Notes on the course on MFG - Course 4

Comments on Section 5 of the course on MFG.

The aim of this presentation is to simplify the lecture of Section 5 of the course devoted to the master equation. We only focus here on 5.1.1, the beginning of 5.1.2, and 5.2.

We are first interested in the differentiability of maps defined on the space of measures. Following Definition 5.1.1 of the notes, we say that a map $U : \mathcal{P}_2 \to \mathbb{R}^k$ is C^1 in the L^2 sense if there exists a bounded continuous map $\frac{\delta U}{\delta m} : \mathcal{P}_2 \times \mathbb{T}^d \to \mathbb{R}^k$ such that, for any $m, m' \in \mathcal{P}_2$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y)\; d(m'-m)(y).$$

We say that $\delta U/\delta m$ is the L^2-derivative of U. Note that $\frac{\delta U}{\delta m}$ is defined only up to an additive constant, that we choose such that

$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m,y) m(dy) = 0 \qquad \forall m \in \mathcal{P}_2.$$

Exercice 1. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous map with a compact support. Show that the map

$$U(m) = \int_{\mathbb{R}^d} \phi(x) m(dx) \qquad m \in \mathcal{P}_1,$$

is C^1 in the L^2 and compute its derivative. Same qu

Exercice 2. Assume that $U : \mathcal{P}_2 \to \mathbb{R}$ is C^1 in the L^2 sense. Show that

$$\frac{\delta U}{\delta m}(m,y) = \lim_{h\to 0^+} \frac{1}{h}\; (U((1-h)m + h\delta_y) - U(m)).$$

When a map $U : \mathcal{P}_2 \to \mathbb{R}$ is C^1 in the L^2 sense, its intrinsic derivative $D_m U : \mathcal{P}_2 \times \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$D_m U(m,y) := D_y \frac{\delta U}{\delta m}(m,y).$$

We say that U has an intrinsic derivative when this quantity is continuous and bounded. The most important property that we have to keep in mind about the intrinsic derivative is the following Itô's formula (described in Proposition 5.1.9 of the note and written here as an exercise).

Exercice 3 (An Itô's formula)**.** We consider an Itô process of the form

$$dX_t = b_t dt + \sigma_t dB_t,$$

where B is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, with associated filtration $(\mathcal{F}_t)$, (b_t) and (σ_t) are progressively measurable processes, such that, for any $T > 0$,

$$\mathbb{E}\left[\int_0^T |b_s|^2 + |\sigma_s|^2\; ds\right] < +\infty.$$

We also assume that X_0 is random, independent of B and satisfies $\mathbb{E}[|X_0|^2] < +\infty$. We denote by $[X_t]$ the law of X_t. s

1. Show that $[X_t]$ belongs to $\mathcal{P}_2$ for any $t \in [0,T]$ and that the map $t \to [X_t]$ is continuous in $\mathcal{P}_2$.

   From now on we fix a map $U : \mathcal{P}_2 \to \mathbb{R}$ which is C^1 in the intrinsic sense and assume that $D_y D_m U$ exists and is continuous and bounded. Our goal is to show that, for any $t \in [0,T]$,

$$(1) \qquad U([X_t]) = U([X_0]) + \int_0^t \mathbb{E}\left[D_m U([X_s], X_s) \cdot b_s + \frac{1}{2}\mathrm{Tr}(\sigma_s \sigma_s^* D_y D_m U([X_s], X_s))\right] ds.$$

2. Fix $s,t>0$. Setting $m^{s,t}(r):=(1-r)[X_s]+r[X_t]$ for $r\in[0,1]$, check that
$$U([X_t])=U([X_s])+\int_0^1\int_s^t \mathbb{E}\left[D_mU(m^{s,t}(r),X_l)+\frac{1}{2}\mathrm{Tr}(\sigma_l\sigma_l^* D_yD_mU(m^{s,t}(r),X_l))\right]dldr.$$

3. Prove (1) by decomposing the interval $[0,t]$ into small intervals $[t_k,t_{k+1}]$ where $t_k=kt/n$ and $n\in\mathbb{N}$ is large, applying the previous step on which subinterval and letting $n\to+\infty$.

4. Assume now that $U:[0,T]\times\mathcal{P}_2\to\mathbb{R}$ is such that ∂_tU exists and is continuous and bounded, that U is C^1 in m in the intrinsic sense and that D_yD_mU exists and is continuous and bounded. Show that
$$U([X_t])=U([X_0])+\int_0^t \mathbb{E}\left[\partial_tU(s,[X_s])+D_mU(s,[X_s],X_s)\cdot b_s+\frac{1}{2}\mathrm{Tr}(\sigma_s\sigma_s^* D_yD_mU(s,[X_s],X_s))\right]ds.$$

We now come back to our simple MFG system.

$$(2)\qquad \left\{\begin{array}{rll}(i) & -\partial_t u-\Delta u+\frac{1}{2}|Du|^2=F(x,m) & \text{in } (0,T)\times\mathbb{R}^d\\ (ii) & \partial_t m-\Delta m-\mathrm{div}\,(m\; Du)=0 & \text{in } (0,T)\times\mathbb{R}^d\\ (iii) & m(0)=m_0\;,\; u(x,T)=G(x,m(T)) & \text{in } \mathbb{R}^d\end{array}\right.$$

Again we assume that F and G satisfy the assumption of given at the beginning of Section 4.2 and are monotone. We know that the optimal control of a small player in this problem is given by $\alpha^*(t,x)=-Du(t,x)$. Note that it depends on time and space, but not on the distributions of the other players m. If this simplifies the expression, it does not allow for mistakes in the evaluation of the evaluation of the flow of measures.

To overcome this issue, one relies on the master equation. It takes the (horrible) form

$$(3)\qquad \left\{\begin{array}{l}-\partial_tU(t,x,m)-\Delta_xU(t,x,m)+\frac{1}{2}|D_xU(t,x,m)|^2-\int_{\mathbb{R}^d}\mathrm{div}_yD_mU(t,x,m,y)\;m(dy)\\ \qquad +\int_{\mathbb{R}^d}D_mU(t,x,m,y)\cdot D_xU(t,y,m)\;m(dy)=F(x,m)\\ \qquad\qquad\qquad \text{in } (0,T)\times\mathbb{R}^d\times\mathcal{P}_2\\ U(T,x,m)=G(x,m)\qquad \text{in } \mathbb{R}^d\times\mathcal{P}_2\end{array}\right.$$

where the unknown is now a map $U:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2\to\mathbb{R}$. Note that the part $-\partial_tU(t,x,m)-\Delta_xU(t,x,m)+\frac{1}{2}|D_xU(t,x,m)|^2-F(x,m)$ is basically the Hamilton-Jacobi equation. The other terms $-\int_{\mathbb{R}^d}\mathrm{div}_yD_mU(t,x,m,y)\;m(dy)+\int_{\mathbb{R}^d}D_mU(t,x,m,y)\cdot D_xU(t,y,m)\;m(dy)$ describe the impact of the evolution of the measure on U.

The following remark explains the link between the master equation and the MFG system (2).

Exercice 4. Assume that U is a classical solution to (3) (meaning that all derivatives appearing in (3) exist, are continuous and bounded in all variables). Fix X_0 be a random variable on $\mathbb{R}^d$ with law $m_0\in\mathcal{P}_2$ and $X=(X_t)$ be a solution to McKean-Vlasov equation
$$X_t=X_0-\int_0^t D_xU(s,X_s,[X_s])ds$$
where $[X_s]$ is the law of X_s.

1. Explain why such an equation has a unique solution (see Section 3.3.1 in the notes).

2. Set $m(t)=[X_t]$ and $u(t,x)=U(t,x,m(t))$. Using Exercise 3, show that the pair (u,m) solves the MFG system (2).