THE KINETIC FOKKER-PLANCK EQUATION IN A DOMAIN

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Abstract. We first briefly recall the De Giorgi-Nash-Moser theory for parabolic equation. We next consider the Kinetic Fokker-Planck (FKP) equation in the whole space and in a domain for which we establish some ultracontractivity estimates. We present some applications to the asymptotic convergence issue and the well-posedness issue. These notes are based on some works in collaboration with K. Carrapatoso, C. Fonte, P Gabriel and R. Medina.

Contents

1. The parabolic equation

1.1. The heat equation and the smooth coefficients case.

▷ classical

1.2. Nash's proof of ultracontractivity.

▷ Nash 1958 [32]

1.3. A variant of Nash's proof.

▷ Carrapatoso-M. arxiv 2024 [9] or before ?

1.4. Other methods.

- De Giorgi iterative method ▷ De Giorgi 1957 [10]
- Moser iterative method and concave method ▷ Moser 1960, 1964 [30, 31]
- Sobolev regularity iterative method ▷ Boccardo, Gallouet 1989 [4]
- kernel method ▷ Guérand, Mouhot 2022 [18] or before ?

1.5. Complements.

• weigthed versions ▷ Fabes, Stroock 1986 [12], Gualdani, M., Mouhot 2017 [17], M., Mouhot 2016 [29], Kavian, M., Ndao 2021 [27]

1.6. Application : fundamental solution.

- mostly classical ▷ see for instance Stampacchia 1965 [34]

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2. The kinetic Fokker-Planck equation in $\mathbb{R}^d$ and in $\mathbb{T}^d$

2.1. smooth coefficients.

- Kolmogorov kernel ▷ Kolmogorov 1934 [28]
- Fourier and commutator techniques ▷ Hormander 1967 [26], Bouchut 2002 [5], Hérau, Pravda-Starov 2011 [23]
- Energy technique ▷ Hérau 2007 [22], Villani 2009 [35]

2.2. De Giorgi-Nash-Moser techniques.

▷ Pascucci, Polidoro 2004 [33], Golse, Imbert, Mouhot, Vasseur 2019 [16], Guérand, Mouhot 2022 [18]

2.3. complements.

▷ local Holder continuity estimate in Golse, Imbert, Mouhot, Vasseur 2019 [16], global in Anceschi, Eleuteri, Polidoro 2019 [1], see also Villani 2009 [35, A.22]

2.4. Hypocoercivity.

- Poincaré inequality ▷ Bakry, Barthe, Cattiaux, Guillin 2008 [2] for instance
- twisted H^1 method ▷ Nier, Hérau 2004 [25], Helffer, Nier 2005 [21], Villani 2009 [35]
- twisted L^2 method ▷ Hérau 2006 [24], Dolbeault, Mouhot, Schmeiser 2015 [11]

2.5. Extension.

- converegence in L^p_ω space ▷ Gallay, Wayne 2002 [15] and more recently [17, 29, 27]

3. The kinteic Fokker-Planck equation in a domain

3.1. Ultracontractivity.

▷ Carrapatoso, Fonte, Gabriel, Medina, M. [13], [9], [8], [14], [7]

3.2. Hypocoercivity.

▷ Guo [19], Bernou, Carrapatoso, M., Tristani [3]

▷ mostly Carrapatoso, M. arxiv 2024 [9], Carrapatoso , M. in preparation [8]

3.3. Doblin-Harris.

▷ Hairer, Mattingly 2011 [20], Cañizo, M. 2023 [6], Fonte, Gabriel, M. arxiv 2023 [13], Carrapatoso, Gabriel, Medina, M. in preparation [7], Fonte, M. in preparation [14]

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