A GENERALIZATION OF THE CSISZÁR-KULLBACK INEQUALITY

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PROPOSITION 0.1. [2] *Let f and g be two nonnegative functions in $L^1 \cap L^p(\Omega)$, $p \in [1,2]$ and consider a nonnegative strictly convex function $\psi \in C^2(0,+\infty)$ such that $\psi(1) = \psi'(1) = 0$. If $A := \inf_{s\in(0,\infty)} s^{2-p}\psi''(s) > 0$, then*

$$\begin{aligned} e_\psi[\,f|g\,] &:= \int_\Omega (\psi(f) - \psi(g) - \psi'(g)(f-g))\,dx \\ &\geq \frac{A}{2^{2/p}}\,\min\left(\|f\|^{p-2}_{L^p(\Omega)}, \|g\|^{p-2}_{L^p(\Omega)}\right)\|f-g\|^2_{L^p(\Omega)}\,. \end{aligned} \tag{0.1}$$

Proof. The case $p = 1$ is the well known Csiszár-Kullback inequality (see for instance [1]). Assume first that $f > 0$. By a Taylor expansion at order two, we get

$$e_\psi[\,f|g\,] = \frac{1}{2}\int_\Omega \psi''(\xi)|f-g|^2\,dx \geq \frac{A}{2}\int_\Omega \xi^{p-2}|f-g|^2\,dx \tag{0.2}$$

where ξ lies between f and g. By Hölder's inequality, for any $h > 0$ and for any measurable set $\mathcal{A} \subset \Omega$, we get

$$\int_{\mathcal{A}} |f-g|^p\,h^{-\alpha}\,h^{\alpha}\,dx \leq \left(\int_{\mathcal{A}} |f-g|^2\,h^{p-2}\,dx\right)^{p/2}\left(\int_{\mathcal{A}} h^{\alpha s}\,dx\right)^{1/s}$$

with $\alpha = p\,(2-p)/2$, $s = 2/(2-p)$. Thus

$$\left(\int_{\mathcal{A}} |f-g|^2\,h^{p-2}\,dx\right)^{p/2} \geq \left(\int_{\mathcal{A}} |f-g|^p\,dx\right)\left(\int_{\mathcal{A}} h^p\,dx\right)^{(p-2)/2}.$$

We apply this formula to two different sets.

i) On $\mathcal{A} = \mathcal{A}_1 = \{x \in \Omega : f(x) > g(x)\}$, use $\xi^{p-2} > f^{p-2}$ and take $h = f$:

$$\left(\int_{\mathcal{A}_1} |f-g|^2\xi^{p-2}\,dx\right)^{p/2} \geq \left(\int_{\mathcal{A}_1} |f-g|^p\,dx\right)\|f\|^{-(2-p)\,p/2}_{L^p(\Omega)}\,.$$

ii) On $\mathcal{A} = \mathcal{A}_2 = \{x \in \Omega : f(x) \leq g(x)\}$, use $\xi^{p-2} \geq g^{p-2}$ and take $h = g$:

$$\left(\int_{\mathcal{A}_2} |f-g|^2\xi^{p-2}\,dx\right)^{p/2} \geq \left(\int_{\mathcal{A}_2} |f-g|^p\,dx\right)\|g\|^{-(2-p)\,p/2}_{L^p(\Omega)}\,.$$

To prove (0.1) in the case $f > 0$, we just add the two previous inequalities in (0.2) and use the inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ for any $a, b \geq 0$ and $r \geq 1$. To handle the case $f \geq 0$, we proceed by a density argument: apply (0.1) to $f_\epsilon(x) = f(x) + \epsilon\,e^{-|x|^2-|v|^2}$ and let $\epsilon \to 0$ using Lebesgue's convergence theorem. ∎

REFERENCES

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