Shilnikov bifurcations in the Hopf-zero singularity

Geometry and Dynamics in interaction

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The Hopf-zero singularity

The Hopf-zero (or central) singularity consists on a vector field $X^* : \mathbb{R}^3 \to \mathbb{R}^3$ such that:

- $X^*(0,0,0) = 0$
- $DX^*(0,0,0)$ has eigenvalues $\pm\alpha^*$, 0.
- After a linear change of variables one can assume that:

$$DX^*(0,0,0) = \begin{pmatrix} 0 & -\alpha^* & 0 \\ \alpha^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(0,0,0)$ is a bifurcation point for X^* or that X^* is a singularity.

The Hopf-zero singularity: unfoldings

We want to study the qualitative behavior of the unfoldings of X^* near the origin $(0, 0, 0)$.

That is, vector fields X_ξ depending on parameters $\xi \in \mathbb{R}^k$ such that $X_0 = X^*$.

- **General case**: X^* has codimension two: We need to parameters: $\xi = (\mu, \nu)$
- **Conservative case**: Since $\mathrm{tr} DX^*(0, 0, 0) = 0$, it has sense to consider conservative unfoldings. In this case X^* has codimenions one and we just need one parameter: $\xi = \mu$
- We will consider the general setting, since the conservative one is just a particular case. Hence, we will study a family of vector fields $X_{\mu,\nu}$ such that $X_{0,0} = X^*$.

The normal form

We will perform changes of variables to $X_{\mu,\nu}$, to write the vector field in the simplest possible form up to some order (some degree in its Taylor expansion).
Then, one studies the effects of the non symmetric (higher order terms) in the dynamics.

J. Guckenheimer On a codimension two bifurcation, *Dynamical Systems and Turbulence*, Warwick 1980.

The second order normal form

Under generic conditions on the singularity X^* and for generic unfoldings, after some scaling of variables we obtain: $X_{\mu,\nu} = X^2_{\mu,\nu} + F^2_{\mu,\nu}$

$$\begin{aligned} \frac{d\bar{x}}{d\bar{t}} &= \bar{x}\,(\nu - \beta_1 \bar{z}) + \bar{y}\alpha^* + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \frac{d\bar{y}}{d\bar{t}} &= -\bar{x}\alpha^* + \bar{y}\,(\nu - \beta_1 \bar{z}) + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \frac{d\bar{z}}{d\bar{t}} &= -\mu + \bar{z}^2 + \gamma_2(\bar{x}^2 + \bar{y}^2) + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \end{aligned}$$

with $\beta_1 \neq 0$, $\gamma_2 \neq 0$ and $F^2_{\mu,\nu} = \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu)$.
As:

$$\mathrm{tr} DX_{\mu,\nu}(\bar{x}, \bar{y}, \bar{z}) = 2(\nu - \beta_1 \bar{z}) + 2\bar{z}.$$

Conservative case: $\nu = 0$ and $\beta_1 = 1$.
There are six topological types of singularities of codimension two depending of the choice of the parameters $\beta_1 \neq 0$ and $\gamma_2 \neq 0$.
The only one that it is not completely understood is the case:

$$\beta_1 > 0, \quad \gamma_2 > 0.$$

(Guckenheimer, Gavrilov, Holmes, Takens, Dumortier, Kutnetsov)

The second order normal form $\beta_1 > 0, \quad \gamma_2 > 0.$

The second order normal form:

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \bar{x}\,(\nu - \beta_1 \bar{z}) + \bar{y}\alpha^* \\
\frac{d\bar{y}}{d\bar{t}} &= -\bar{x}\alpha^* + \bar{y}\,(\nu - \beta_1 \bar{z}) \\
\frac{d\bar{z}}{d\bar{t}} &= -\mu + \bar{z}^2 + \gamma_2(\bar{x}^2 + \bar{y}^2)
\end{aligned}$$

Use cilindrical coordinates :

$$\bar{x} = r\cos\theta, \;\; \bar{y} = r\sin\theta, \;\; \bar{z} = z$$

to get:

$$\begin{aligned}
\frac{dr}{d\bar{t}} &= r\,(\nu - \beta_1 z) \\
\frac{d\theta}{d\bar{t}} &= \alpha^* \\
\frac{dz}{d\bar{t}} &= -\mu + z^2 + \gamma_2 r^2
\end{aligned}$$

bifurcation diagram of the second order normal form: $\beta_1 > 0, \quad \gamma_2 > 0.$

- If $\mu < 0$ the system has no equilibrium points and the dynamics is known.
- At $\mu = 0$ the system has an equilibrium at the orgin which bifurcates,
- for $\mu > 0$, the system has two equilibrium points $\bar{S}^2_\pm = (0, 0, \pm\sqrt{\mu})$.
- For $\mu > 0$, the linearization $DX^2_{\mu,\nu}(0, 0, \pm\sqrt{\mu})$ has eigenvalues:

$$\lambda_1^\pm = \nu \mp \beta_1\sqrt{\mu} + \mathrm{i}\alpha^*, \ \lambda_2^\pm = \overline{\lambda_1^\pm}, \ \lambda_3^\pm = \pm 2\sqrt{\mu}.$$

  Therefore:
  - $\nu > \beta_1\sqrt{\mu}$, $\bar{S}^2_+$ is a repellor and $\bar{S}^2_-$ is a saddle-focus.
  - $\nu < -\beta_1\sqrt{\mu}$, $\bar{S}^2_+$ is a saddle-focus and $\bar{S}^2_-$ is an attractor.
  - $-\beta_1\sqrt{\mu} < \nu < \beta_1\sqrt{\mu}$, $\bar{S}^2_+$ and $\bar{S}^2_-$ are saddle-focus. (conservative case: $\nu = 0$, $\beta_1 = 1$)
- The dynamics of $X^2_{\mu,\nu}$ as well as the one of $X_{\mu,\nu}$ are well known in the first two cases. Here we study the last case, not completely understood and we will take $(\mu, \nu) \in U$, being

$$U = \{(\mu, \nu) \in \mathbb{R}^2 : \mu > 0, \ |\nu| < \beta_1\sqrt{\mu}\}.$$

Dynamics of the second order normal form: $(\mu, \nu) \in U$

equations: $\frac{dr}{d\bar{t}} = r(\nu - \beta_1 z)$, $\frac{dz}{d\bar{t}} = -\mu + z^2 + \gamma_2 r^2$

- It has two saddle-focus critical points $S_\pm(\mu, \nu) = (0, 0, \pm\sqrt{\mu})$
- The segment between $S_\pm(\mu, \nu)$ on the z-axis:

$$W_1 = \{x = y = 0, \ -\sqrt{\mu} \le z \le \sqrt{\mu}\}$$

  is a heteroclinic connection between the two points.

- When $\nu = 0$, the two-dimensional stable manifold of $\bar{S}^2_+$ also coincides with the two-dimensional unstable manifold of $\bar{S}^2_-$, giving rise to a two-dimensional heteroclinic surface:

$$W_2 = \left\{ z^2 + \frac{\gamma_2}{\beta_1 + 1} r^2 = \mu \right\}.$$

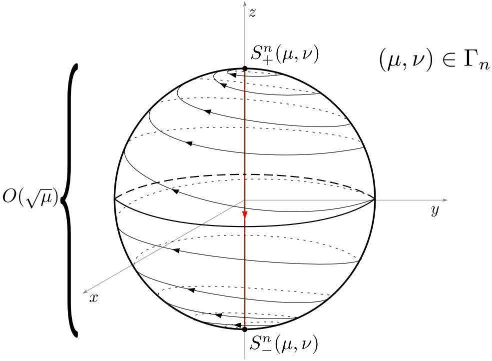
- For $\nu = 0$ the system has a first integral in the general (both conservative and non conservative) case:

$$H(r, z) = r^{\frac{2}{\beta_1}} \left(-\mu + z^2 + \frac{\gamma_2}{\beta_1 + 1} r^2 \right).$$

Dynamics of the second order normal form: $(\mu, \nu) \in U$

equations: $\frac{dr}{d\bar{t}} = r\,(\nu - \beta_1 z)$, $\frac{dz}{d\bar{t}} = -\mu + z^2 + \gamma_2 r^2$

- When $\nu \neq 0$, the one dimensional heteroclinic connection W_1 persists, but not the two-dimensional heteroclinic surface.
  The intersection of these manifolds with the plane $z = 0$ are two curves C^u, C^s such that C^u is inside the interior of C^s or viceversa depending on the sign of ν.
- In the conservative setting, the two dimensional invariant manifolds of the critical points coincide for all values of μ.



phase portrait of the second order normal form for $\nu = 0$

We expect that the terms of order three break the connections and there will be the possibility of having homoclinic orbits to one of the points!!

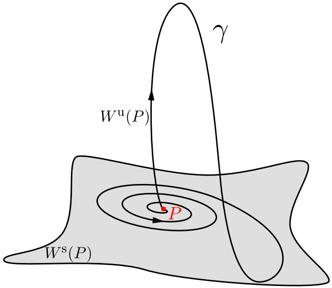
The Shilnikov Bifurcation

Consider a vector field:

$$X : \mathbb{R}^3 \to \mathbb{R}^3$$

such that $X(P) = 0$. We say that a Shilnikov Bifurcation occurs if:

- $DX(P)$ has eigenvalues $-\rho \pm i\omega$ and λ.
- $\lambda > \rho > 0$, $\omega \neq 0$.
- There exists a homoclinic orbit $\gamma \in W^s(P) \cap W^u(P)$.



Shilnikov, 1965

There are countably many periodic orbits in a neighborhood of γ.

Shilnikov, L.P. A case of the existence of a denumerable set of periodic motions, *Dokl. Akad. Nauk SSSR*, 1965.

Very complex dynamics arise from this bifurcation.

We want to study the occurrence of such bifurcation in generic unfoldings of the Hopf-zero singularity.

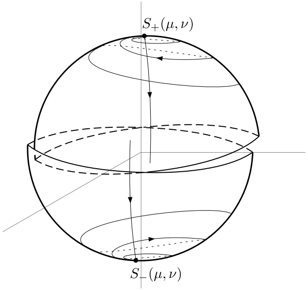
In order to proof that the homoclinc orbit exists, one has to check:

- The one-dimensional heteroclinic connection disappears
- The two-dimensional stable and unstable manifolds do not coincide

Remember:

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \bar{x}\,(\nu-\beta_1\bar{z})+\bar{y}\alpha^*+\mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu)\\ \frac{d\bar{y}}{d\bar{t}} &= -\bar{x}\alpha^*+\bar{y}\,(\nu-\beta_1\bar{z})+\mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu)\\ \frac{d\bar{z}}{d\bar{t}} &= -\mu+\bar{z}^2+\gamma_2(\bar{x}^2+\bar{y}^2)+\mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu)\end{aligned}$$

As everything happens in a neighborhood of $(0,0,0)$ of size $O(\sqrt{\mu})$, the terms of order three are at least $O(\mu^{3/2})$. One expects that these terms destroy the heteroclinic connections and create homoclinic ones!!.



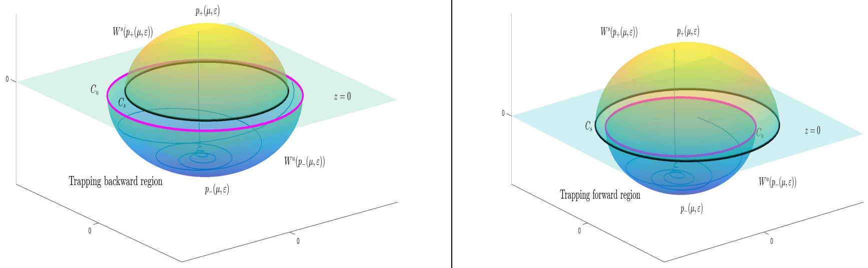
If we are able to see that this is true and give some quantitative results one can apply a result of Dumortier-Ibáñez-Kokubu-Simó

Proof of the existence of Shilnikov orbits

Idea of the proof (general case)

- Proof the the one-dimensional manifolds split
- Proof the the two-dimensional manifolds split
- Proof that the one dimensional unstable manifold of S_+ intersects again the plane $z = 0$ in a point "very close to" the unstable manifold of S_-.

Moving parameters we see that:



At some values of the parameters we can have an homoclinic orbit!

Proof of the existence of Shilnikov orbits

Idea of the proof (conservative case)

- Proof the the one-dimensional manifolds split
- Proof the the two-dimensional manifolds split
- Proof that the one dimensional unstable manifold of S_+ intersects again the plane $z = 0$ in a point “very close to” the unstable manifolds of S_-. Call $\varphi_1(\delta)$ the angle of this point.
- Call $\varphi_2(\delta)$ the angle of the intersection point between the two-dimensional manifolds in $z = 0$.

if $\varphi_1(\delta) - \varphi_2(\delta) \to \infty$ as $\delta \to 0$, we can prove the existence of homolinic orbits for a sequence $\delta_n \to 0$.

Dynamics of the normal form of ordre n when $(\mu, \nu) \in U$

- It has two saddle-focus critical points $S_{\pm}(\mu,\nu) = (0,0,z_{\pm}(\mu,\nu))$, with $z_{\pm}(\mu,\nu) = \pm\sqrt{\mu} + O((\mu^2+\nu^2)^{1/2})$.
- The segment between $S_{\pm}(\mu,\nu)$ on the z-axis is a heteroclinic connection.
- The distance between the 2-dimensional invariant manifolds measured at their intersection with the plane $z = 0$ is of the form:

$$c_1\nu + c_2\mu + \mathcal{O}\left(\nu\frac{\nu}{\sqrt{\mu}}, \mu^{3/2}, \nu\sqrt{\mu}\right), \qquad c_1 \neq 0,$$

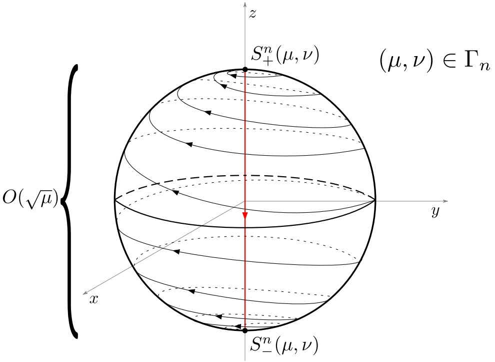
  Therefore, if ν is not of order μ, this distance can not be zero and the two dimensional manifolds of $\bar{S}^n_{\pm}$ do not intersect.

- Moreover, if the parameters (ν,μ) belong to a curve Γ_n of the form:

$$\Gamma_n = \left\{(\mu,\nu) \in U \,:\, \nu = -\frac{c_1}{c_2}\mu + \mathcal{O}(\mu^{3/2})\right\},$$

  there exists a two dimensional heteroclinic surface for *any* finite order n.

- The heteroclinic surface exists for any value of the parameter μ in the conservative case.



phase portrait of the normal form of any ordre for $(\mu, \nu) \in \Gamma_n$

The normal form of order n and the whole vector field

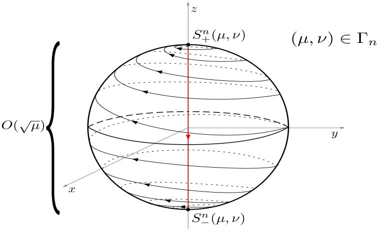
Fix $n \in \mathbb{N}$. There exists an analytic change of coordinates after which $X_{\mu,\nu} = X^n_{\mu,\nu} + F^n_{\mu,\nu}$, where:

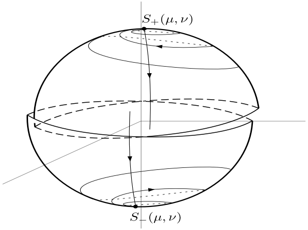
$$F^n_{\mu,\nu}(x,y,z) = O_{n+1}(x,y,z,\mu,\nu).$$

and $X^n_{\mu,\nu}$ is the **truncation of the normal form** of $X_{\mu,\nu}$ at order n.

$$F^n_{\mu,\nu}(x,y,z) = O_{n+1}(x,y,z,\mu,\nu) = O(\mu^{\frac{n+1}{2}})$$

For $\nu = O(\mu)$ (and therefore near Γ_n) this is a phenomenon beyond all orders.





Previous results:

$\mathcal{C}^\infty$ **unfoldings** Broer-Vegter Subordinate Šil'nikov bifurcations near some singularities of vector fields having low codimension, *Ergodic Theory and Dynamical Systems*, 1984.

- They show that the normal form procedure can be done up to "infinite order". After a $\mathcal{C}^\infty$ change of variables (Borel-Ritt theorem):

$$X_{\mu,\nu} = X^\infty_{\mu,\nu} + F^\infty_{\mu,\nu},$$

  where $X^\infty_{\mu,\nu}$ has an analogous form as $X^n_{\mu,\nu}$ and $F^\infty_{\mu,\nu}$ is flat.

- **Main result**: There exist flat perturbations $p_{\mu,\nu}$ such that:

$$\tilde{X}_{\mu,\nu} = X^\infty_{\mu,\nu} + p_{\mu,\nu}$$

  has Shilnikov bifurcations at $(\mu,\nu) = (\mu_n, \nu(\mu_n)) \in \Gamma$, $n \in \mathbb{N}$, with $\mu_n \to 0$.

Differences between our goals and the result by Broer and Vegter:

- The result by Broer and Vegter is an existence theorem. Our goal is to give conditions that we can "check" to see if the analytic unfoldings $X_{\mu,\nu}$ have Shilnikov bifurcations.

- Moreover, the fields for which they prove the existence of Shilnikov bifurcations are $\mathcal{C}^\infty$ fields that are **not** analytic.

Previous results: Analytic unfoldings

Dumortier-Ibáñez-Kokubu-Simó About the unfolding of a Hopf-zero singularity, *Discrete and Continuous Dynamical Systems*, 2013.

Main result: Assuming some quantitative information on the splitting of the heteroclinic manifolds, there exist infinitely many Shilnikov bifurcations at parameter points $(\mu_n, \nu(\mu_n)) \in \Gamma$, $n \in \mathbb{N}$, with $\mu_n \to 0$.

They give numerical computations of the splitting of the 1-dimensional and 2-dimensional heteroclinic connections.

Study of the splitting of the heteroclinic connections

If we deal with **analytic unfoldings**, it is sufficient to study the normal form of order two.

We define $\delta = \sqrt{\mu}$ and $\sigma = \delta^{-1}\nu$. After some rescaling, $\bar{x} = \sqrt{\mu}x$, $\bar{y} = \sqrt{\mu}y$, $\bar{z} = \sqrt{\mu}z$ and scaling also time one gets:

$$\begin{aligned}
\frac{dx}{dt} &= x(\sigma - \beta_1 z) + \frac{\alpha^*}{\delta} y + \delta^{-2} f(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\
\frac{dy}{dt} &= -\frac{\alpha^*}{\delta} x + y(\sigma - \beta_1 z) + \delta^{-2} g(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\
\frac{dz}{dt} &= -1 + \gamma_2(x^2 + y^2) + z^2 + \delta^{-2} h(\delta x, \delta y, \delta z, \delta, \delta\sigma),
\end{aligned}$$

where $f, g, h = O_3(\delta x, \delta y, \delta z, \delta, \delta\sigma)$, $\delta > 0$, $|\sigma| < d$, $b > 0$, $\alpha^* > 0$.

Now $f = g = h = 0$ gives the dynamics of the second order normal form but the "size" of the points, the connections etc of $O(1)$ and the "perturbation" is of order δ.

Write the previous system as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = X^2_{\delta,\sigma}(x,y,z) + \delta^{-2} F(x,y,z,\delta,\sigma).$$

where $F(x,y,z,\delta,\sigma) = (f,g,h)(\delta x, \delta y, \delta z, \delta, \delta\sigma)$.
We add an artificial parameter ε:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = X^2_{\delta,\sigma}(x,y,z) + \varepsilon\delta^{-2} F(x,y,z,\delta,\sigma).$$

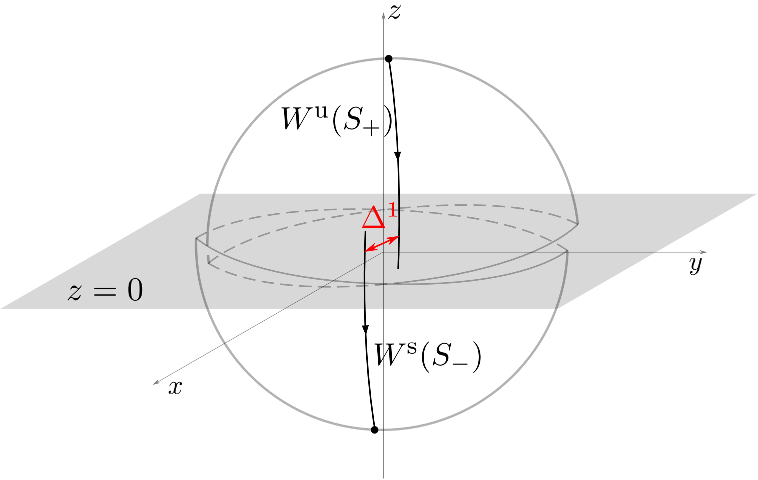
and apply perturbation theory to our system to find the perturbed manifolds and their distance.

- **Regular case:** ε small parameter ($\varepsilon = \delta^k$ for some $k > 0$ non-generic unfoldings)
- **Singular case:** $\varepsilon = 1$ (generic unfoldings)

Results: 1D manifolds

1. **Regular case** (non-generic unfoldings):
   - Baldomá-S. Breakdown of heteroclinic orbits for some analytic unfoldings of the Hopf-zero singularity, *Journal of Nonlinear Science*, 2006.

2. **Singular case** (generic unfoldings):
   - Baldomá- Castejón -S. Exponentially small heteroclinic breakdown in the generic Hopf-zero singularity, *Journal of Dynamics and Differential Equations*, 2013.


z
$W^{u}(S_+)$
Δ^1
y
$z = 0$
$W^{s}(S_-)$
x

Splitting formula in both the regular and the singular case

- theorem

  For all $0 \leq \varepsilon \leq 1$, the distance between the 1-dimensional manifolds in the plane $z = 0$ is given asymptotically by:

$$\Delta^1 = \varepsilon C(\varepsilon) \delta^{-(1+\beta_1)} e^{-\frac{\alpha^* \pi}{2\delta} + \frac{\pi}{2}(\alpha^* h_0)} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right).$$

  where

  $h_0 = -\lim_{z\to 0} z^{-3} h(0,0,z,0,0)$.

- conservative case: $\beta_1 = 1$

$$\Delta^1 = \varepsilon C(\varepsilon) \delta^{-2} e^{-\frac{\alpha^* \pi}{2\delta} + \frac{\pi}{2}(\alpha^* h_0)} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right).$$

Remark

$e^{-\frac{\alpha^* \pi}{2\delta}} = O(\delta^n)$, $\forall n$!!

The constant $C(\varepsilon)$: The regular case

- $C(0)$ is a good approximation of $C(\varepsilon)$ in the **regular case**.
- $C(0)$ is determined by a Melnikov integral:

$$\int_{-\infty}^{+\infty} \frac{e^{\frac{i\alpha^* s}{\delta}}}{\cosh s}(f+ig)(\phi_{\rm het}(s))ds$$

where $\phi_{\rm het} = (0, 0, -\tanh t)$ is the heteroclinic connection of the second order normal form

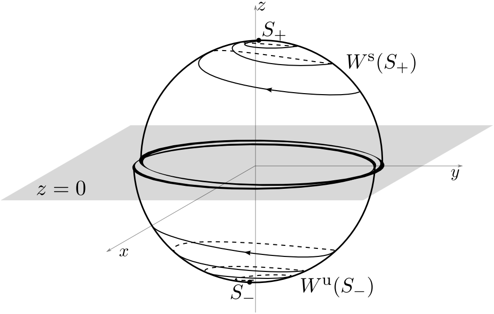
- In fact one has: $C(0) = 2\pi \hat{m}(i\alpha^*)|$, where $\hat{m}$ is the Borel transform of the function:

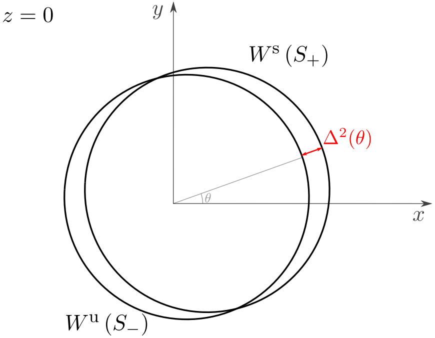
$$m(u) = [f(0,0,-u,0,0) + ig(0,0,-u,0,0)] .$$

The constant $C(1)$: The singular case

- For $\varepsilon = 1$ (**singular case**), one has that $C(1) = C^*$ is a constant determined by the so-called inner equation.
- We use this equation to obtain good approximations of the invariant manifolds and its **difference** in suitable domains near the singularities $\pm i\frac{\pi}{2}$ of the heteroclinic connection of the unperturbed system.
- The inner equation is independent of the parameters δ and σ, and it is determined by the Hopf-zero singularity $X_{0,0} = X^*$.
- Unlike $C(0)$, we do not have a closed formula for C^*. It can be computed numerically (Dumortier-Ibáñez-Kokubu-Simó).

Results: 2D manifolds





Results: 2D manifolds

Remark We know that if σ is not $\mathcal{O}(\delta)$ the difference is **not** exponentially small.

This is just classical perturbation theory

Theorem Let $\sigma = \mathcal{O}(\delta)$. For $0 \leq \varepsilon \leq 1$, the distance between the 2-dimensional invariant manifolds on the plane $z = 0$ is given asymptotically by:

$$\Delta^2(\theta) = \varepsilon\delta^{\frac{-2(1+\beta_1)}{\beta_1}} \left[\Upsilon^{[0]}(\delta,\sigma,\varepsilon) + e^{-\frac{\alpha^*\pi}{2\beta_1\delta}}\left(C_1(\varepsilon)\sin\left(\theta - \frac{L_0}{\beta_1}\log\delta\right) + C_2(\varepsilon)\cos\left(\theta - \frac{L_0}{\beta_1}\log\delta\right) + \mathcal{O}\left(\frac{e^{-\frac{\alpha^*\pi}{2\beta_1\delta}}}{\log(1/\delta)}\right)\right)\right].$$

(conservative case: $\sigma = 0$, $\beta_1 = 1$).

- Breakdown of a 2D heteroclinic connection in the Hopf-zero singularity (I) (I. Baldomá, O. Castejón, T. M. S)
- Breakdown of a 2D heteroclinic connection in the Hopf-zero singularity (II) (I. Baldomá, O. Castejón, T. M. S)

To appear in JNLS

- $C_i(0)$ are good approximations of $C_i(\varepsilon)$ in the regular case, and they are determined by a suitable Melnikov function.

- Again, they depend on the Borel transforms of some functions.

- For $\varepsilon = 1$, one has that $C_i(1) = C_i^*$ are constants determined also by the inner equation.

- We do not have closed formulas for C_i^*. They can be computed numerically (Dumortier-Ibáñez-Kokubu-Simó).

The coefficient $\Upsilon^{[0]}$: The general case

$$\Delta^2(\theta) = \varepsilon\delta^{\frac{-2(1+\beta_1)}{\beta_1}} \left[\Upsilon^{[0]}(\delta,\sigma,\varepsilon) + e^{-\frac{\alpha^*\pi}{2\beta_1\delta}} \left(C_1(\varepsilon)\sin\theta + C_2(\varepsilon)\cos\theta \right) + \cdots \right]$$

and we have a formula for $\Upsilon^{[0]}$

$$\Upsilon^{[0]} = \Upsilon_0^{[0]} + \mathcal{O}(\varepsilon\delta^2), \qquad \Upsilon_0^{[0]} = \sigma I + \varepsilon\delta\bar{J}(\delta,\sigma),$$

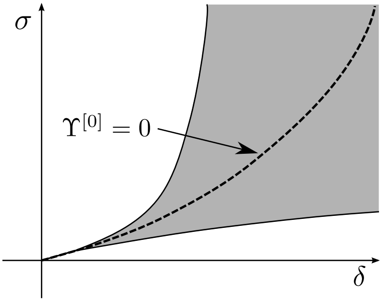
I, $\bar{J}$ have formulas.

Lemma There exists a curve $\Gamma = \{\sigma = \frac{\bar{J}}{I}\varepsilon\delta + O(\varepsilon\delta^2)$ such that

$\Upsilon^{[0]}(\delta,\sigma(\delta),\varepsilon) = 0.$

The coefficient $\Upsilon^{[0]}$: The general case

In general, there exists a wedge-shaped domain in the parameter space where $\Upsilon^{[0]}(\delta, \sigma, \varepsilon)$ is exponentially small.



Where the parameters are in this domain, the 2D invariant manifolds intersect and the Shilnikov phenomenon takes place.

The coefficient $\Upsilon^{[0]}$: The conservative case

- Conservative case: $\sigma = 0$ and $\beta_1 = 1$.
- Now we just have one parameter, so we cannot impose the value of $\Upsilon^{[0]}$.
- One can see that $\Upsilon^{[0]}(\delta, \varepsilon) = 0$ for all δ sufficiently small, $0 \leq \varepsilon \leq 1$.
- Therefore in all these cases we have the Shilnikov bifurcation in the unfoldings.

Main features of our methods

- We use that the invariant manifolds are graphs in suitable variables. This gives adequate parameterizations φ^u and φ^s.
- In classical perturbation theory one proves that the perturbed manifolds are $\varepsilon\delta$-close to the unperturbed heteroclinic ones.
  - One obtains that the manifolds are solutions of a fix point equation in a Banach space $\mathcal{X}$:

$$\varphi^u = \mathcal{F}^u(\varphi^u)$$

  - The functional $\mathcal{F}^u : B_R \subset \mathcal{X} \to B_R \subset \mathcal{X}$, $R = O(\varepsilon\delta)$ small Lipchitz constant $K = O(\varepsilon\delta)$
  - Then $\varphi^u = \varphi^u_0 + \varphi^u_1$, where $\varphi^u_0 = \mathcal{F}^u(0) = O(\varepsilon\delta)$ and $\varphi^u_1 = O(\varepsilon\delta)^2$
  - One obtais analogous estimates for φ^s
- Finally $\varphi^u - \varphi^s = \varphi^u_0 - \varphi^s_0 + O(\varepsilon\delta)^2$ and $\varphi^u_0 - \varphi^s_0 = \mathcal{F}^u(0) - \mathcal{F}^u(0)$ is given by the Melnikov formula!

- **Problem**: $\Delta_0 = \varphi_0^u - \varphi_0^s$ is exponentially small, so a priori it is not the dominant term.
- To improve the bounds of the error ne needs to extend the domain of definition of the manifolds to complex domains. Close to the singularities of the unperturbed heteroclinic orbit. This presents dificulties in the proof.
- **Regular case**:
  - One can prove that the error term is smaller (classical perturbation theory works).
  - Δ_0 is indeed the dominant term of the difference Δ.
- **Singular case**:
  - We cannot take the same φ_0^u and φ_0^s, since then Δ_0 is not the dominant term.
  - In this case, we have to take $\tilde{\varphi}_0^u$ and $\tilde{\varphi}_0^s$ suitable solutions of the inner equation. They are the dominant part of φ^u and φ^snear the singularities.
  - Then $\tilde{\Delta}_0$ is the dominant term of the difference Δ.