Near extreme eigenvalues of large random Gaussian matrices and applications

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A. Perret, Y. V. Fyodorov, G. S., in preparation

Extreme value statistics

- Statement of the problem

$X_1, X_2, \cdots, X_N : N$ random variables, $P_{\rm joint}(X_1, X_2, \cdots, X_N)$

$X_{\max} = \max_{1 \leq i \leq N} X_i, F_N(M) = \mathbb{P}(X_{\max} \leq M)$ Q: $N \to \infty$?

- Fully understood for i.i.d. random variables

Three different universality classes depending on the pdf of X_i : Gumbel, Fréchet, Weibull

Extreme value statistics

- Statement of the problem

$X_1, X_2, \cdots, X_N : N$ random variables, $P_{\rm joint}(X_1, X_2, \cdots, X_N)$

$X_{\max} = \max_{1 \leq i \leq N} X_i, F_N(M) = \mathbb{P}(X_{\max} \leq M)$ Q: $N \to \infty$?

- Very few exact results for strongly correlated variables

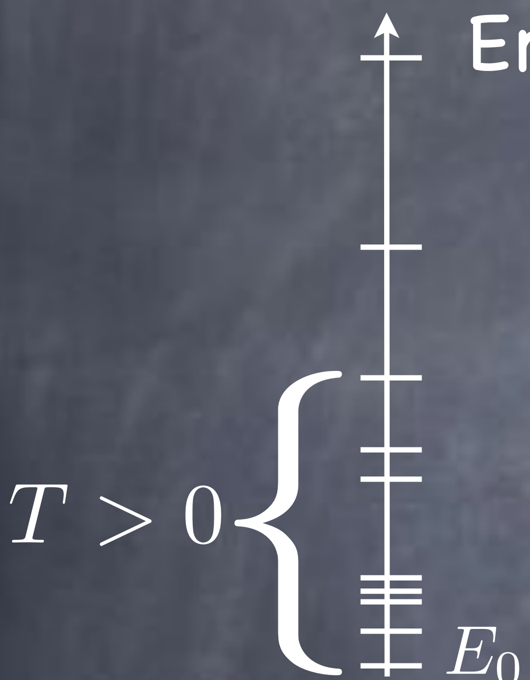
  ✓ Random walks

  ✓ Random matrices

- $X_{\max}$ is interesting BUT concerns a single variable among N

Statistics of Near-Extremes

- Statistical Physics



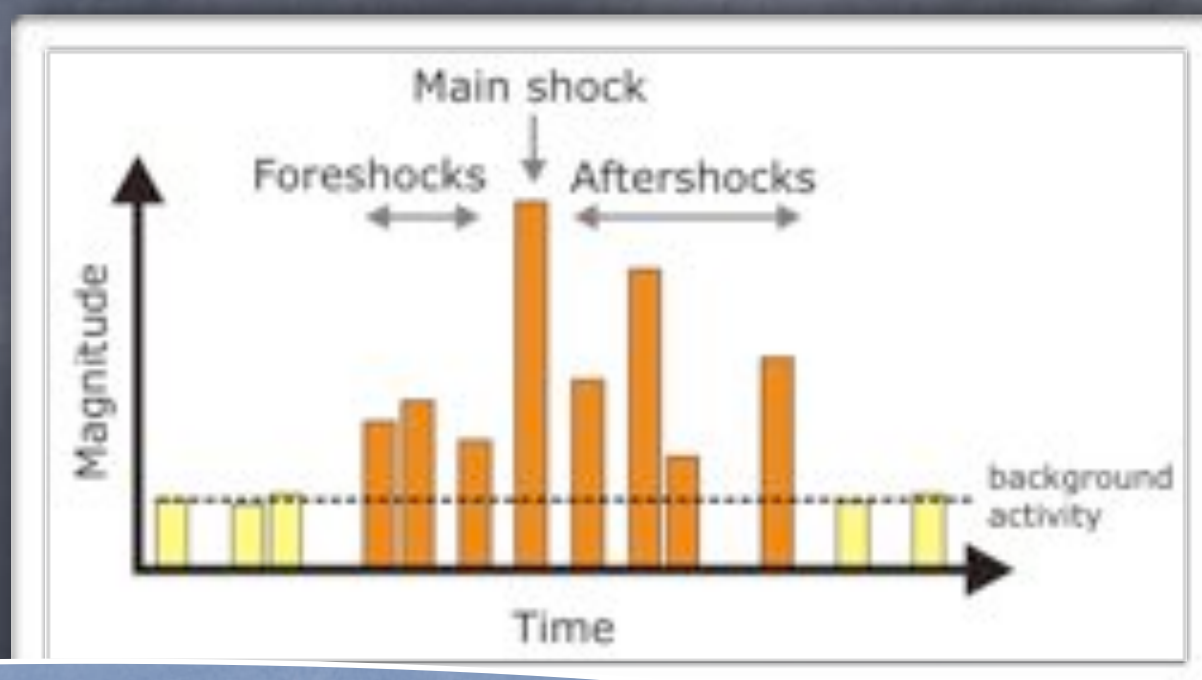
Energy levels

Ground state $T = 0$

see also:

- ✓ Branching Brownian motion
- ✓ 1/f noise
- ✓ Brownian motion
- ✓ ...

- Natural sciences (e.g. seismology)



Crowding near the extremes

Statistics of Near-Extremes

How to quantify the crowding close to extreme values ?

- Look at (higher) order statistics: k[th] maximum

$$X_{\max} = M_{1,N} > M_{2,N} > \cdots > M_{N,N} = X_{\min}$$

and in particular the spacings (gaps) $d_{k,N} = M_{k,N} - M_{k+1,N}$

- Consider the density of near-extremes Sabhapandit, Majumdar '07

$$\rho_{\mathrm{DOS}}(r,N) = \frac{1}{N-1} \sum_{i=1, i\neq \mathrm{imax}}^{N} \overline{\delta(X_{\max} - X_i - r)}$$

Near-extreme eigenvalues of random matrices

- Let J be a real symmetric (or complex Hermitian) $N \times N$ random matrix

- The matrix J has N real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ which are strongly correlated

- Largest eigenvalue $\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$

- Density of near extreme eigenvalues $$\rho_{\rm DOS}(r, N) = \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max} - \lambda_i - r)}$$

- A related quantity is the gap between the two largest eigenvalues

see also Witte, Bornemann, Forrester '13

Application: minimizing a quadratic form on the sphere

- Quadratic form on the N-dimensional sphere S_N

$$H[\,\vec{s}\,] = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\, s_i s_j \ , \ (\vec{s}\,)^2 = \sum_{i=1} s_i^2 = N$$

$\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of J with $\lambda_{\max} = \max\limits_{1 \le i \le N} \lambda_i$

- Minimisation of this quadratic form on the sphere

$\Longrightarrow$ introduce a Lagrange multiplier z

$$\tilde{H}[\,\vec{s}\,, z] = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\, s_i s_j + z\left(\sum_{i=1}^{N} s_i^2 - N\right)$$

$$\min_{\vec{s},z} \tilde{H}[\,\vec{s}\,, z] = \tilde{H}[\,\vec{s}_{\max}\,, z_{\max}] = -N\frac{\lambda_{\max}}{2} \quad \text{with} \quad \begin{cases} J\,\vec{s}_{\max} = \lambda_{\max}\,\vec{s}_{\max} \\ z_{\max} = \dfrac{\lambda_{\max}}{2} \end{cases}$$

Application: minimizing a quadratic form on the sphere

$$\tilde{H}[\,\vec{s}\,, z] = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij} s_i s_j + z\left(\sum_{i=1}^{N} s_i^2 - N\right)$$

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- Eigenvalues of the Hessian matrix at the minimum $\vec{s}_{\max}, z_{\max}$

spectrum of $\left.\dfrac{\delta^2 \tilde{H}}{\delta s_i \delta s_j}\right|_{\vec{s}_{\max}, z_{\max}}$ is $\{0, \lambda_{\max} - \lambda_1, \lambda_{\max} - \lambda_2, \cdots, \lambda_{\max} - \lambda_N\}$

Reminding that $\rho_{\mathrm{DOS}}(r, N) = \dfrac{1}{N-1}\displaystyle\sum_{\substack{i=1 \\ i \neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max} - \lambda_i - r)}$

$\Longrightarrow$ $\rho_{\mathrm{DOS}}(r, N)$ is the mean eigenvalue density of $\left.\dfrac{\delta^2 \tilde{H}}{\delta s_i \delta s_j}\right|_{\vec{s}_{\max}, z_{\max}}$

Application to spherical fully connected spin-glasses

- Dynamics of the spherical fully connected spin-glass model (spherical Sherrington-Kirkpatrick model)

Cugliandolo, Dean '95
Ben Arous, Dembo, Guionnet '06

$$\frac{\partial s_i(t)}{\partial t} = -\frac{\delta}{\delta s_i(t)} \tilde{H}[\vec{s}(t), z(t)] + h_i(t) + \zeta_i(t)$$

temp. T

infinitesimal mag. field

where $\tilde{H}[\vec{s}, z] = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij} s_i s_j + z \left(\sum_{i=1}^{N} s_i^2 - N \right)$

and J belongs to the GOE ensemble of RMT with variance $\overline{J_{ij}{}^2} = \frac{1}{N}$

- Random initial condition at time $t = 0$: $\vec{s}(t=0) = \sum_{\alpha=1}^{N} \vec{s}_\alpha$

normalized eigenvectors of J: $\vec{s}_\alpha \mid J \, \vec{s}_\alpha = \lambda_\alpha \, \vec{s}_\alpha$

- Relaxational dynamics characterized by two-time quantities

Application to spherical fully connected spin-glasses

$$\frac{\partial s_i(t)}{\partial t} = \sum_{j=1}^{N} J_{ij} s_j(t) - z(t) s_i(t) + h_i(t)\ ,\ T = 0$$

- Relaxational dynamics characterized by two-time quantities $t > t'$

Correlation function

$$C(t,t') = \frac{1}{N} \sum_{i=1}^{N} \overline{s_i(t) s_i(t')}$$

Response function

$$R(t,t') = \frac{1}{N} \sum_{i=1}^{N} \overline{\frac{\delta s_i(t)}{\delta h_i(t')}}\Bigg|_{h=0}$$

- In the quasi-stationary regime, for $t, t' \gg 1$, $t - t' = \tau$ fixed

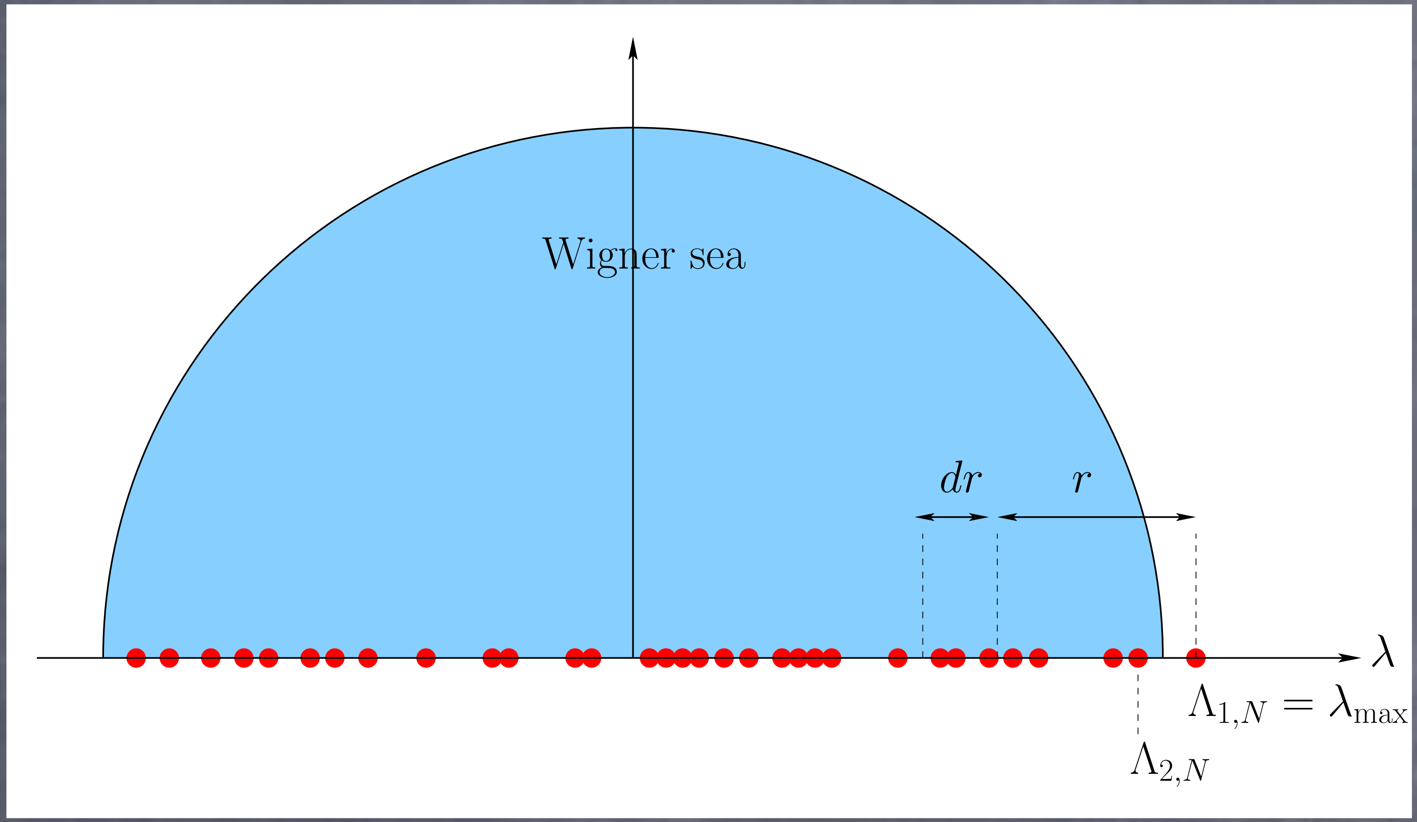
$$\boxed{R(t,t') \sim \int_0^\infty e^{-r(t-t')} \rho_{\mathrm{DOS}}(r,N)\, dr}$$

Perret, Fyodorov, G. S. '14

see also Kurchan, Laloux '96

Near-extreme eigenvalues of random matrices

$$P_{\text{joint}}(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \, e^{-N \frac{\beta}{2} \sum_{i=1}^N \lambda_i^2}$$



✓ Density of near extreme eigenvalues

$$\rho_{\text{DOS}}(r, N) = \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max} - \lambda_i - r)}$$

✓ 1st gap

$$\rho_{\text{GAP}}(r, N)dr = \mathbb{P}[(\Lambda_{1,N} - \Lambda_{2,N}) \in [r, r+dr]]$$

Density of near extreme eigenvalues in GUE

$$\rho_{\rm DOS}(r,N) = \frac{1}{N-1} \sum_{\substack{i=1 \\ i\neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max} - \lambda_i - r)}$$

- Fluctuations of the largest eigenvalue $\lambda_{\max} = \max_{1\leq i\leq N} \lambda_i$

$$\lambda_{\max} = \sqrt{2} + N^{-2/3}\frac{1}{\sqrt{2}}\underbrace{\chi_\beta}_{\text{Tracy-Widom}}$$

- Depending on (r,N) one expects two different regimes

  ✓ $r = O(N^0)$ bulk regime

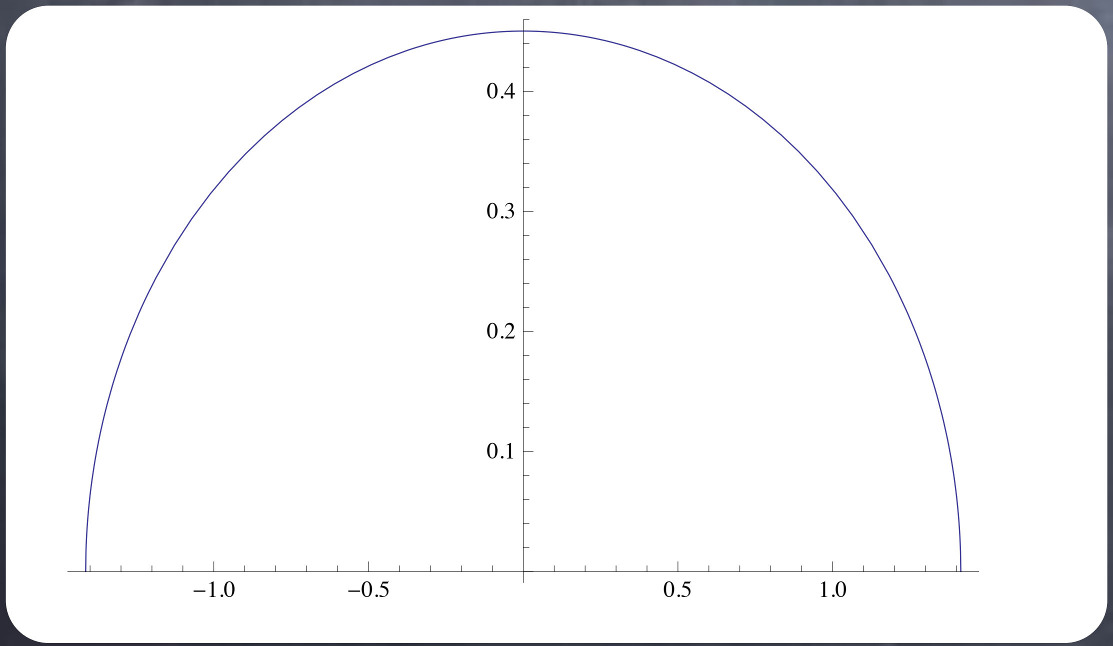
  ✓ $r = O(N^{-2/3})$ edge regime

A detour by the density of eigenvalues of random matrices

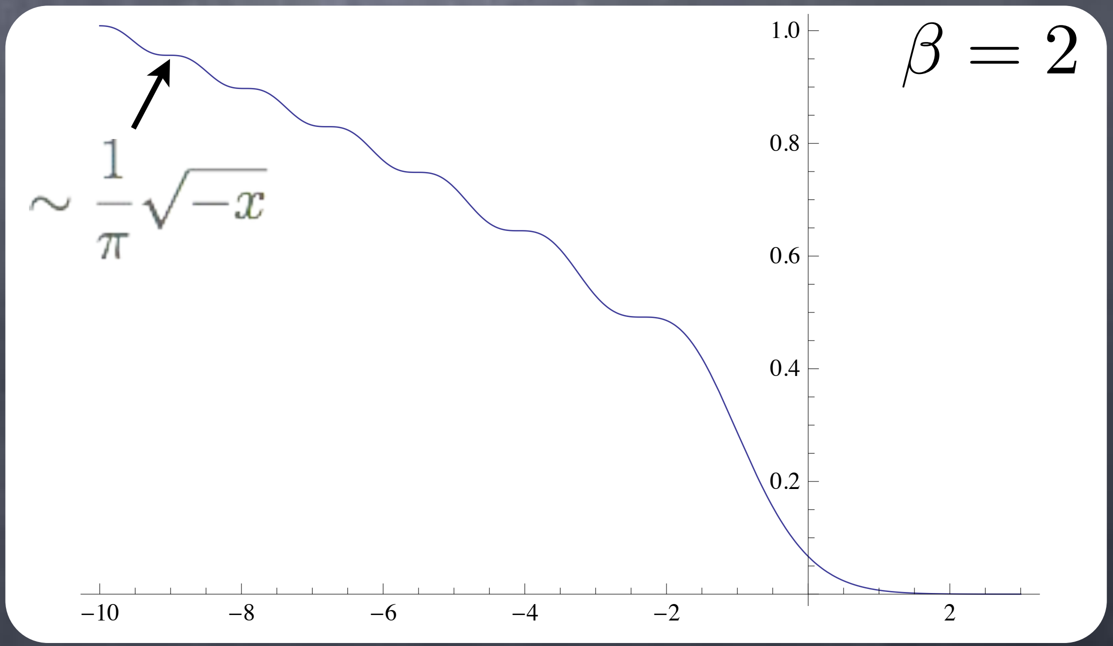
$$\rho(\lambda, N) = \frac{1}{N}\sum_{i=1}^{N} \overline{\delta(\lambda - \lambda_i)}$$

- Two regimes: bulk and edge regime   Bowick & Brézin '91 , Forrester '93

$$\rho(\lambda, N) \sim \begin{cases} \rho_{\text{bulk}}(\lambda)\ ,\ \lambda = O(N^0)\ \&\ |\lambda| < \sqrt{2} \\ \\ \sqrt{2}N^{-1/3}\rho_{\text{edge}}[(\lambda - \sqrt{2})\sqrt{2}N^{2/3}]\ ,\ |\lambda - \sqrt{2}| = O(N^{-2/3}) \end{cases}$$



$$\rho_{\text{bulk}}(x) = \frac{1}{\pi}\sqrt{2 - x^2}$$



$$\rho_{\text{edge}}(x) = [\text{Ai}'(x)]^2 - x\text{Ai}^2(x)$$

A detour by the density of eigenvalues of GUE random matrices

$$\rho(\lambda, N) = \frac{1}{N} \sum_{i=1}^{N} \overline{\delta(\lambda - \lambda_i)}$$

- Two regimes: bulk and edge regime

$$\rho(\lambda, N) \sim \begin{cases} \rho_{\text{bulk}}(\lambda) \ , \ \lambda = O(N^0) \ \& \ |\lambda| < \sqrt{2} \\ \\ \sqrt{2} N^{-1/3} \rho_{\text{edge}}[(\lambda - \sqrt{2})\sqrt{2} N^{2/3}] \ , \ |\lambda - \sqrt{2}| = O(N^{-2/3}) \end{cases}$$

- Matching between the bulk and edge regimes

$$\rho_{\text{edge}}(x) \sim \begin{cases} \frac{1}{\pi}\sqrt{-x} \ , \ x \to -\infty & \text{matching with Wigner semi-circle} \\ \\ e^{-\frac{2\beta}{3} x^{3/2}} \ , \ x \to +\infty & \text{coincides with the right tail of TW} \end{cases}$$

Density of near extreme eigenvalues: results

$$\rho_{\rm DOS}(r,N) = \frac{1}{N-1} \sum_{\substack{i=1 \\ i\neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max} - \lambda_i - r)}$$

$$\boxed{\rho_{\rm DOS}(r,N) \sim \begin{cases} \tilde{\rho}_{\rm bulk}(r) \;,\; r = O(N^0) \;\&\; 0 < r < 2\sqrt{2} \\ \sqrt{2} N^{-1/3} \tilde{\rho}_{\rm edge}(r\sqrt{2}N^{2/3}) \;,\; r = O(N^{-2/3}) \end{cases}}$$

A. Perret, G. S. '14

- In the bulk $r \sim O(N^0)$:
  - ✓ $\rho_{\rm DOS}(r,N)$ is insensitive to the fluctuations of $\lambda_{\max} \sim \sqrt{2}$
  - ✓ $\rho_{\rm DOS}(r,N) \sim \frac{1}{N} \sum_{i=1}^{N} \overline{\delta(\sqrt{2} - \lambda_i - r)} = \rho(\sqrt{2} - r, N)$
  - ⇒ ✓ $\boxed{\tilde{\rho}_{\rm bulk}(x) = \frac{1}{\pi}\sqrt{x(2\sqrt{2} - x)}}$ shifted Wigner semi-circle

Density of near extreme eigenvalues: results

$$\rho_{\rm DOS}(r,N) = \frac{1}{N-1}\sum_{\substack{i=1\\ i\neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max}-\lambda_i-r)}$$

$$\rho_{\rm DOS}(r,N) \sim \begin{cases} \tilde\rho_{\rm bulk}(r) \;,\; r = O(N^0) \;\&\; 0 < r < 2\sqrt{2} \\ \\ \sqrt{2}N^{-1/3}\tilde\rho_{\rm edge}(r\sqrt{2}N^{2/3}) \;,\; r = O(N^{-2/3}) \end{cases}$$

- At the edge $r \sim O(N^{-2/3})$, a non trivial function

$$\tilde\rho_{\rm edge}(x) \sim \begin{cases} a_\beta x^\beta \;,\; x \to 0 \\ \\ \dfrac{\sqrt{x}}{\pi} \;,\; x \to \infty \end{cases}$$

A. Perret, G. S. '14

Consequences on the dynamics of the spherical fully connected spin-glass model

$$\frac{\partial s_i(t)}{\partial t} = \sum_{j=1}^{N} J_{ij} s_j(t) - z(t) s_i(t) + h_i(t) \ , \ T = 0$$

- In the quasi-stationary regime, for $t, t' \gg 1$, $t - t' = \tau$

$$R(t,t') \sim \int_0^\infty e^{-r(t-t')} \rho_{\mathrm{DOS}}(r,N)\, dr$$

two regimes: $$\rho_{\mathrm{DOS}}(r,N) \sim \begin{cases} \tilde{\rho}_{\mathrm{bulk}}(r) \ , \ r = O(N^0) \ \& \ 0 < r < 2\sqrt{2} \\ \\ \sqrt{2} N^{-1/3} \tilde{\rho}_{\mathrm{edge}}(r\sqrt{2}N^{2/3}) \ , \ r = O(N^{-2/3}) \end{cases}$$

- Two temporal regimes for the response function

✓ $1 \ll t, t' \ll N^{2/3}$ — Cugliandolo, Dean '95; Ben Arous, Dembo, Guionnet '06

✓ $\boxed{t, t' = O(N^{2/3})}$ — Perret, Fyodorov, G. S. '15

Consequences on the dynamics of the spherical fully connected spin-glass model

$$R(t,t') \sim \int_0^\infty e^{-r(t-t')} \rho_{\rm DOS}(r,N)\, dr$$

- For $\tau = (t-t') \sim \mathcal{O}(N^{2/3})$

$$R(t,t') \sim \frac{1}{N} f_R\left(\frac{t-t'}{N^{2/3}}\right) \;,\; f_R(x) = \int_0^\infty e^{-\frac{r}{\sqrt{2}}x} \tilde{\rho}_{\rm edge}(x)\, dx$$

universal !

- Asymptotic behaviors

$$f_R(x) \sim \begin{cases} x^{-3/2} \;,\; x \to 0 \\ \\ x^{-2} \;,\; x \to \infty \end{cases}$$

- Can one compute the full universal function $\tilde{\rho}_{\rm edge}(x)$?

$\Longrightarrow$ an exact calculation for GUE   Perret, G. S. '14

Density of near extreme eigenvalues for GUE

Perret, G. S. '14

$$\boxed{\tilde{\rho}_{\rm edge}(\tilde{r}) = \frac{2^{1/3}}{\pi} \int_{-\infty}^{\infty} \left(\tilde{f}^2(\tilde{r},x) - \left(\int_x^{\infty} q(u) \tilde{f}(\tilde{r},u) du \right)^2 \right) \mathcal{F}_2(x)\, dx}$$

$$\mathcal{F}_2(x) = \exp\left[- \int_x^{\infty} (u-x) q^2(u) du \right] \, , \quad \begin{cases} q'' = 2q^3 + x\, q \, , \\ q(x) \sim \mathrm{Ai}(x) \text{ for } x \to \infty \, , \end{cases}$$

$$\partial_x^2 \tilde{f}(\tilde{r},x) - [x + 2q^2(x)] \tilde{f}(\tilde{r},x) = -\tilde{r} \tilde{f}(\tilde{r},x) \, , \; \tilde{f}(\tilde{r},x) \underset{x\to\infty}{\sim} 2^{-1/6} \sqrt{\pi} \mathrm{Ai}(x-\tilde{r})$$

$\tilde{f}(\tilde{r},x)$ is related to a solution of the Lax pair associated to Painlevé XXXIV

Density of near extreme eigenvalues for GUE

$$\boxed{\tilde{\rho}_{\rm edge}(\tilde{r}) = \frac{2^{1/3}}{\pi}\int_{-\infty}^{\infty}\left(\tilde{f}^2(\tilde{r},x) - \left(\int_x^{\infty} q(u)\tilde{f}(\tilde{r},u)du\right)^2\right)\mathcal{F}_2(x)\,dx}$$

$$\partial_x^2 \tilde{f}(\tilde{r},x) - [x + 2q^2(x)]\tilde{f}(\tilde{r},x) = -\tilde{r}\tilde{f}(\tilde{r},x)\ ,\ \tilde{f}(\tilde{r},x) \underset{x\to\infty}{\sim} 2^{-1/6}\sqrt{\pi}\mathrm{Ai}(x-\tilde{r})$$

$\tilde{f}(\tilde{r},x)$ is related to a solution of the Lax pair associated to Painlevé XXXIV

$$\frac{\partial}{\partial \tilde{r}}\begin{pmatrix}\tilde{f}(\tilde{r},x)\\ \tilde{g}(\tilde{r},x)\end{pmatrix} = \tilde{\mathbf{A}}\begin{pmatrix}f(\tilde{r},x)\\ g(\tilde{r},x)\end{pmatrix}\ ,\ \frac{\partial}{\partial x}\begin{pmatrix}\tilde{f}(\tilde{r},x)\\ \tilde{g}(\tilde{r},x)\end{pmatrix} = \tilde{\mathbf{B}}\begin{pmatrix}\tilde{f}(\tilde{r},x)\\ \tilde{g}(\tilde{r},x)\end{pmatrix}$$

$$\tilde{\mathbf{A}} = \begin{pmatrix} -\dfrac{q'(x)}{q(x)} & 1+\dfrac{q^2(x)}{\tilde{r}} \\ -\tilde{r} - \dfrac{\int_x^{\infty} q^2(u)du}{q^2(x)} & \dfrac{q'(x)}{q(x)}\end{pmatrix}\ ,\ \tilde{\mathbf{B}} = \begin{pmatrix}\dfrac{q'(x)}{q(x)} & -1 \\ \tilde{r} & -\dfrac{q'(x)}{q(x)}\end{pmatrix}$$

$$\tilde{f}(\tilde{r},x) \underset{\tilde{r}\to\infty}{\sim} 2^{-1/6}\tilde{r}^{-1/4}\sin\left(\frac{2}{3}\tilde{r}^{3/2} - x\sqrt{\tilde{r}} + \frac{\pi}{4}\right) + \mathcal{O}(\tilde{r}^{-3/4})\ ,$$

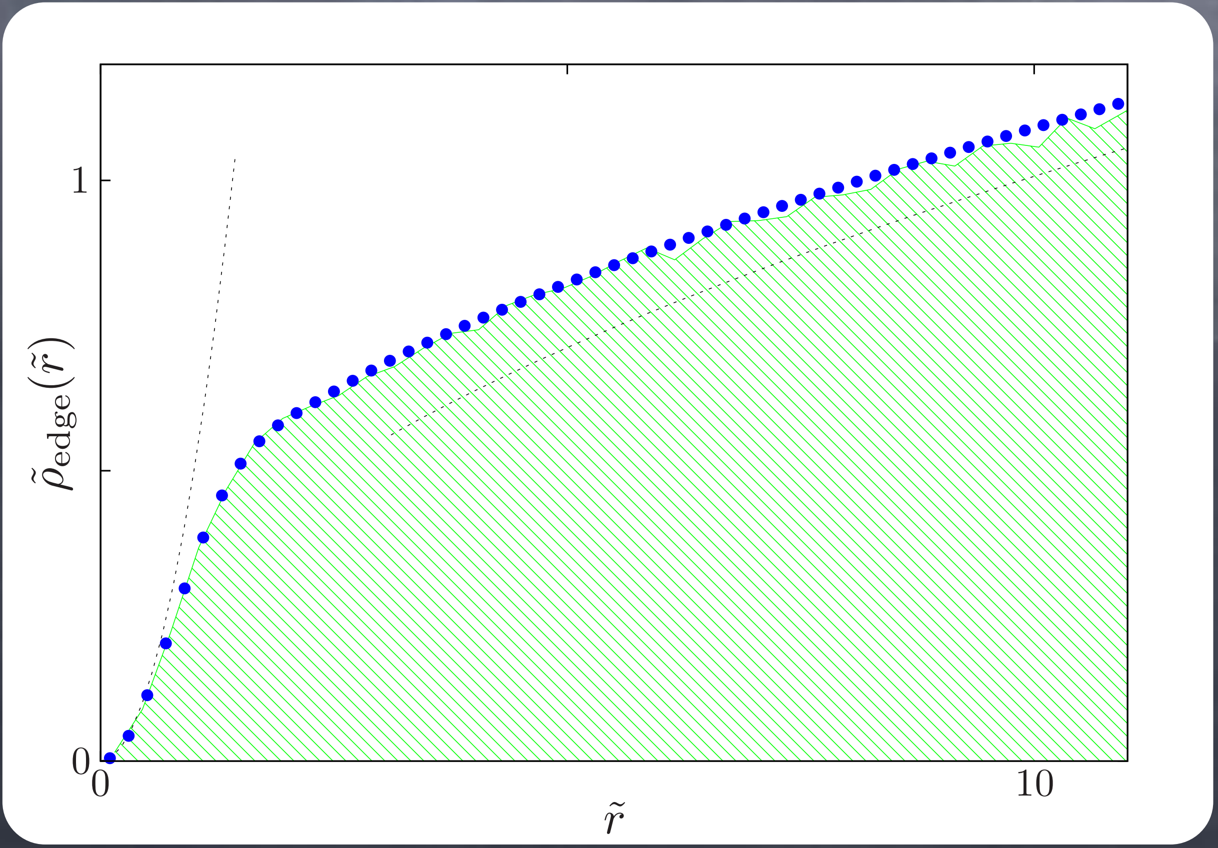
$$\tilde{g}(\tilde{r},x) \underset{\tilde{r}\to\infty}{\sim} 2^{-1/6}\tilde{r}^{1/4}\cos\left(\frac{2}{3}\tilde{r}^{3/2} - x\sqrt{\tilde{r}} + \frac{\pi}{4}\right) + \mathcal{O}(\tilde{r}^{-1/4})$$

Density of near extreme eigenvalues: results

$$\boxed{\tilde{\rho}_{\rm edge}(\tilde{r}) = \frac{2^{1/3}}{\pi} \int_{-\infty}^{\infty} \left(\tilde{f}^2(\tilde{r},x) - \left(\int_x^{\infty} q(u) \tilde{f}(\tilde{r},u) du \right)^2 \right) \mathcal{F}_2(x)\, dx}$$

- Asymptotic behaviors

$$\tilde{\rho}_{\rm edge}(\tilde{r}) \sim \begin{cases} \frac{1}{2}\tilde{r}^2 + a_4 \tilde{r}^4 + \mathcal{O}(\tilde{r}^6) & , \; \tilde{r} \to 0 \\ \\ \dfrac{\sqrt{\tilde{r}}}{\pi} & , \; \tilde{r} \to \infty \end{cases}$$



Outline

- Exact formulas for $\rho_{\mathrm{DOS}}(r,N)$ & $\rho_{\mathrm{GAP}}(r,N)$ for finite N
- Asymptotic analysis for large N
- Comparison with existing results
- Conclusion and related open problems

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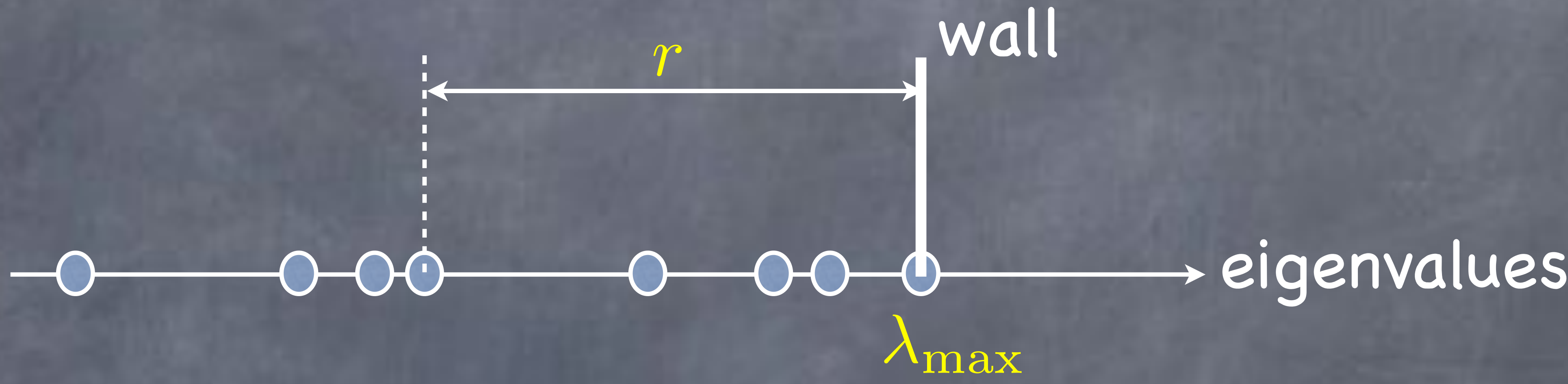
An exact formula for $\rho_{\text{DOS}}(r, N)$

$$\rho_{\text{DOS}}(r,N) = \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max} - \lambda_i - r)}$$



$$\rho_{\text{DOS}}(r,N) = N \int_{-\infty}^{\infty} \mathrm{d}y \underbrace{\int_{-\infty}^{y} \mathrm{d}\lambda_1 \int_{-\infty}^{y} \mathrm{d}\lambda_2 \cdots \int_{-\infty}^{y} \mathrm{d}\lambda_{N-2}\, P_{\text{joint}}(\lambda_1, \lambda_2, ..., \lambda_{N-2}, y-r, y)}_{\text{two-point correlations for conditioned eigenvalues}}$$

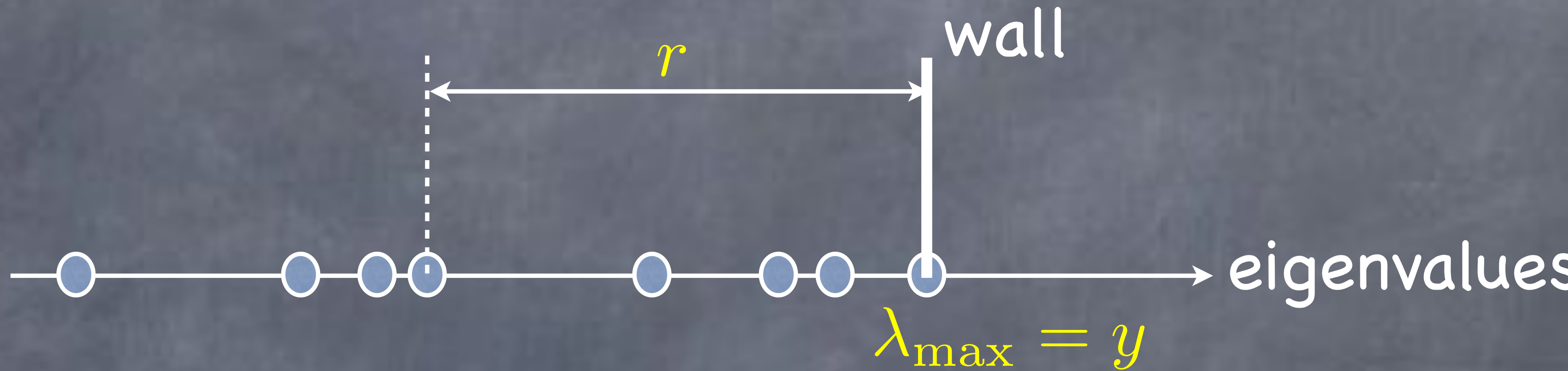
- After some manipulations one obtains

$$\rho_{\text{DOS}}(r,N) = \frac{N(N-2)!}{Z_N} \int_{-\infty}^{\infty} \mathrm{d}y \prod_{k=0}^{N-1} h_k(y) \begin{vmatrix} K_N(y-r, y-r) & K_N(y-r, y) \\ K_N(y, y-r) & K_N(y,y) \end{vmatrix}$$

wih the kernel $$K_N(\lambda, \lambda') = \sum_{k=0}^{N-1} \frac{1}{h_k(y)} \pi_k(\lambda, y) \pi_k(\lambda', y) e^{-\frac{\lambda^2 + \lambda'^2}{2}}$$

An exact formula for $\rho_{\mathrm{DOS}}(r,N)$

$$\rho_{\mathrm{DOS}}(r,N) = \frac{1}{N-1}\sum_{\substack{i=1\\ i\neq i_{\max}}}^{N} \overline{\delta(\lambda_{\max}-\lambda_i - r)}$$



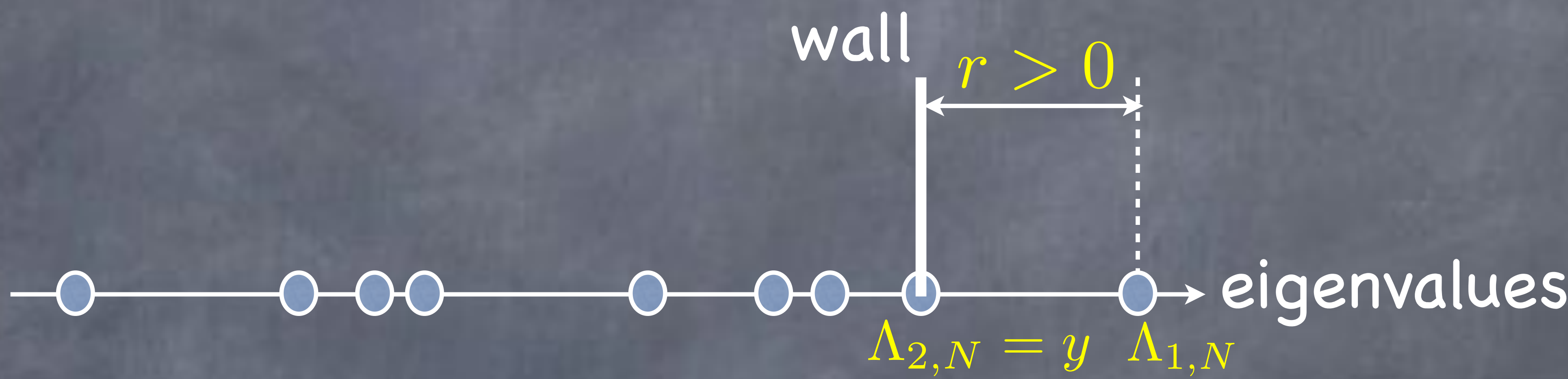
$$\rho_{\mathrm{DOS}}(r,N) = \frac{N(N-2)!}{Z_N}\int_{-\infty}^{\infty} \mathrm{d}y\, \underbrace{\prod_{k=0}^{N-1} h_k(y)}_{\propto F_N(y) = \mathbb{P}(\lambda_{\max}\le y)} \underbrace{\begin{vmatrix} K_N(y-r,y-r) & K_N(y-r,y) \\ K_N(y,y-r) & K_N(y,y) \end{vmatrix}}_{\text{2-point correlations}}$$

- Finally a useful formula is

$$\rho_{\mathrm{DOS}}(r,N) = \frac{1}{N-1}\int_{-\infty}^{\infty} dy\,\left[F_N'(y) K_N(y-r,y-r) - F_N(y) K_N^2(y,y-r)\right]$$

An exact formula for the PDF of the first gap

$$\rho_{\text{GAP}}(r,N)dr = \mathbb{P}[(\Lambda_{1,N} - \Lambda_{2,N}) \in [r, r+dr]]$$



$$\rho_{\text{GAP}}(r,N) = N(N-1)\int_{-\infty}^{+\infty} dy \int_{-\infty}^{y} d\lambda_1 \int_{-\infty}^{y} d\lambda_2 \cdots \int_{-\infty}^{y} d\lambda_{N-2} P_{\text{joint}}(\lambda_1, \cdots, \lambda_{N-2}, y, y+r)$$

Reminding that

$$\rho_{\text{DOS}}(r,N) = N \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{y} \mathrm{d}\lambda_1 \int_{-\infty}^{y} \mathrm{d}\lambda_2 \cdots \int_{-\infty}^{y} \mathrm{d}\lambda_{N-2}\, P_{\text{joint}}(\lambda_1, \lambda_2, ..., \lambda_{N-2}, y-r, y)$$

one gets

$$\boxed{\rho_{\text{GAP}}(r,N) = (N-1)\rho_{\text{DOS}}(-r,N)}$$

Outline

- Exact formulas for $\rho_{\mathrm{DOS}}(r,N)$ & $\rho_{\mathrm{GAP}}(r,N)$ for finite N
- Orthogonal polynomials (OPs) on the semi-infinite real line
- Comparison with existing results
- Conclusion and related open problems

Orthogonal polynomials

$$P_{\text{joint}}(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \, e^{-\sum_{i=1}^N \lambda_i^2}$$

$$\lambda_{\max} = \max_{1 \le i \le N} \lambda_i$$

- Cumulative distribution function of $\lambda_{\max}$

$$F_N(y) = \mathrm{Pr.}(\max_{1 \le i \le N} \lambda_i \le y)$$

$$F_N(y) = \frac{1}{Z_N} \int_{-\infty}^{y} d\lambda_1 \cdots \int_{-\infty}^{y} d\lambda_N \prod_{i<j} (\lambda_i - \lambda_j)^2 \, e^{-\sum_{i=1}^N \lambda_i^2}$$

- Orthogonal polynomials on the semi-infinite real line

$$\langle \pi_k, \pi_{k'} \rangle = \int_{-\infty}^{y} \mathrm{d}\lambda \, \pi_k(\lambda, y) \pi_{k'}(\lambda, y) e^{-\lambda^2} = \delta_{k,k'} h_k(y)$$

$$\pi_k(\lambda, y) = \lambda^k + ...$$

$$F_N(y) = \frac{N!}{Z_N} \prod_{j=0}^{N-1} h_j(y)$$

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Physical picture associated to the OP sytem

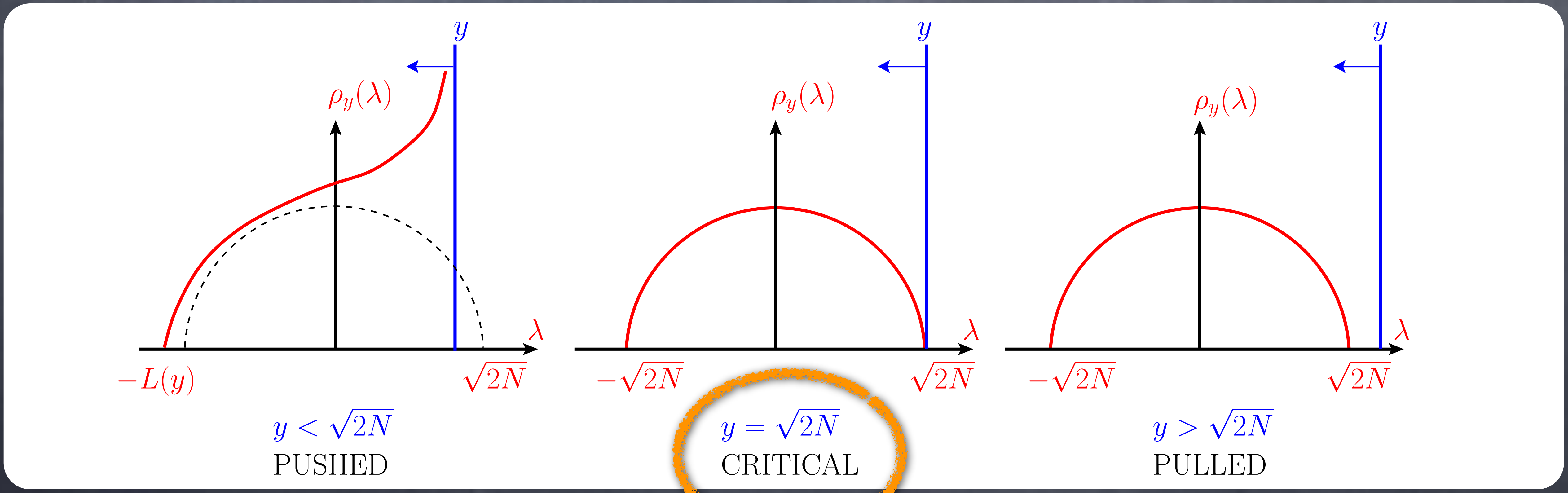
$$F_N(y) = \mathrm{Pr.}(\max_{1\le i\le N} \lambda_i \le y)$$

$$F_N(y) = \frac{1}{Z_N}\int_{-\infty}^{y} d\lambda_1 \cdots \int_{-\infty}^{y} d\lambda_N \prod_{i<j}(\lambda_i-\lambda_j)^2\, e^{-\sum_{i=1}^N \lambda_i^2}$$

- Coulomb gas with a wall located in y

for a review see S. N. Majumdar, G. S. '13

- Moving the wall through the edge: the density $\rho_y(\lambda)$



described by a double scaling limit

see also T. Claeys, A. Kuijlaars '08, «... *when the soft edge meets the hard edge*»

Large N analysis of $\rho_{\rm DOS}(N,r)$

$$\rho_{\rm DOS}(r,N) = \frac{1}{N-1}\sum_{\substack{i=1 \\ i\neq i_{\rm max}}}^{N} \langle \delta(\lambda_{\rm max} - \lambda_i - r)\rangle$$

- Edge regime : $r = \mathcal{O}(N^{-\frac{1}{6}})$

$$\rho_{\rm DOS}(r,N) = \frac{1}{N-1}\int_{-\infty}^{\infty} dy \left[F'_N(y) K_N(y-r, y-r) - F_N(y) K_N^2(y, y-r)\right]$$

- Analysis of the kernel in the double scaling limit

$$K_N(\lambda,\lambda') = \sum_{k=0}^{N-1} \psi_k(\lambda,y)\psi_k(\lambda',y) \quad \text{where} \quad \psi_k(\lambda,y) = \frac{1}{\sqrt{h_k(y)}}\pi_k(\lambda,y) e^{-\frac{\lambda^2}{2}}$$

Find a solution of the recurrence in the double scaling limit

$$\lambda\,\psi_N(\lambda,y) = \sqrt{R_{N+1}(y)}\psi_{N+1}(\lambda,y) + S_N(y)\psi_N(\lambda,y) + \sqrt{R_N(y)}\psi_{N-1}(\lambda,y)$$

Large N analysis of $\rho_{\rm DOS}(N,r)$

$$\rho_{\rm DOS}(r,N) = \frac{1}{N-1}\sum_{\substack{i=1\\ i\neq i_{\max}}}^{N} \langle \delta(\lambda_{\max} - \lambda_i - r)\rangle$$

$$\lambda\,\psi_N(\lambda,y) = \sqrt{R_{N+1}(y)}\psi_{N+1}(\lambda,y) + S_N(y)\psi_N(\lambda,y) + \sqrt{R_N(y)}\psi_{N-1}(\lambda,y)$$

- In the double scaling limit, one has C. Nadal, S. N. Majumdar '13

$$\begin{cases} R_N(y) &= \dfrac{N}{2}\left(1 - N^{-\frac{2}{3}} q^2(x) + \mathcal{O}(N^{-1})\right) \\ S_N(y) &= -\dfrac{N^{-\frac{1}{6}}}{\sqrt{2}} q^2(x) + \mathcal{O}(N^{-\frac{1}{2}}) \end{cases} \quad \text{with} \quad \begin{cases} q'' = 2q^3 + x\,q\,, \\ q(x) \sim \mathrm{Ai}(x) \text{ for } x \to \infty\,, \end{cases}$$

$$x = \sqrt{2}N^{\frac{1}{6}}\left(y - \sqrt{2N}\right)$$

- In the double scaling limit the recurrence relation is solved by

$$\boxed{\begin{gathered}\psi_N(y-r,y) = \frac{2^{1/4}}{\sqrt{\pi}} N^{-\frac{1}{12}} G\left(\sqrt{2}N^{1/6}r, \sqrt{2}N^{1/6}(y-\sqrt{2N})\right) + \mathcal{O}(N^{-\frac{5}{12}}) \\ -\partial_x^2 G(\tilde r, x) + [x + 2q^2(x)]G(\tilde r, x) = \tilde r\, G(\tilde r, x)\end{gathered}}$$

A. Perret, G. S. '14

Large N analysis of $\rho_{\rm DOS}(N,r)$

- After some more computations...

$$K_N\left(y-\frac{\tilde r}{\sqrt2}N^{-1/6},y-\frac{\tilde r'}{\sqrt2}N^{-1/6}\right)\underset{N\to\infty}{\sim}N^{1/6}2^{5/6}\frac{\tilde f(\tilde r,x)\tilde g(\tilde r',x)-\tilde f(\tilde r',x)\tilde g(\tilde r,x)}{\pi(\tilde r-\tilde r')}$$

where $\tilde f(\tilde r,x)$, $\tilde g(\tilde r,x)$ solve the Lax pair for Painlevé XXXIV

$$\frac{\partial}{\partial\tilde r}\begin{pmatrix}\tilde f(\tilde r,x)\\ \tilde g(\tilde r,x)\end{pmatrix}=\tilde{\mathbf A}\begin{pmatrix}f(\tilde r,x)\\ g(\tilde r,x)\end{pmatrix}\,,\;\frac{\partial}{\partial x}\begin{pmatrix}\tilde f(\tilde r,x)\\ \tilde g(\tilde r,x)\end{pmatrix}=\tilde{\mathbf B}\begin{pmatrix}\tilde f(\tilde r,x)\\ \tilde g(\tilde r,x)\end{pmatrix}$$

$$\tilde{\mathbf A}=\begin{pmatrix}-\dfrac{q'(x)}{q(x)} & 1+\dfrac{q^2(x)}{\tilde r}\\ -\tilde r-\dfrac{\int_x^\infty q^2(u)du}{q^2(x)} & \dfrac{q'(x)}{q(x)}\end{pmatrix}\,,\;\tilde{\mathbf B}=\begin{pmatrix}\dfrac{q'(x)}{q(x)} & -1\\ \tilde r & -\dfrac{q'(x)}{q(x)}\end{pmatrix}$$

$$\tilde f(\tilde r,x)\underset{\tilde r\to\infty}{\sim}2^{-1/6}\tilde r^{-1/4}\sin\left(\frac23\tilde r^{3/2}-x\sqrt{\tilde r}+\frac\pi4\right)+\mathcal O(\tilde r^{-3/4})\,,$$

$$\tilde g(\tilde r,x)\underset{\tilde r\to\infty}{\sim}2^{-1/6}\tilde r^{1/4}\cos\left(\frac23\tilde r^{3/2}-x\sqrt{\tilde r}+\frac\pi4\right)+\mathcal O(\tilde r^{-1/4})$$

see T. Claeys, A. Kuijlaars '07 for a (rigorous) derivation using RH

Density of near extreme eigenvalues: results

$$\boxed{\tilde{\rho}_{\mathrm{edge}}(\tilde{r}) = \frac{2^{1/3}}{\pi} \int_{-\infty}^{\infty} \left(\tilde{f}^2(\tilde{r},x) - \left(\int_x^{\infty} q(u) \tilde{f}(\tilde{r},u) du \right)^2 \right) \mathcal{F}_2(x)\, dx}$$

$$\partial_x^2 \tilde{f}(\tilde{r},x) - [x + 2q^2(x)] \tilde{f}(\tilde{r},x) = -\tilde{r} \tilde{f}(\tilde{r},x) \ , \ \tilde{f}(\tilde{r},x) \underset{x\to\infty}{\sim} 2^{-1/6} \sqrt{\pi} \mathrm{Ai}(x - \tilde{r})$$

$\tilde{f}(\tilde{r},x)$ is related to a solution of the Lax pair associated to Painlevé XXXIV

$$\frac{\partial}{\partial \tilde{r}} \begin{pmatrix} \tilde{f}(\tilde{r},x) \\ \tilde{g}(\tilde{r},x) \end{pmatrix} = \tilde{\mathbf{A}} \begin{pmatrix} f(\tilde{r},x) \\ g(\tilde{r},x) \end{pmatrix} , \ \frac{\partial}{\partial x} \begin{pmatrix} \tilde{f}(\tilde{r},x) \\ \tilde{g}(\tilde{r},x) \end{pmatrix} = \tilde{\mathbf{B}} \begin{pmatrix} \tilde{f}(\tilde{r},x) \\ \tilde{g}(\tilde{r},x) \end{pmatrix}$$

$$\tilde{\mathbf{A}} = \begin{pmatrix} -\dfrac{q'(x)}{q(x)} & 1 + \dfrac{q^2(x)}{\tilde{r}} \\ -\tilde{r} - \dfrac{\int_x^{\infty} q^2(u) du}{q^2(x)} & \dfrac{q'(x)}{q(x)} \end{pmatrix} , \ \tilde{\mathbf{B}} = \begin{pmatrix} \dfrac{q'(x)}{q(x)} & -1 \\ \tilde{r} & -\dfrac{q'(x)}{q(x)} \end{pmatrix}$$

$$\tilde{f}(\tilde{r},x) \underset{\tilde{r}\to\infty}{\sim} 2^{-1/6} \tilde{r}^{-1/4} \sin\left(\frac{2}{3} \tilde{r}^{3/2} - x\sqrt{\tilde{r}} + \frac{\pi}{4} \right) + \mathcal{O}(\tilde{r}^{-3/4}) ,$$

$$\tilde{g}(\tilde{r},x) \underset{\tilde{r}\to\infty}{\sim} 2^{-1/6} \tilde{r}^{1/4} \cos\left(\frac{2}{3} \tilde{r}^{3/2} - x\sqrt{\tilde{r}} + \frac{\pi}{4} \right) + \mathcal{O}(\tilde{r}^{-1/4})$$

Typical fluctuations of the gap: results

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using $\rho_{\rm GAP}(r,N) = (N-1)\rho_{\rm DOS}(-r,N)$

we obtain $\rho_{\rm GAP}(r,N) = \sqrt{2}N^{1/6}\tilde{\rho}_{\rm typ}(r\sqrt{2}N^{1/6})$

$$\boxed{\tilde{\rho}_{\rm typ}(\tilde{r}) = \frac{2^{1/3}}{\pi}\int_{-\infty}^{\infty}\left(\tilde{f}^2(-\tilde{r},x) - \left(\int_x^{\infty} q(u)\tilde{f}(-\tilde{r},u)du\right)^2\right)\mathcal{F}_2(x)\,dx}$$

$$\partial_x^2\tilde{f}(-\tilde{r},x) - [x+2q^2(x)]\tilde{f}(-\tilde{r},x) = \tilde{r}\tilde{f}(-\tilde{r},x)\ ,\ \ \tilde{f}(-\tilde{r},x) \underset{x\to\infty}{\sim} 2^{-1/6}\sqrt{\pi}\mathrm{Ai}(x+\tilde{r})$$

from which we obtain the asymptotic behaviors

$$\tilde{\rho}_{\rm typ}(\tilde{r}) = \begin{cases} \frac{1}{2}\tilde{r}^2 + a_4\tilde{r}^4 + \mathcal{O}(\tilde{r}^6)\ , & \tilde{r}\to 0 \\ A\ \exp\left(-\frac{4}{3}\tilde{r}^{3/2} + \frac{8}{3}\sqrt{2}\,\tilde{r}^{3/4}\right)\tilde{r}^{-21/32}\left(1 - \frac{1405\sqrt{2}}{1536}\tilde{r}^{-3/4} + \mathcal{O}(\tilde{r}^{-3/2})\right)\ , & \tilde{r}\to +\infty \end{cases}$$

with $A = 2^{-91/48}e^{\zeta'(-1)}/\sqrt{\pi}$

and $a_4 = -0.393575...$ complicated integral involving $q(x)$

Outline

- Exact formulas for $\rho_{\mathrm{DOS}}(r,N)$ & $\rho_{\mathrm{GAP}}(r,N)$ for finite N

- Orthogonal polynomials (OPs) on the semi-infinite real line

- Comparison with existing results

- Conclusion and related open problems

Relations with existing results

- Density of near extreme eigenvalues was not studied in RMT (to my knowledge)

- Previous studies of the PDF of the first gap

  ✓ an expression in terms of a Fredholm determinant Forrester '93

  ✓ an expression in terms of Painlevé transcendents Witte, Bornemann, Forrester '13

  and a numerical computation of the formula in terms of a Fredholm determinant

PDF of the first gap: the formula of Witte, Bornemann, Forrester

$$\tilde{\rho}_{\rm typ}(\tilde{r}) = \int_{-\infty}^{\infty} \mathrm{p}^{\rm soft}_{(2)}(t, t-\tilde{r})\, dt$$

$$\mathrm{p}^{\rm soft}_{(2)}(t, t-\tilde{r}) = \frac{t^{-5/2}}{4\pi} \mathrm{p}^{\rm soft}_{(1)}(t) \exp\left(-\frac{4}{3} t^{3/2}\right) \exp\left(\int_{2^{1/3}t}^{\infty} \mathrm{d}y \left\{ (2q_{3/2} + \frac{4}{p_{3/2}})(-y) - \sqrt{2y} - \frac{5}{2y} \right\}\right)$$
$$\times (\mathrm{U}\partial_x \mathrm{V} - \mathrm{V}\partial_x \mathrm{U})(-2^{1/3}\tilde{r}; -2^{1/3}t)$$

with

$$\mathrm{p}^{\rm soft}_{(1)}(t) = \mathrm{K}^{\rm soft}(t,t) \exp\left(-\int_s^{\infty} \mathrm{d}t \left(\sigma_{\rm II}(t) - \frac{\mathrm{d}}{\mathrm{d}t} \log \mathrm{K}^{\rm soft}(t,t)\right)\right)$$ **and** $$\mathrm{K}^{\rm soft}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}$$

$$\sigma_{\rm II}(t) = -2^{1/3} H(-2^3 t)$$ **with** $$H = -\frac{1}{2}\left(2q_\alpha^2 - p_\alpha + t\right) p_\alpha - 2q_\alpha\,, \alpha = 3/2$$

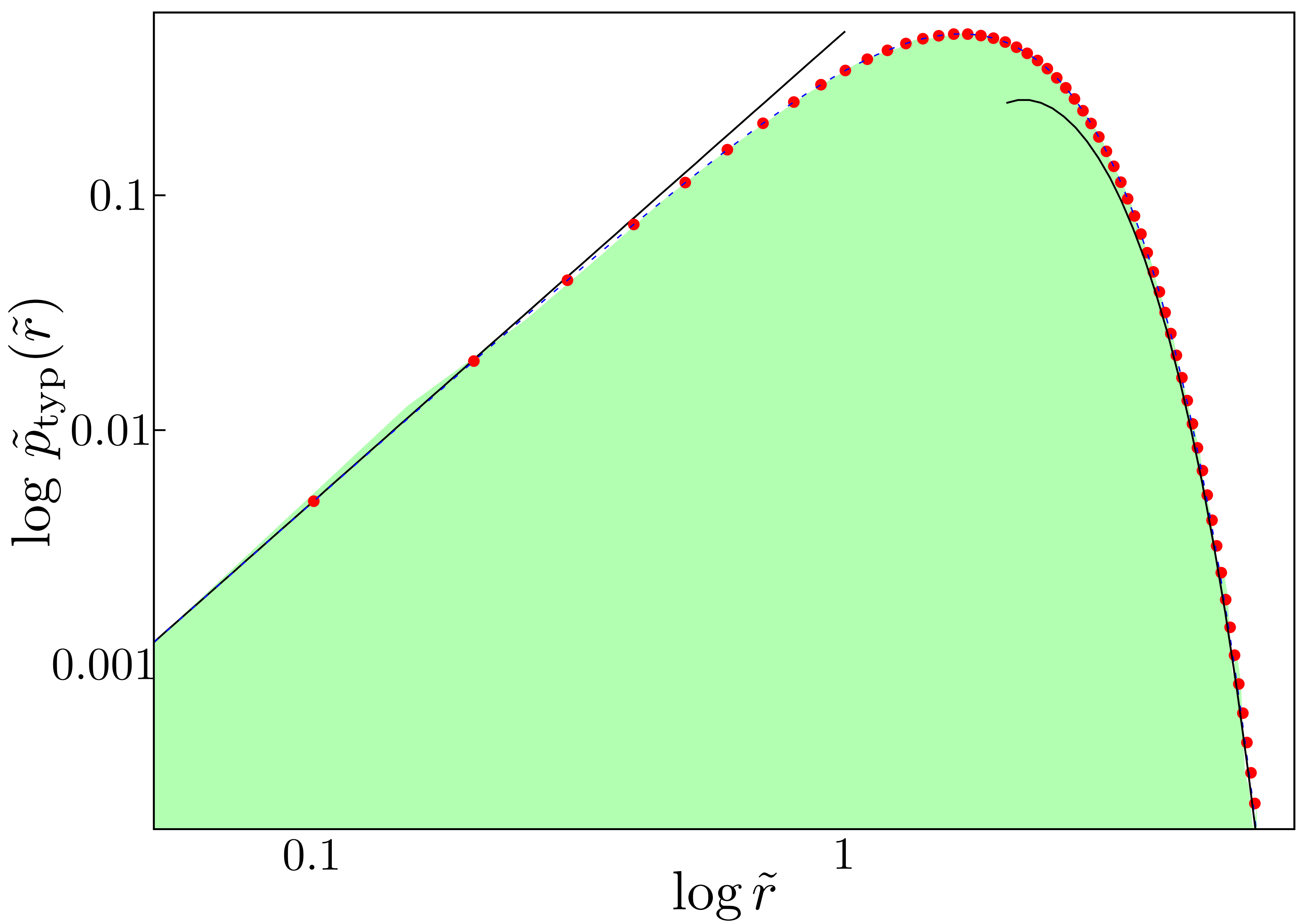
$$\partial_t q_\alpha = \partial_{p_\alpha} H = p_\alpha - q_\alpha^2 - \frac{1}{2} t\,,\ \partial_t p_\alpha = \partial_{q_\alpha} H = 2 q_\alpha p_\alpha + 2$$

$$\partial_x \begin{pmatrix} \mathrm{U} \\ \mathrm{V} \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} -q_{3/2} - \frac{2}{p_{3/2}} & -1 \\ \frac{1}{2}(t - p_{3/2}) + [q_{3/2} + \frac{2}{p_{3/2}}]^2 & q_{3/2} + \frac{2}{p_{3/2}} \end{pmatrix} + \begin{pmatrix} 1 & p_{3/2} \\ 0 & -1 \end{pmatrix} \frac{1}{x} \right\} \begin{pmatrix} \mathrm{U} \\ \mathrm{V} \end{pmatrix},$$

$$\partial_t \begin{pmatrix} \mathrm{U} \\ \mathrm{V} \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 0 & -2[q_{3/2} + \frac{2}{p_{3/2}}] \end{pmatrix} \right\} \begin{pmatrix} \mathrm{U} \\ \mathrm{V} \end{pmatrix}$$

Showing that this formula coincides with ours is still challenging...

PDF of the first gap: the numerical evaluation of Witte, Bornemann, Forrester



Conclusion and related open questions

- Applications to the relaxational dynamics of mean-field spin glass

- Exact results for the statistics of near extreme eigenvalues of GUE

- A new formula for the PDF of the first gap in terms of Painlevé transcendents (precise asymptotics)

$$\tilde{\rho}_{\rm typ}(\tilde{r}) = \begin{cases} \frac{1}{2}\tilde{r}^2 + a_4 \tilde{r}^4 + \mathcal{O}(\tilde{r}^6) \;, & \tilde{r} \to 0 \\ A \;\exp\left(-\frac{4}{3}\tilde{r}^{3/2} + \frac{8}{3}\sqrt{2}\,\tilde{r}^{3/4}\right)\tilde{r}^{-21/32}\left(1 - \frac{1405\sqrt{2}}{1536}\tilde{r}^{-3/4} + \mathcal{O}(\tilde{r}^{-3/2})\right) \;, & \tilde{r} \to +\infty \end{cases}$$

- What about GOE and GSE (skew orthogonal polynomials) ?

- What about Laguerre-Wishart matrices ?

ÉCOLE DE PHYSIQUE
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Session CIV

Stochastic processes and random matrices

July 6-31, 2015

http://lptms.u-psud.fr/workshop/randmat/

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Herbert Spohn (Universität München)	The Kardar-Parisi-Zhang equation – a statistical physics perspective
Bálint Virág (University of Toronto)	β-ensembles and random Schrödinger operators

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