We prove here the following: Let X be a Banach space and let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous convex function and $D = \operatorname{dom} f$. We assume that D has non empty interior, $f \in C^1(\operatorname{int}(D))$. We prove here that if $\lim_{n\to\infty} \|df(x_n)\|_* = \infty$ for any sequence $x_n \to \partial D$, then $\partial f(x) = \emptyset$ for any $x \in \partial D$. (In finite dimension, one can replace "interior" by "relative interior" and the same is true at the relative boundary, as can easily deduced by restricting the problem to the affine subspace where D has non empty interior.)

Proof: We first observe that f is locally bounded from above in the interior of the domain, because we have assumed it is C^1. (If we want to relax this assumption, it also relies on Baires's theorem, using that for y in the interior we have $D = \cup_{n\geq 1} y + n[\{x : f(x) \leq f(y)+1\} - y]$ so that the set $\{f(x) \leq f(y)+1\}$ contains a ball, then one can show that f is bounded in some ball around y.)

We consider y, δ with $B(y,\delta) \subset \operatorname{int}(D)$ and $\sup_{B(y,\delta)} f < +\infty$. We let for $x \in D$, $t \in]0,1[$, $|z| < \delta$,

$$f_t(z) = \frac{1}{t}(f((1-t)x + t(y+z)) - f(x)).$$

By convexity, $f_t(z) \leq f(y+z) - f(x) \leq \sup_{B(y,\delta)} f - f(x) < +\infty$.

Assume $x \in \partial D$ with $\partial f(x) \neq \emptyset$. Then if $p \in \partial f(x)$,

$$f((1-t)x + t(y+z)) - f(x) \geq (p, (1-t)x + t(y+z) - x) = t(p, y+z-x)$$

so that for $|z| < \delta$, $f_t(z) \geq (p, y-x) - \delta\|p\|_*$. Letting $C = \max\{\sup_{B(y,\delta)} f - f(x), \delta\|p\|_* - (p, y-x)\}$, we find that $|f_t(z)| \leq C$ in $B(0,\delta)$. For $L = 4C/\delta$, we deduce that f_t is L-Lipschitz in $B(0,\delta/2)$. In particular,

$$\|df(x + t(y-x))\|_* = \|df_t(0)\|_* \leq L$$

for any $t > 0$, which contradicts the assumption. We could even remove the assumption that f is C^1 in the interior of the domain and assume that for (x_n, p_n) with $p_n \in \partial f(x_n)$ and $x_n \to \partial D$, $\|p_n\|_* \to \infty$, and would get a contradiction as well.