Solutions exercices sur le différentielle

Exercice 1.53 (Valeurs approchées)

1. $(1.02)^{3.01} \sim 1.061418$. On choisit $f(x,y) = x^y = e^{y\ln(x)}$ et $M_0 = (1,3)$. La fonction f est de classe $\mathcal{C}^1$ sur $]0,\infty[\times\mathbb{R}$, et $f(1,3) = 1$. En outre

$$\frac{\partial f}{\partial x}(x,y) = yx^{y-1}, \quad \frac{\partial f}{\partial y}(x,y) = x^y\ln(x), \quad \frac{\partial f}{\partial x}(1,3) = 3, \quad \frac{\partial f}{\partial y}(1,3) = 0 \tag{1}$$

   Alors $f(1.02, 3.01) \simeq \hat{f}_{(1,3)}(1+0.02, 3+0.01)$ et $\hat{f}_{(1,3)}(1+0.02, 3+0.01) = 1 + 0.02\times 3 + 0.01\times 0 = 1.06$

2. $\sqrt{5.7\times 6.2} \sim 5.9447$. On choisit $f(x,y) = \sqrt{xy}$ et $M_0 = (6,6)$. La fonction f est de classe $\mathcal{C}^1$ sur $]0,\infty[\times]0,\infty[$, et $f(6,6) = 6$. En outre

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2}\sqrt{\frac{y}{x}}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{1}{2}\sqrt{\frac{x}{y}}, \quad \frac{\partial f}{\partial x}(6,6) = \frac{1}{2}, \quad \frac{\partial f}{\partial y}(6,6) = \frac{1}{2} \tag{2}$$

   Alors $f(5.7, 6.2) \simeq \hat{f}_{(6,6)}(6-0.3, 6+0.2)$ et $\hat{f}_{(6,6)}(6-0.3, 6+0.2) = 6 - 0.3\times\frac{1}{2} + 0.2\times\frac{1}{2} = 5.95$

3. $\ln(1.02\times 0.9) \sim -0.08555$ On choisit $f(x,y) = \ln(xy)$ et $M_0 = (1,1)$. La fonction f est de classe $\mathcal{C}^1$ sur $D_f = \{(x,y)\in\mathbb{R}^2 : x\neq 0, y\neq 0, xy>0\}$, et $f(1,1) = 0$. En outre

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{1}{y}, \quad \frac{\partial f}{\partial x}(1,1) = 1, \quad \frac{\partial f}{\partial y}(1,1) = 1 \tag{3}$$

   Alors $f(1.02, 0.9) \simeq \hat{f}_{(1,1)}(1+0.02, 1-0.1)$ et $\hat{f}_{(1,1)}(1+0.02, 1-0.01) = 0 + 0.02\times 1 - 0.1\times 1 = -0.08$

Exercice 1.54 (Approximation affine)

(i) $f(x,y) = \frac{x+y}{1+x+y}$ au voisinage de $(1,0)$ et $(0,0)$

La fonction f est de classe $\mathcal{C}^1$ sur $D_f = \{(x,y)\in\mathbb{R}^2 : 1+x+y>0\}$. En outre

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{(x+y+1)^2}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{1}{(x+y+1)^2} \tag{4}$$

(a) On calcule

$$f(0,1) = \frac{1}{2}, \quad \frac{\partial f}{\partial x}(1,0) = \frac{1}{4}, \quad \frac{\partial f}{\partial y}(1,0) = \frac{1}{4} \tag{5}$$

l'approximation affine de f au voisinage du point $(1,0)$ est

$$\begin{aligned}\hat{f}_{(1,0)}(1+h,k) &= f(1,0) + h\frac{\partial f}{\partial x}(1,0) + k\frac{\partial f}{\partial y}(1,0) = f(1,0) + df_{(1,0)}(h,k)\\ &= \frac{1}{2} + \frac{h}{4} + \frac{k}{4}\end{aligned} \tag{6}$$

(b) On calcule

$$f(0,0) = 1,\ \frac{\partial f}{\partial x}(0,0) = 1,\ \frac{\partial f}{\partial y}(0,0) = 1 \tag{7}$$

l'approximation affine de f au voisinage du point $(0,0)$ est

$$\begin{aligned}\hat{f}_{(0,0)}(h,k) &= f(0,0) + h\frac{\partial f}{\partial x}(0,0) + k\frac{\partial f}{\partial y}(0,0) = f(0,0) + df_{(0,0)}(h,k) \\ &= 1 + h + k\end{aligned} \tag{8}$$

(ii) $g(x,y) = x\exp\left(x^2 - y^2\right)$ au voisinage de $(1,1)$

La fonction g est de classe $\mathcal{C}^1$ sur $\mathbb{R}^2$. En outre

$$\frac{\partial g}{\partial x}(x,y) = \left(2x^2+1\right)e^{x^2-y^2},\ \ \frac{\partial g}{\partial y}(x,y) = -2xye^{x^2-y^2} \tag{9}$$

et

$$g(1,1) = 1,\ \frac{\partial g}{\partial x}(1,1) = 3,\ \ \frac{\partial g}{\partial y}(x,y) = -2 \tag{10}$$

l'approximation affine de f au voisinage du point $(1,1)$ est

$$\begin{aligned}\hat{g}_{(1,1)}(h,k) &= g(1,1) + h\frac{\partial g}{\partial x}(1,1) + k\frac{\partial g}{\partial y}(1,1) = g(1,1) + dg_{(1,1)}(h,k) \\ &= 1 + 3h - 2k\end{aligned} \tag{11}$$