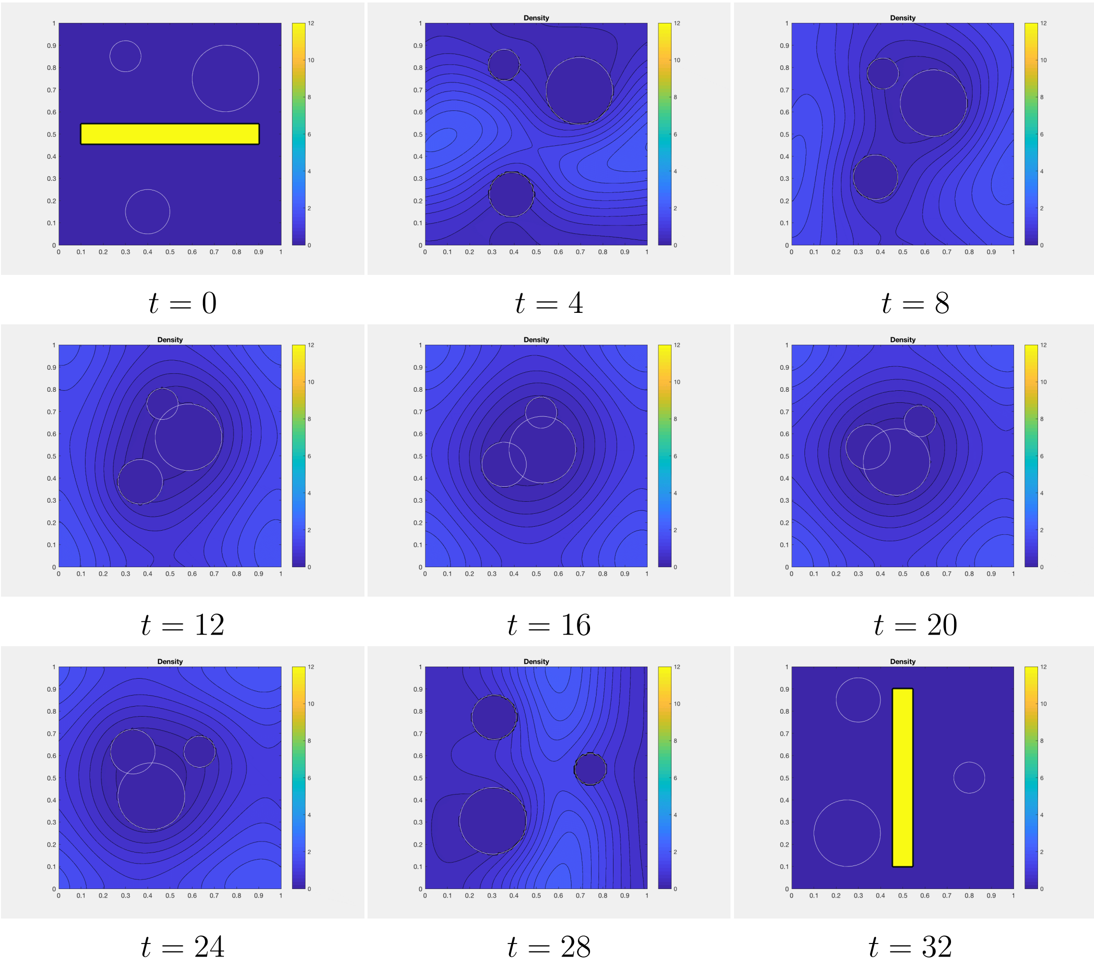

Figure 4: Planning MFG on the torus with moving obstacles, fixed initial and final densities similar to figure 1, Densities at different time steps. $\varepsilon = 1$ and 32 time steps.

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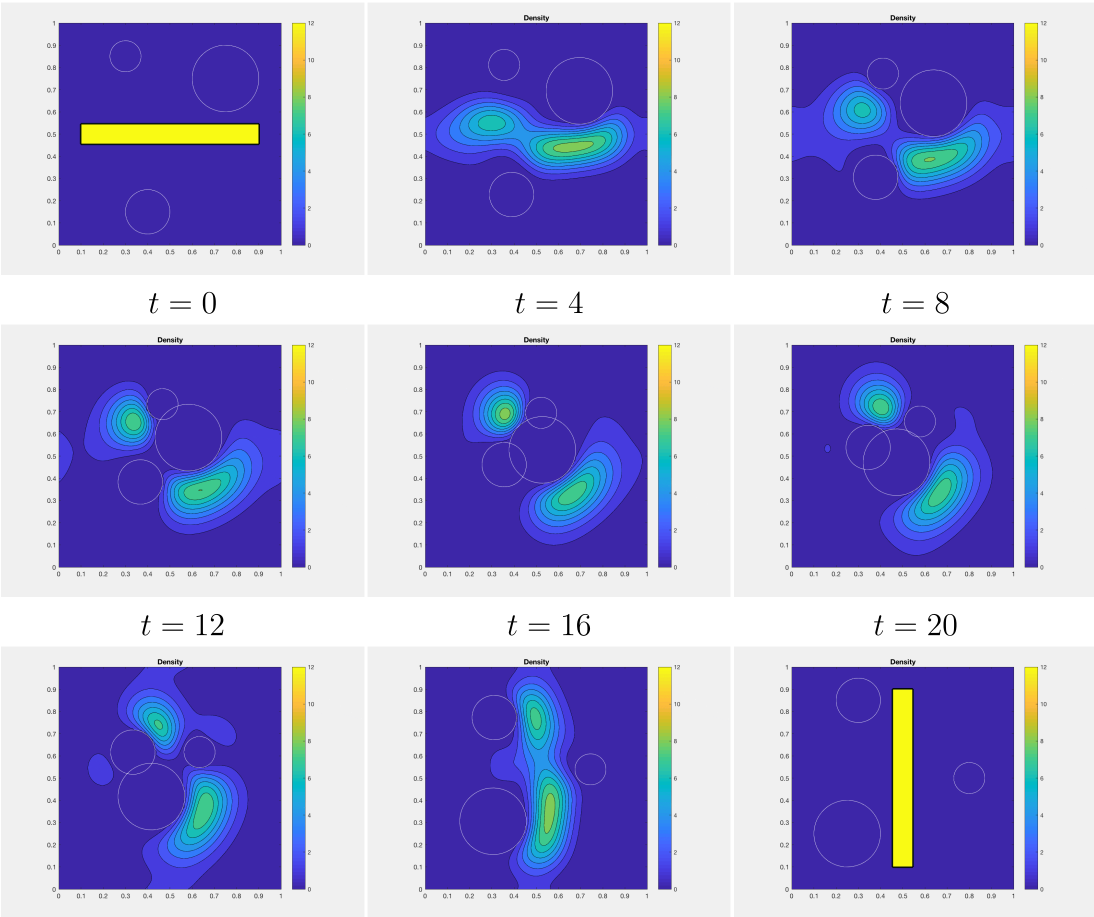

$t = 0$ $t = 4$ $t = 8$

$t = 12$ $t = 16$ $t = 20$

$t = 24$ $t = 28$ $t = 32$

Figure 5: Planning MFG on the torus with moving obstacles, fixed initial and final densities similar to figure 1, Densities at different time steps. $\varepsilon = 0.1$ and 32 time steps.

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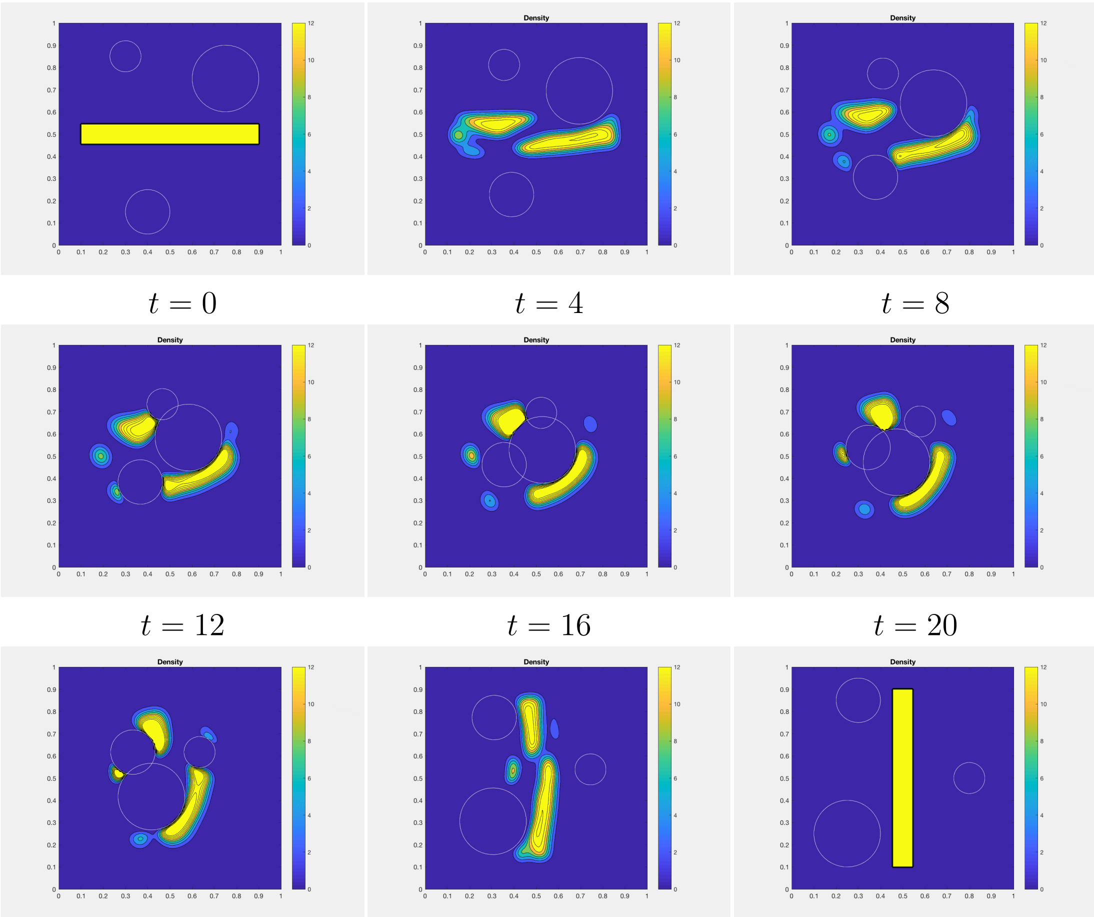

$t = 0$ $t = 4$ $t = 8$

$t = 12$ $t = 16$ $t = 20$

$t = 24$ $t = 28$ $t = 32$

Figure 6: Planning MFG on the torus with moving obstacles, fixed initial and final densities similar to figure 1, Densities at different time steps. $\varepsilon = 0.01$ and 32 time steps.

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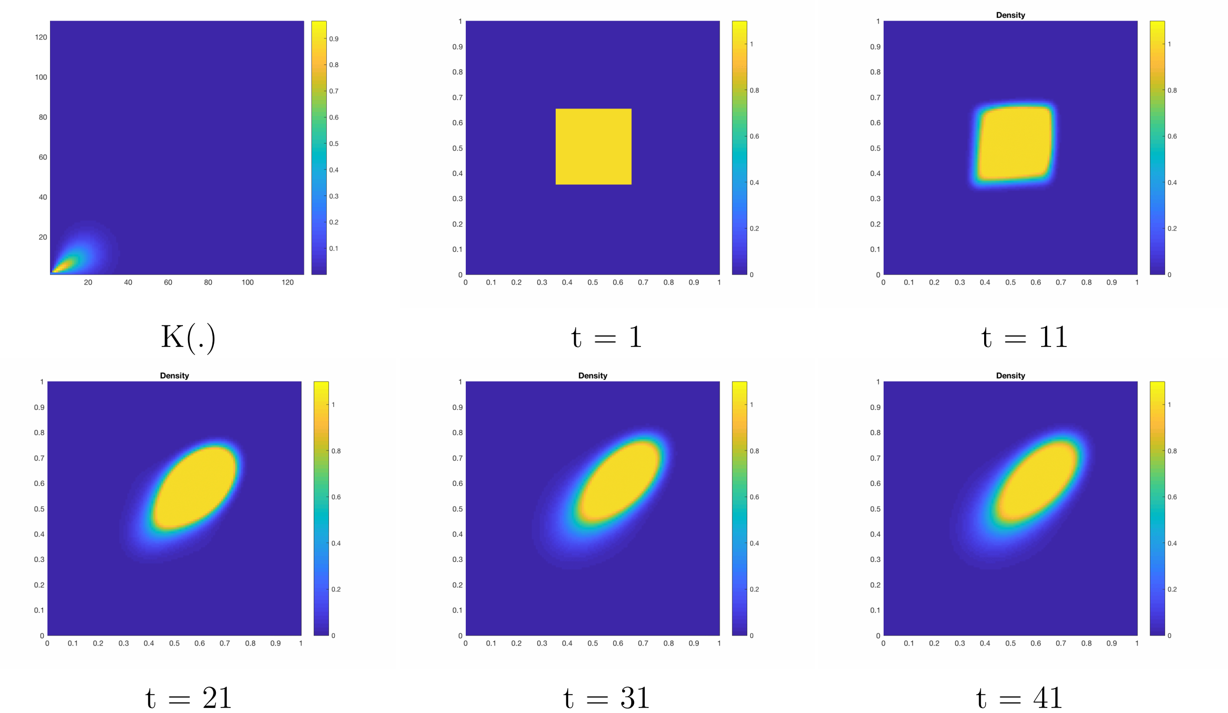

K(.) t = 1 t = 11

t = 21 t = 31 t = 41

Figure 7: Non local interaction. Top left : Kernel K. Densities at different time steps. $\varepsilon = 10^{-3}$.

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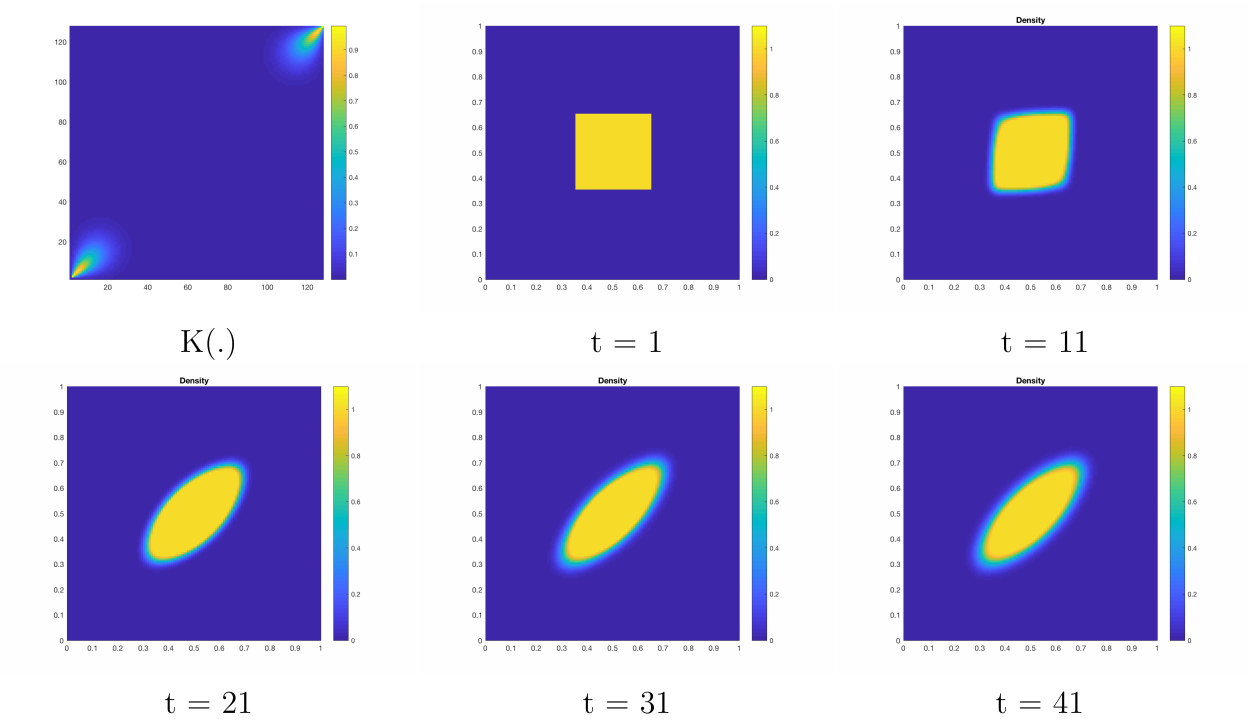

Figure 8: Non local interaction. Top left : potential K. Densities at different time steps. $\varepsilon = 10^{-3}$.

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