École Normale Supérieure de Lyon **Année 2013–2014**

Département de Mathématiques

Option B pour l'agrégation

TD/TP 7 :
Equations aux dérivées partielles: Equations elliptiques

Sur Sobolev.

Exercice 1 (Non-completeness). We define $\Omega := \mathrm{B}(0,1)$. Prove that the space

$$V := \left\{ v \in \mathcal{C}^1(\overline{\Omega}), v = 0 \text{ sur } \partial\Omega \right\}$$

is not complete for the scalar product of $\mathrm{H}^1_0(\Omega)$, $\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$. For this purpose, you may consider the following sequences:

If $n = 1$, the following regularized absolute value

$$u_n(x) := \begin{cases} -x - 1, \qquad x \leq -\frac{1}{n}, \\ \frac{n}{2}x^2 - 1 + \frac{1}{2n}, \qquad -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ x - 1, \qquad x \geq \frac{1}{n}. \end{cases}$$

If $n = 2$,

$$u_n(x) := \left| \ln\left(\frac{|x|^2}{4} + n^{-1} \right) \right|^{\frac{\alpha}{2}} - \left| \ln\left(\frac{1}{4} + n^{-1} \right) \right|^{\frac{\alpha}{2}}.$$

If $n > 2$,

$$u_n(x) := \frac{1}{(|x|^2 + n^{-1})^{\beta/2}} - \frac{1}{(1 + n^{-1})^{\beta/2}}.$$

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Exercice 2 (Poincaré-Wirtinger). Let $\Omega \in \mathbb{R}^d$ be an open, bounded domain. Show that there exists a constant $C(\Omega)$, depending on the domain, such that

$$\forall\, u \in \mathrm{H}^1(\Omega), \qquad \int_\Omega |u - \bar{u}|^2 \, \mathrm{d}x \leq C(\Omega) \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

where $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x) \, \mathrm{d}x$.

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Exercice 3 (Rellich). Let $\Omega \in \mathbb{R}^d$ be open and bounded. Prove that any uniformly bounded sequence in $\mathrm{H}^1_0(\Omega)$ is relatively compact in $\mathrm{L}^2(\Omega)$.

Hint 1: This means that if $\{u_n\} \subset \mathrm{H}^1_0(\Omega)$ *is a sequence such that* $\|u_n\|_{\mathrm{H}^1(\Omega)} \leq C$ *for some constant* C *independant of* n*, then there exists a subsequence* $\{u_{\varphi(n)}\}$ *(with* $\varphi : \mathbb{N} \to \mathbb{N}$ *strictly increasing) and a limit* $u \in \mathrm{L}^2(\Omega)$ *such that*

$$\left\|u_{\varphi(n)} - u\right\|_{\mathrm{L}^2(\Omega)} \to 0 \qquad \textit{as} \;\; n \to \infty.$$

Hint 2: Show that $u \in \mathrm{H}^1(\mathbb{R}^d)$ *iff* $u \in \mathrm{L}^2(\mathbb{R}^d)$ *and*

$$\exists M \geq 0,\ \forall h \in \mathbb{R}^d, \quad \|u(\cdot + h) - u\|_{\mathrm{L}^2} \leq M\|h\|.$$

Hint 3: We recall the Riesz-Fréchet-Kolmogorov theorem:

Let $\Omega \in \mathbb{R}^d$ *be open. First consider* $\omega \subset\subset \Omega$*, i.e.* ω *open with* $\bar{\omega} \subset \Omega$*. Consider* $\mathrm{G} \subset \mathrm{L}^2(\Omega)$*. Suppose*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \delta < d(\omega, \partial\Omega),\ \forall h \in \mathbb{R}^d,\ |h| < \delta,\ \ \forall\, u \in \mathrm{G}, \quad \|u(\cdot + h) - u(\cdot)\|^2_{\mathrm{L}^2(\omega)} \leq \varepsilon. \tag{1}$$

Then $\mathrm{G}|_\omega$ *is relatively compact in* $\mathrm{L}^2(\omega)$*. Second, assume in addition to* (1) *that*

$$\forall \varepsilon > 0, \exists\, \omega \subset\subset \Omega,\ \ \forall\, u \in \mathrm{G}, \qquad \|u\|_{\mathrm{L}^2(\Omega\setminus\omega)} < \varepsilon.$$

Then G *is relatively compact in* $\mathrm{L}^2(\Omega)$*.*

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Du Lax-Milgram en veux-tu en voilà!

Exercice 4. Let $u \in \mathrm{H}^1(\Omega)$ be a weak solution of the following Neumann problem:

$$\begin{cases} -\nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u = f & \text{in } \Omega, \\ -A(x)\nabla u \cdot n = g & \text{on } \partial\Omega. \end{cases} \tag{2}$$

where $f \in \mathrm{L}^2(\Omega)$, $g \in \mathrm{H}^1(\Omega)$. The coefficient A is elliptic and $b \in L^\infty(\Omega)$ satisfies $\nabla \cdot b = 0$ in Ω and $b \cdot n = 0$ on $\partial\Omega$. Prove that (2) has a unique weak solution up to addition of a constant if and only if the source terms satisfy the following compatibility condition:

$$\int\limits_\Omega f(x)\,\mathrm{d}x = \int\limits_{\partial\Omega} g(x)\,\mathrm{d}\sigma(x). \tag{3}$$

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Exercice 5 (Robin). Let $\sigma \geq 0$ and $f \in \mathrm{L}^2(\mathbb{R}^n_+), g \in \mathrm{L}^2(\mathbb{R}^{n-1})$. We study the following problem with *so-called* Robin boundary conditions:

$$\begin{cases} -\Delta u(x) + u(x) = f(x), & x \in \mathbb{R}^n_+, \\ -\partial_1 u(0,y) + \sigma u(0,y) = g(y), & y \in \mathbb{R}^{n-1}. \end{cases} \tag{4}$$

1. Give a definition of strong solution of (4) and of weak solution of (4).

2. Show existence and uniqueness of a weak solution of (4).
3. Prove that is $g = 0$ *a.e.*, the weak solution of (4) is in $\mathrm{H}^2(\mathbb{R}^n_+)$.
4. Assuming that $g = 0$ *a.e.*, we denote by u_n the strong solution associated to $\sigma = n$. Prove that u_n converges in $\mathrm{H}^1(\Omega)$ towards a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u(x) + u(x) = f(x), & x \in \mathbb{R}^n_+, \\ u(0, y) = 0, & y \in \mathbb{R}^{n-1}. \end{cases} \tag{5}$$

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Régularité et rigidité elliptique.

Exercice 6 (Cacciopoli - Interior regularity). Suppose $\Omega \in \mathbb{R}^d$ be open. Let $x_0 \in \Omega$ and $0 < \rho < \bar{\rho}$ such that $B(x_0, \bar{\rho}) \subset \Omega$. Suppose $u \in \mathrm{H}^1(\Omega)$ satisfies

$$-\Delta u + b \cdot \nabla u + a\, u = 0 \text{ in } \Omega,$$

where $a, b_i \in \mathbb{R}$ for $1 \leq i \leq d$. Show that there exists a constant C such that

$$\int\limits_{B(x_0,\rho)} |\nabla u|^2 \,\mathrm{d}x \leq \frac{C}{(\bar{\rho} - \rho)^2} \int\limits_{B(x_0,\bar{\rho})} |u|^2 \,\mathrm{d}x.$$

Take $a = b_i = 0$. Deduce that

$$\forall\, k \in \mathbb{N}, \quad \|u\|^2_{\mathrm{H}^k(B(x_0,\rho))} \leq C(\rho, \bar{\rho}, k)\|u\|^2_{\mathrm{L}^2(B(x_0,\bar{\rho}))}, \quad \|u\|^2_{C^k(B(x_0,\rho))} \leq C(\rho, \bar{\rho}, k)\|u\|^2_{\mathrm{L}^2(B(x_0,\bar{\rho}))}.$$

What do you conclude?

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Exercice 7 (Maximum Principle - Divergence Form). Also, let $c(x) \in L^\infty(\Omega)$ and $c(x) \geq -\lambda$ with

$$\lambda := \inf \left\{ \int\limits_\Omega |\nabla v(x)|^2 \,\mathrm{d}x \;;\; v \in \mathrm{H}^1_0(\Omega),\ \|v\|_{\mathrm{L}^2(\Omega)} = 1 \right\}.$$

Suppose $u \in \mathrm{H}^1(\Omega)$ verifies in the weak sense

$$-\Delta u + c(x)u \geq 0$$

on Ω, which means in an explicit way we have:

$$\forall\, \phi \in C^\infty_0(\Omega), \phi \geq 0, \qquad \int\limits_\Omega \nabla u \cdot \nabla \phi \,\mathrm{d}x + \int\limits_\Omega c\, u\phi \,\mathrm{d}x \geq 0. \tag{6}$$

Show that $\inf_{x \in \Omega} u(x) = \inf_{x \in \partial\Omega} u(x)$.
Hint: Use the density of $C^\infty_0(\Omega)$ in $\mathrm{H}^1_0(\Omega)$. Take $-(u - \inf\limits_{x \in \partial\Omega} u(x))^-$ as test function.

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Exercice 8 (Strong Maximum Principle). Let $\Omega \in \mathbb{R}^d$ be a connected open domain. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and a_{ij}, b_i, c are smooth enough, a is uniformly elliptic and $c(x) \geq 0$ in Ω.

1. *(Preliminary)* Prove that if (α_{ij}) and (β_{ij}) are two positive definite matrices then we have $\sum_{i,j} \alpha_{ij}\beta_{ij} \geq 0$.
2. Show that if u is a subsolution of the elliptic operator:

$$-\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u \leq 0 \quad \text{in } \Omega,$$

   such that

$$u(x_0) = \max_{x \in \bar{\Omega}} u(x) \geq 0,$$

   then u is constant in Ω.
3. Write (and prove ☺) a similar statement for a supersolution.

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Exercice 9 (Harnack's inequality for harmonic functions). Suppose that u solves

$$-\Delta u = 0, \qquad \forall\, x \in D(0, R).$$

for some $R > 0$. We recall that u is thus analytic.

1. Prove the following Poisson representation formula:

$$\forall z \in D(0, R), \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} u(Re^{i\theta})d\theta.$$

2. Prove that for $\theta \in [0, 2\pi]$, $z \in B(0, R), |z| = r$, we have

$$\frac{R-r}{R+r} \leq \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \leq \frac{R+r}{R-r}$$

3. Deduce from the above the following so-called *Harnack inequality* when u is nonnegative:

$$\forall z \in B(0, R), |z| = r, \qquad \left(\frac{R-r}{R+r}\right) u(0) \leq u(z) \leq \left(\frac{R+r}{R-r}\right) u(0).$$

This can be generalized (with pain) to a nonnegative solution u of

$$-\nabla \cdot (A\nabla u) + b \cdot \nabla u + cu = 0 \quad \textit{in } \Omega,$$

on any connected $\omega \subset\subset \Omega$. The Harnack inequality writes

$$\sup_{x \in \omega} u(x) \leq C(\omega) \inf_{x \in \omega} u(x).$$

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Simulations numériques d'équations elliptiques.

Exercice 10 (Batman begins). We consider the following Laplace equation

$$\begin{cases} -u" + c(x)u = f, & x \in (0,1)\,, \\ u(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

1. Solve numerically our problem.
2. When we prescribe $f(x) = \left(1 + 2x - x^2\right) e^x$ and $c(x) = x$, what is the solution u?
3. Compare numerically both solutions and plot the error curves in a 'loglog' scale.

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Exercice 11 (Batman returns). Redo the previous exercise with the following Laplace equation

$$\begin{cases} -u"(x) = f(x), & x \in (0,1)\,, \\ u(0) = 0, \quad u'(1) = 0, \end{cases}$$

where the Neumann boundary condition should be discretized as follows:

$$u_{N+1} = u_N, \qquad \text{and then} \qquad u_{N+1} - u_N = \frac{h^2}{2} f(1).$$

How would you generalize this two schemes to the full system

$$\begin{cases} -u" + c(x)u = 0, & x \in (0,1)\,, \\ u(0) = \alpha, \quad u'(1) = \beta \end{cases} \quad ?$$

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Exercice 12 (Batman & Robin). We now consider Robin (☺) boundary conditions:

$$\begin{cases} -u" + u = f(x), & x \in (0,1)\,, \\ u'(0) + \alpha u(0) = 0, \quad u'(1) + \alpha u(1) = 0, \end{cases}$$

How do you discretize the boundaries? What is the order of the scheme?

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Exercice 13 (Spectral problem). Solve numerically the following spectral problem

$$\begin{cases} -Q_{\theta\theta} + g(\theta)Q = \lambda Q, & \theta \in (0,1)\,, \\ Q_\theta(0) = Q_\theta(1) = 0 \end{cases}$$

with a shooting method. Choose your favorite function g.

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Exercice 14 (Batman Forever). We consider the following Laplace equation with a drift term

$$\begin{cases} -u'' + c(x)u' = f, & x \in (0,1)\,, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

1. Solve numerically our problem with a very naive scheme.
2. We want to get an order 2 scheme. What should we do?

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Exercice 15 (Laplace equation 2D*). We consider the following Laplace equation

$$\begin{cases} -u_{xx} - u_{yy} + c(x,y)u = 0, & x \in (0,1)^2\,, \\ u(0,y) = 1, \quad u(1,y) = 0, \\ u_y(x,0) = 0, u_y(x,1) = 0. \end{cases}$$

1. Design the numerical matrix of the elliptic operator.
2. Go.

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