Shilnikov bifurcations in the Hopf-zero singularity

I. Baldomá, O. Castejón, S. Ibáñez, T. M-Seara.

The so-called Hopf-zero singularity consists in a vector field in $\mathbf{R}^3$ having the origin as a critical point, with a zero eigenvalue and a pair of conjugate purely imaginary eigenvalues. Depending of the sign in the second order Taylor coefficients of the singularity, the dynamics of its unfoldings is not completely understood. If one considers conservative (i.e. one-parameter) unfoldings of such singularity, one can see that the truncation of the normal form at any order possesses two saddle-focus critical points with a one- and a two-dimensional heteroclinic connection. The same happens for non-conservative (i.e. two-parameter) unfoldings when the parameters lie in a certain curve (see for instance [1]).

However, when one considers the whole vector field, one expects these heteroclinic connections to be destroyed. This fact can lead to the birth of a homoclinic connection to one of the critical points, producing thus a Shilnikov bifurcation. For the case of $\mathcal{C}^\infty$ unfoldings, this has been proved before (see [2]), but for analytic unfoldings it is still an open problem.

Our study concerns the splittings of the one and two-dimensional heteroclinic connections (see [3] for the one-dimensional case). Of course, these cannot be detected in the truncation of the normal form at any order, and hence they are expected to be exponentially small with respect to one of the perturbation parameters. In [4] it has been seen that a complete understanding of how the heteroclinic connections are broken is the last step to prove the existence of Shilnikov bifurcations for analytic unfoldings of the Hopf-zero singularity. Our results and [4] give the existence of Shilnikov bifurcations for analytic unfoldings.

References

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