Option B pour l'agrégation

TD/TP 8 :
Equations aux dérivées partielles: Equations de transport

Rule #1: "Never change the deal".

Exercice 1 ("Tranquille Maurice, tranquille..."). For $u^0 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, solve the following transport equations:

$$\begin{cases} u_t(t,x) + cu_x(t,x) = -u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases}$$

$$\begin{cases} u_t(t,x) + xu_x(t,x) = u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases}$$

⋆

Exercice 2 (Burgers with source term). We consider the Burgers equation with a source term

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \alpha u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = \frac{1}{1+|x|} \text{ and then } \frac{1}{1+x^2}, & x \in \mathbb{R}, \end{cases}$$

Solve this equation for sufficiently small times (:= to be precised).

⋆

Exercice 3 (Characteristics for the Vlasov equation). We consider the following kinetic *Vlasov* equation on $\Omega \subset \mathbb{R}^n$ with a space confining potential $\mathcal{V}_\varepsilon$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \mathcal{V}_\varepsilon \cdot \nabla_v f = 0, & (t,x,v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^n, \\ f(0,x,v) := f_0(x,v) \end{cases}$$

1. Write the characteristic ODE satisfied by $(X_\varepsilon(x,v), V_\varepsilon(x,v))$.
2. Prove that the trajectories are Hamiltonian (in the phase space).
3. We consider the special case where $\Omega := \mathbb{R}^2$ and $\mathcal{V}_\varepsilon(x) := C(\varepsilon)\left(1 - \min\left(\frac{x_1}{\varepsilon}, 1\right)\right) e_1$. Write the characteristics and study the convergence when $\varepsilon \to 0$.
4. What do you deduce for f^ε, at least formally, when $\varepsilon \to 0$.

⋆

Exercice 4 (Finite speed of propagation). Consider a scalar conservation law with flux, $f \in C^2$:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

with initial data u^{in}. Show that if u^{in} is compactly supported with, say, $\mathrm{Supp}(u^{in}) \subset [-B, +B]$ for some $B \in \mathbb{R}$, then

$$\mathrm{Supp}(u(t, \cdot)) \subset [-B + t \min_{x \in \mathbb{R}} f'(u^{in}(x)), +B + t \min_{x \in \mathbb{R}} f'(u^{in}(x))].$$

⋆

Rule #2: "No names"

Exercice 5 (Wonderwall). We consider the transport problem in the half space:

$$\begin{cases} \partial_t u + c\partial_x u = 0, & (t, x) \in]0, +\infty[\times]0, +\infty[, \\ u(t = 0, x) = f(x), & \forall x \geq 0, \\ u(t, x = 0) = g(t), & \forall t \geq 0, \end{cases}$$

where f and $g \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$ and $c > 0$. Give some compatibility conditions that ensure the existence of a classical solution. What happens in the case $c < 0$?

⋆

Exercice 6 (Not in the full space). For $u^0 \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$, solve:

$$\begin{cases} x\partial_t u + t\partial_x u = 0, & (t, x) \in \ \Omega = \left\{(t, x) \in \mathbb{R}^2 \mid x > t > 0\right\}, \\ u(0, \cdot) = u_0. \end{cases}$$

⋆

Exercice 7 (Vanishing viscosity). Consider the following Burgers equation with a viscosity term:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \dfrac{\partial u_\varepsilon}{\partial x} - \varepsilon \dfrac{\partial^2 u_\varepsilon}{\partial x^2} = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ u_\varepsilon(0, x) = u^{in}(x) & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a small viscosity constant.

1. Make the following *so-called* Hopf-Cole transform:

$$u_\varepsilon = \frac{-2\varepsilon}{\omega_\varepsilon} \frac{\partial \omega_\varepsilon}{\partial x},$$

   and compute the solution to the initial value problem (1).

2. Let $\{u_\varepsilon\}$ be a sequence of solutions associated with (1) for different values of ε.

   (a) Prove that the sequence u_ε converges almost everywhere to a limit u when $\varepsilon \to 0$.

(b) Suppose that $u^{in} \in C_b(\mathbb{R})$. Show that the limit u thus obtained solves the following Burgers equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 & \text{in } (0,T] \times \mathbb{R} \\ u(0,x) = u^{in}(x) & x \in \mathbb{R}. \end{cases} \tag{2}$$

⋆

Exercice 8 (Two phase flow in (porous) media). Find the solution of the Riemann problem for the following scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{u^2 + (1-u)^2}\right) = 0 \text{ in } (0,\infty) \times \mathbb{R}$$

with initial condition

$$u(0,x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

and then

$$u(0,x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0. \end{cases}$$

⋆

Rule #3: "Never open the package."

Exercice 9 (Back to basics: the transport equation). Enjoy solving the standard free transport equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad \text{in } (0,\infty) \times \mathbb{R}, \tag{3}$$

with the following numerical schemes:

- Schéma décentré:

$$\frac{u_j^{n+1} - u_j^n}{dt} + c\frac{u_j^n - u_{j-1}^n}{dx} = 0$$

- Schéma centré:

$$\frac{u_j^{n+1} - u_j^n}{dt} + c\frac{u_{j+1}^n - u_{j-1}^n}{2dx} = 0$$

- Schéma de Lax-Friedrichs:

$$\frac{u_j^{n+1} - \frac{u_{j+1}^n + u_j^n}{2}}{dt} + c\frac{u_{j+1}^n - u_{j-1}^n}{2dx} = 0$$

- Schéma de Lax-Wendroff:

$$\frac{u_j^{n+1} - u_j^n}{dt} + c\frac{u_{j+1}^n - u_{j-1}^n}{2dx} - c^2 dt\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2dx^2} = 0$$

⋆ One implicit scheme:

$$\frac{u_j^{n+1} - u_j^n}{dt} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2dx} = 0$$

For each scheme, you will wonder about the stability and convergence (namely the order in dt and dx), and plot the l^∞ and L^2 norms of the numerical solution. For simplicity, one can rewrite the schemes with the parameter $\alpha := \frac{cdt}{dx}$.

⋆

Exercice 10 (Conservation laws). Solve numerically:

1. The two-fluids flow equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{u^2+(1-u)^2}\right) = 0 \text{ in } (0,\infty)\times\mathbb{R}$$

2. The standard Burgers equation

3. The equation of road traffic ($f(u) := u(1-u)$)

with a decreasing step initial condition. You are allowed (if it is relevant) to use (and study) the following Lax-Friedrichs scheme

$$\frac{u_j^{n+1} - \frac{u_{j+1}^n + u_j^n}{2}}{dt} + \frac{\Phi^n_{j+\frac{1}{2}} - \Phi^n_{j-\frac{1}{2}}}{dx} = 0,$$

where the numerical flux is given by

$$\Phi_j^n := \frac{f(u_j^n) + f(u_{j+1}^n)}{2}.$$

⋆

Exercice 11 (Hamilton-Jacobi equation). Solve the eikonal equation

$$\partial_t u + |\partial_x u|^2 = 0$$

⋆

Exercice 12 (Numerical diffusivity). We consider the transport equation with $c(x) = 1 - \mathbf{1}_{(-a,a)}(x)$, where a is a small parameter. What is the theoretical behavior of $u_0(x) = \mathbf{1}_{]-\infty,-10]}$? What do you see numerically observe? To see more clearly the effect you can put a growth term on $]a,+\infty[$.

⋆

Exercice 13 (Kinetic travelling waves*). Solve the following kinetic reaction transport equation

$$\partial_t f + v\cdot\partial_x f = \frac{1}{2}\int_{-1}^{1} f(v)dv - f + \left(\int_{-1}^{1} f(v)dv\right)\left(\frac{1}{2} - f\right), \tag{4}$$

with $x\in\mathbb{R}$ and $v\in[-1,1]$. Take a decreasing step initial condition. Be careful of the boundaries!

⋆