Periodic orbits in the restricted 3-body problem and Arnold's $\mathcal{J}^+$-invariant

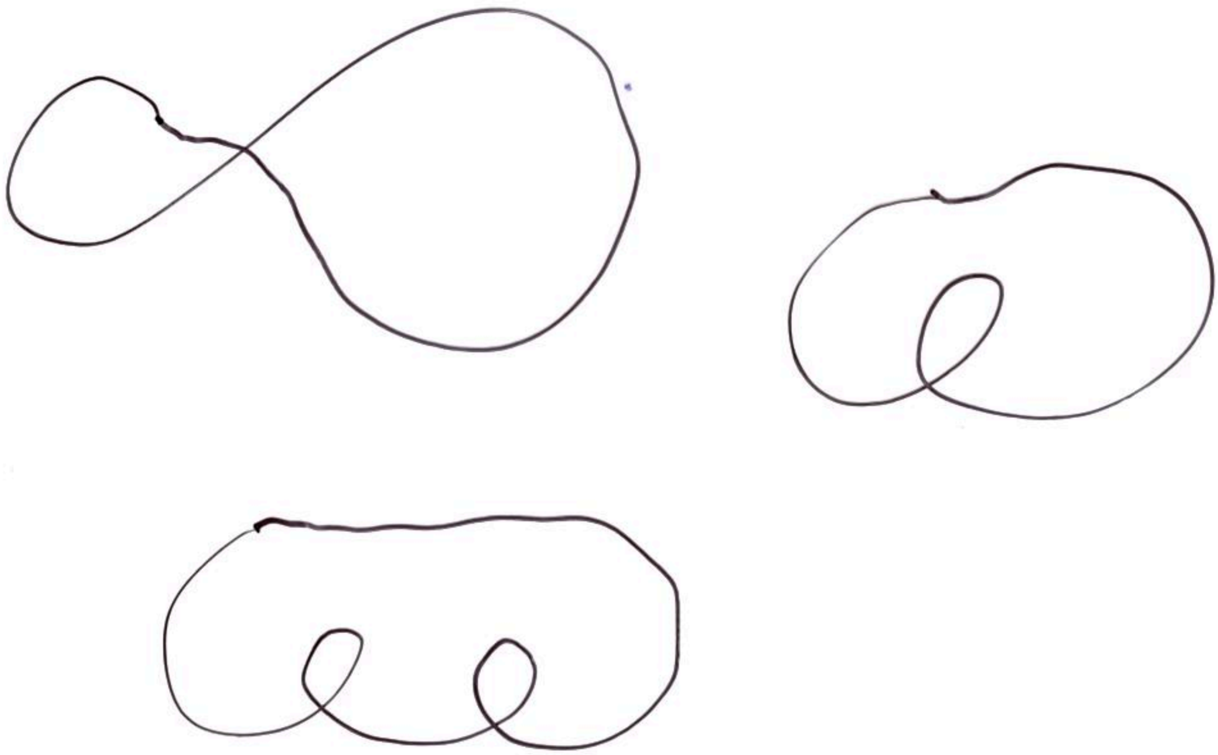
Joint with Kai Cieliebak

Otto van Koert

Immersed Curves

$\gamma : S^1 \longrightarrow \mathbb{R}^2$ smooth

$\gamma'(t) \neq 0 \quad \forall \; t \in S^1$



Invariants : Rotation number

(Whitney index)

$$S^1 \longrightarrow S^1$$

$$t \longmapsto \frac{\gamma'(t)}{|\gamma'(t)|}$$

$$\mathrm{rot}(\gamma) = \deg\left(t \mapsto \frac{\gamma'(t)}{|\gamma'(t)|}\right)$$

Example :

rot = 1   rot = −1   rot = 0   rot = 2

Thm (Whitney - Graustein)

There is a bijection

$$\left\{ \begin{array}{l} \text{Homotopy classes} \\ \text{of immersions} \\ \gamma : S^1 \to \mathbb{R}^2 \end{array} \right\} \cong \mathbb{Z}$$

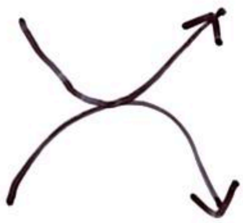
$$[\gamma] \longmapsto \operatorname{rot} \gamma$$

3 disasters along a generic

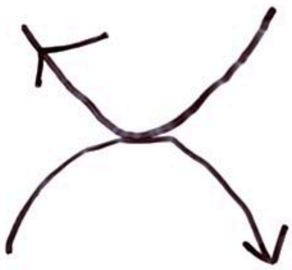
homotopy

triple points

direct self tangencies



inverse self tangencies



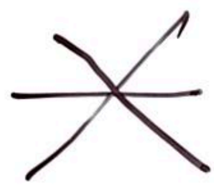
Arnold: Introduces invariants

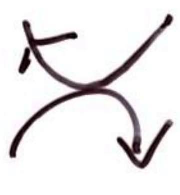
for generically immersed curves

invariant under generic homotopies

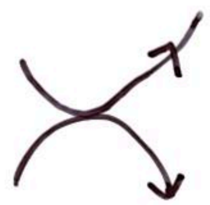
avoiding disasters

J^+ : invariant under

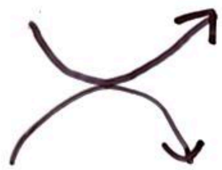




not invariant under

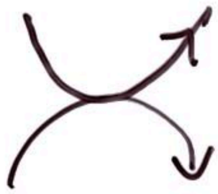


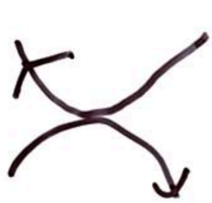
J^- : invariant under 

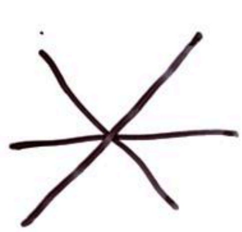


not invariant under 

St (strangeness)

invariant under 



not invariant under 

Definition of J^+

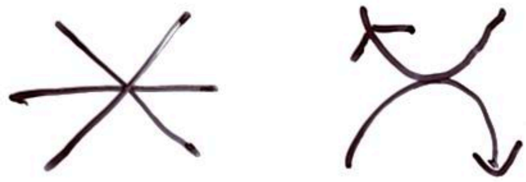
Standard curves

γ_{-2} γ_{-1} γ_0 γ_1 γ_2

rot $= -2$ rot $= -1$ rot $= 0$ rot $= 1$ rot $= 2$

Axioms for J^+

(Invariance) J^+ invariant

under 

(Direct self tangencies)

$+2$

(Normalization) $J^+(\gamma_0) = 0$

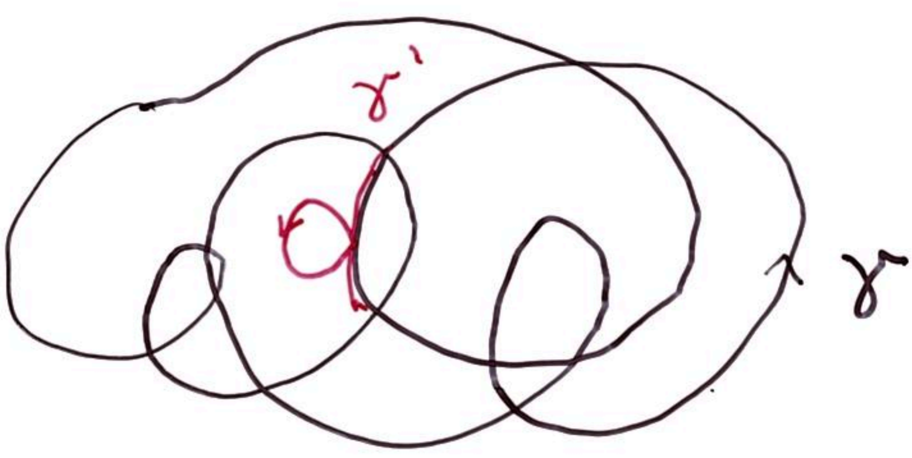
$$J^+(\gamma_i) = -2(|i|-1)$$

$i = \pm 1, \pm 2, \ldots$

Thm (Arnold)

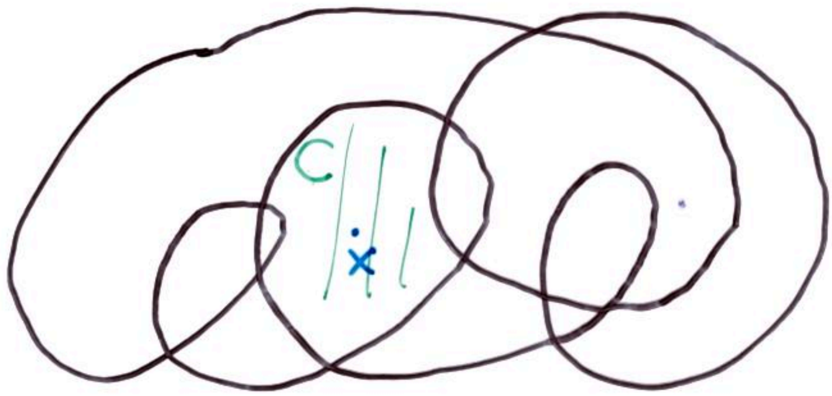
J^+ exists and is uniquely determined by Axioms.

Changing of J^+ under addition of small loops



$\gamma : S^1 \to \mathbb{R}^2$ immersion

$C \subset \mathbb{R}^2 \setminus \operatorname{im}\gamma$ connected component

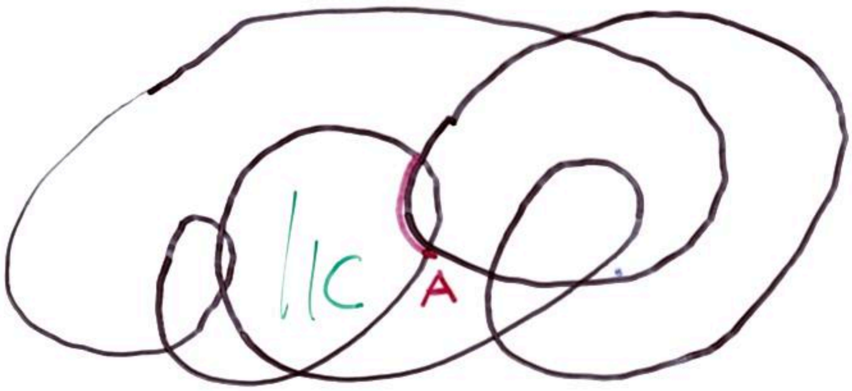


$x \in C$

$w(x, \gamma)$ winding number of γ around x

Independent of $x \in C$

$\Rightarrow \; w(C,\gamma) := w(x,\gamma) \qquad x \in C$



$A \subset \operatorname{im}\gamma$ boundary arc of ∂C

Add small loop in C to arc A

$\rightsquigarrow$ immersed curve γ'

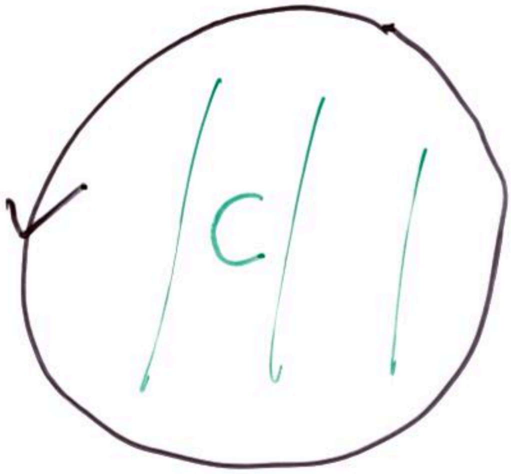
Lemma :

$$\mathcal{I}^+(\gamma') = \mathcal{I}^+(\gamma) + 2\varepsilon w(C, \gamma)$$

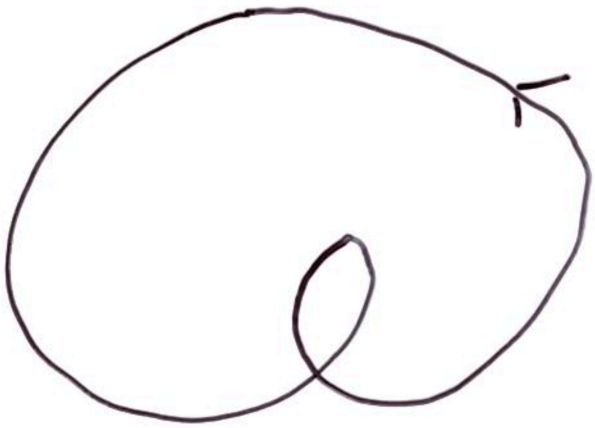
$$\varepsilon = \begin{cases} -1 & \text{orientation of } A \text{ coincides with bdry orientation of } C \\ 1 & \text{else} \end{cases}$$

Example :

1)



$$\gamma = \gamma_1 \qquad w(C,\gamma) = 1$$

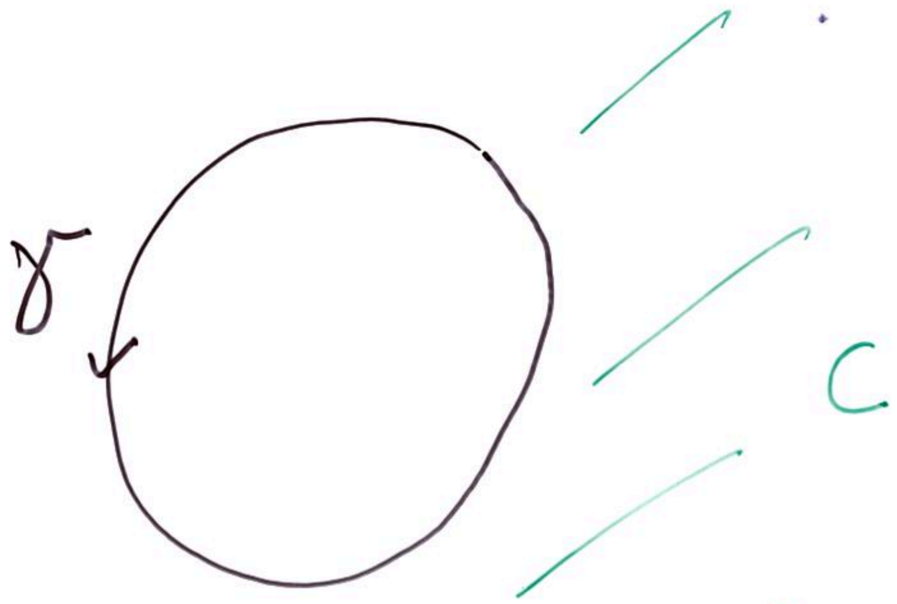


$$\gamma' = \gamma_2$$

$$J^+(\gamma') = J^+(\gamma_2) = -2$$

$$= \underline{J^+(\gamma)} - 2\,w(C,\gamma)$$

2)



$w(C, \gamma) = 0$

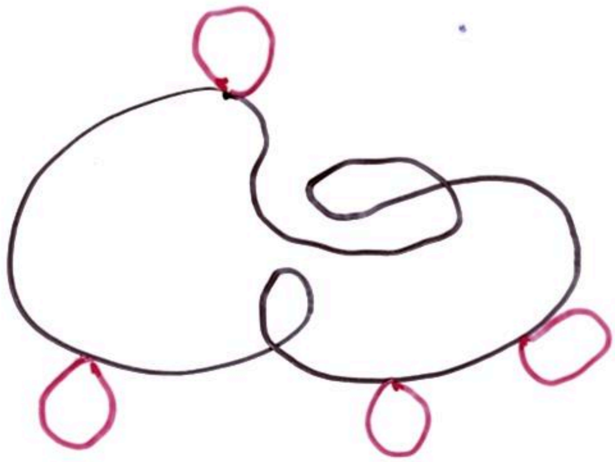
γ'

$=$

γ_0

$$J^+(\gamma') = J^+(\gamma_0) = 0 = J^+(\gamma)$$

Corollary: J^+ does **not** change under addition of exterior loops

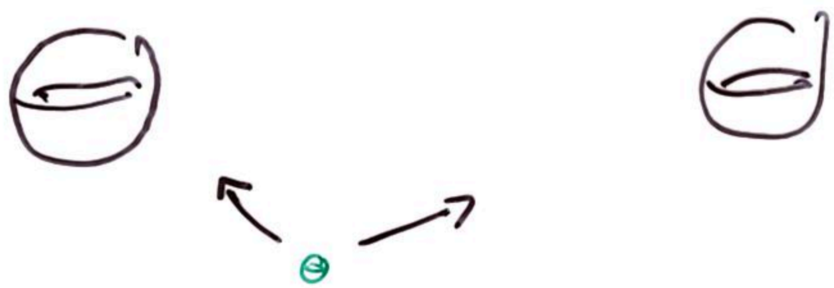


Restricted 3-body problem

2 masses

1 massless body

<u>Goal</u>: Understand dynamics of massless body



Possible interpretations

· Masses: Sun, earth

Massless body: Moon

· Masses: Earth, Moon

Massless body: Satellite

Masses: Double star

Massless body: Planet

μ Mass of Earth

$1-\mu$ " " Sun

<u>Circular case:</u> Earth and Sun move on circles around common center of mass

$$E(t) = (1-\mu)\,(\cos t, \sin t)$$

$$S(t) = -\mu\,(\cos t, \sin t)$$

Hamiltonian for Moon

(in inertial system)

$$H^i(q,p) = \underbrace{\frac{1}{2}p^2}_{\substack{\text{kinetic}\\\text{energy}}} \underbrace{- \frac{\mu}{|q-E(t)|} - \frac{1-\mu}{|q-S(t)|}}_{\substack{\text{potential}\\\text{energy}}}$$

Time dependent

No preservation of energy

Rotating coordinates

Sun, Earth at rest

$E = (1-\mu, 0)$

$S = (-\mu, 0)$

Hamiltonian for Moon

$$H(q,p) = \frac{1}{2} p^2 - \frac{\mu}{|q-E|} - \frac{1-\mu}{|q-S|}$$

$$+ \underbrace{p_1 q_2 - p_2 q_1}_{\text{angular momentum}}$$

angular momentum (generates rotation)

Complete squares

twisted kinetic energy (Coriolis force)

$$H(q,p) = \frac{1}{2}\left((p_1+q_2)^2 + (p_2-q_1)^2\right)$$

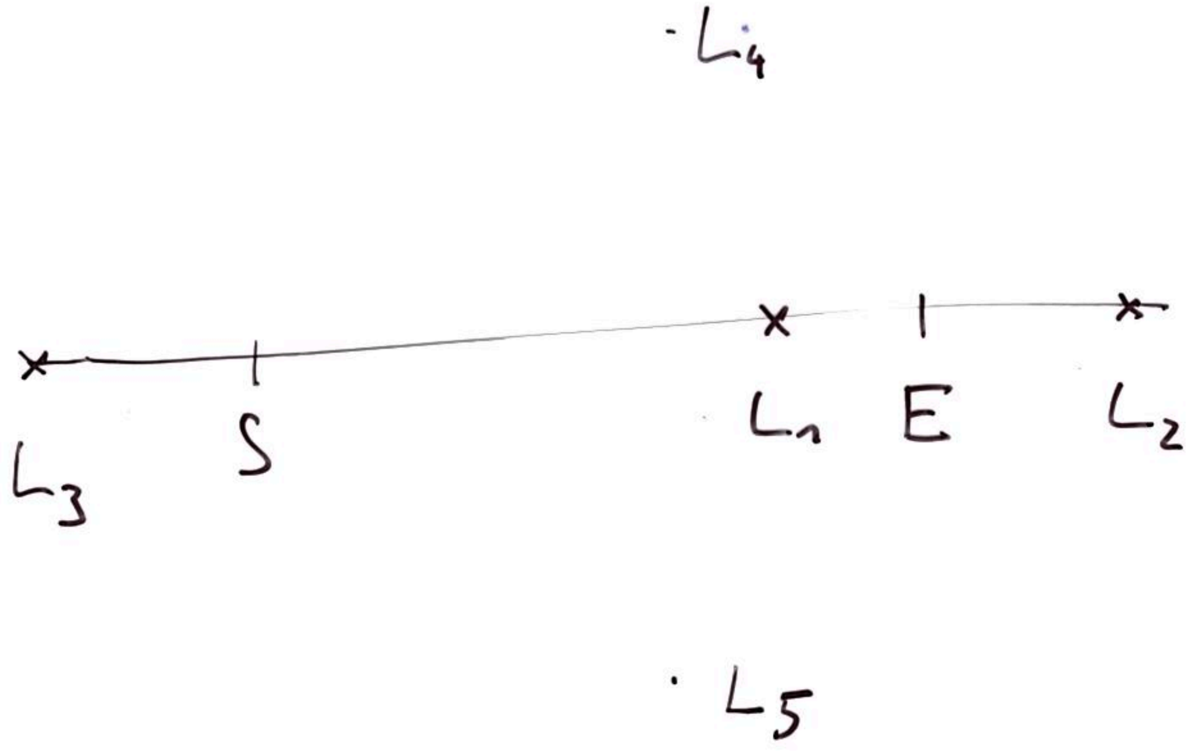
$$\underbrace{-\frac{\mu}{|q-E|} - \frac{1-\mu}{|q-S|} \underbrace{- \frac{1}{2} q^2}_{\to \text{ centrifugal force}}}_{=: U(q)}$$

$=: U(q)$ effective potential

Lagrange points

$$\operatorname{crit} H \cong \operatorname{crit} U \supset \text{Lagrange points}$$

$$(q,p) \longmapsto q$$



Hill's region

$c \in \mathbb{R}$

level set

$\Sigma_c = H^{-1}(c)$

$\subset T^*\mathbb{R}^2 = \mathbb{R}^4$

$\pi : T^*\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(q,p) \longmapsto q$

$\mathfrak{K}_c := \pi(\Sigma_c)$

$= \{ q \in \mathbb{R}^2 \setminus \{E, S\} : U(q) \leq c \}$

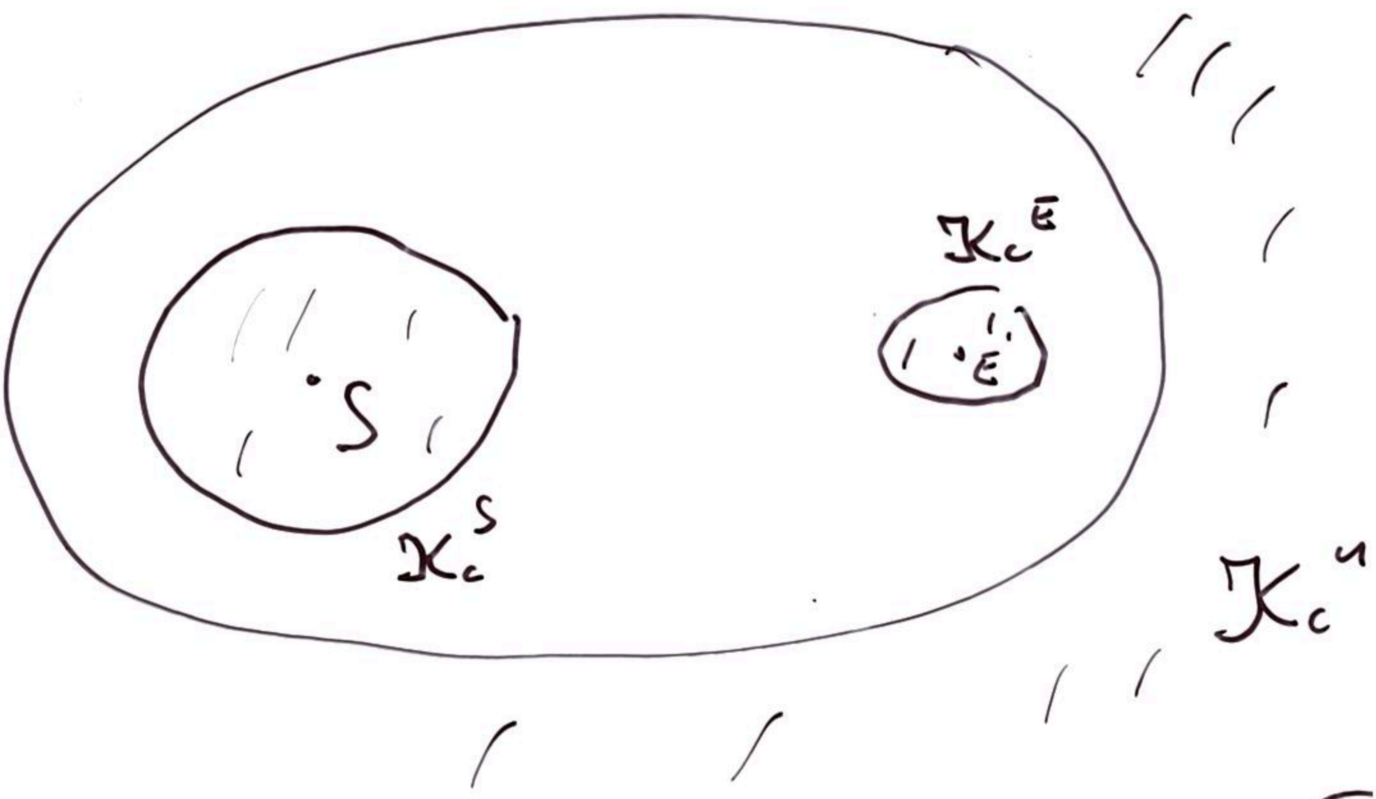
Smallest critical value

$$c_1 := H(L_1)$$

1. Lagrange point

$$c < c_1$$

$\mathcal{K}_c$ 3 connected components

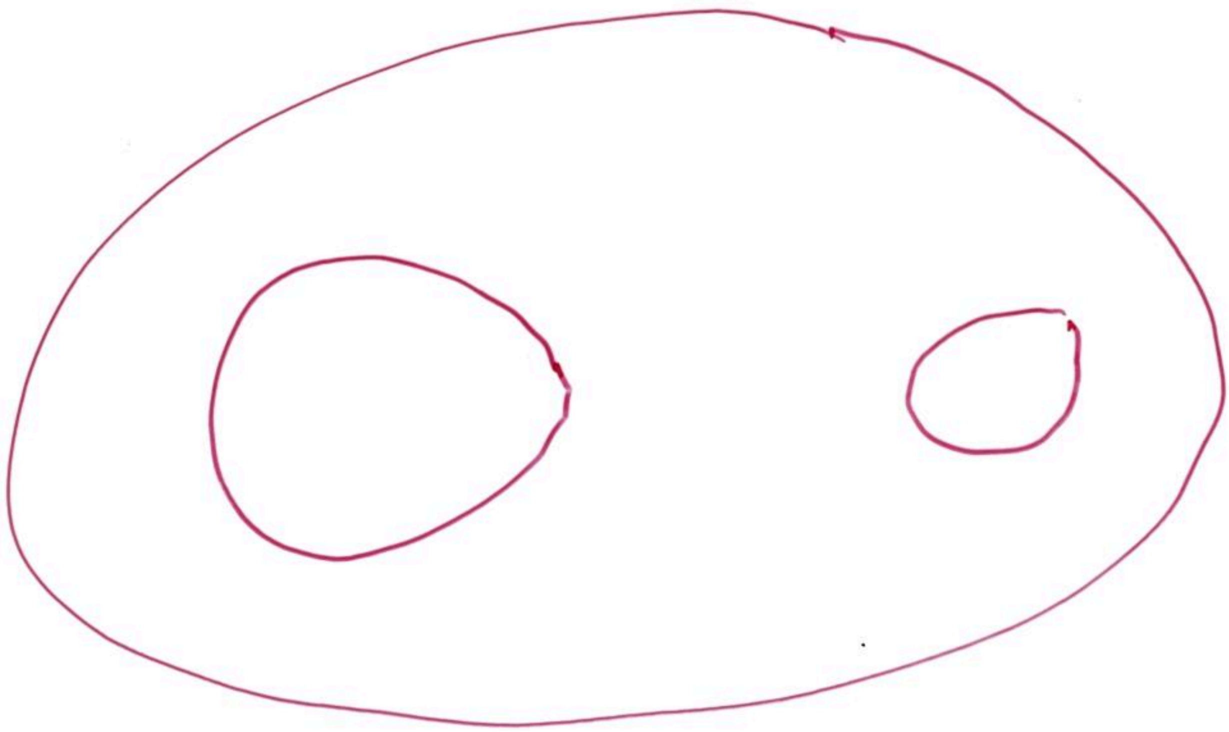


Zero velocity curves

= Boundary of Hill's region

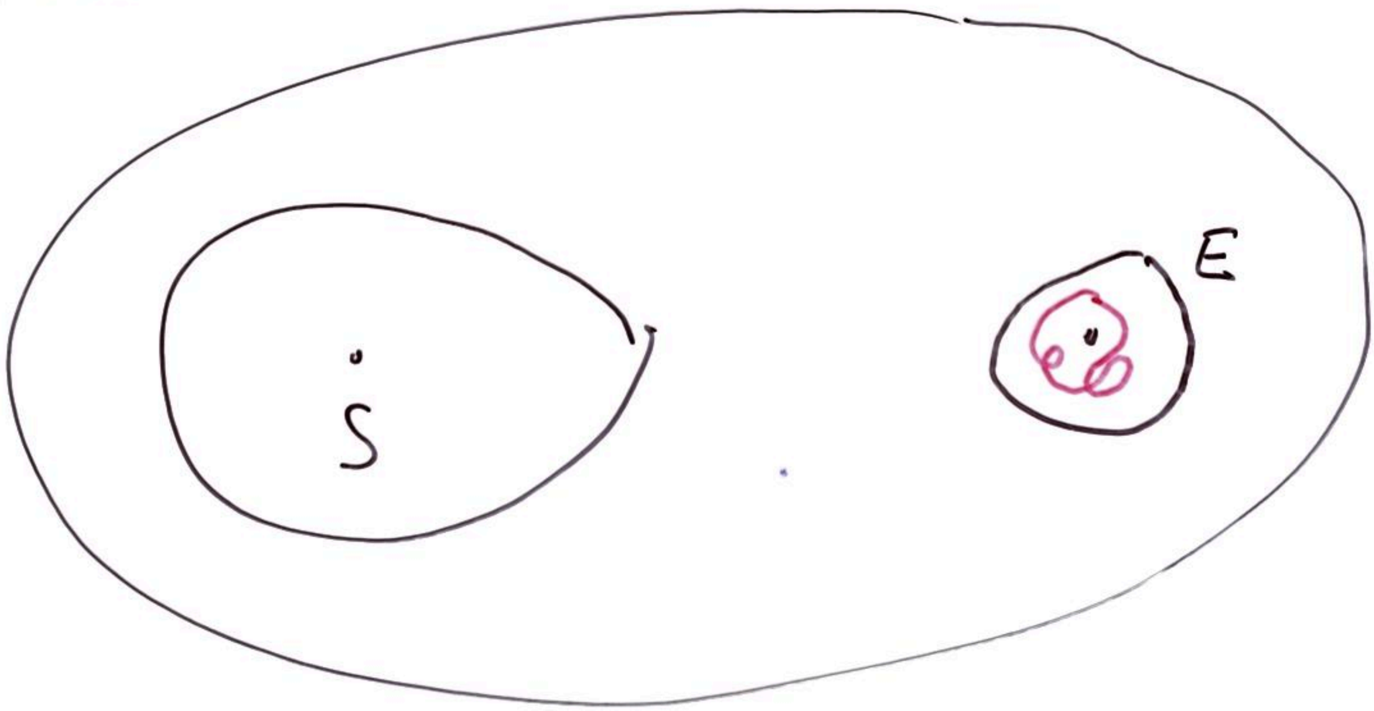
$= \partial \mathfrak{K}_c$

$= \{ q \in \mathbb{R}^2 : U(q) = c \}$



Moon of maximal lunarity

Moon: Periodic orbit close to E



If Moon $\subset \mathring{\mathfrak{K}}_c^E$

$\uparrow$ interior

$\longrightarrow$ immersed curve

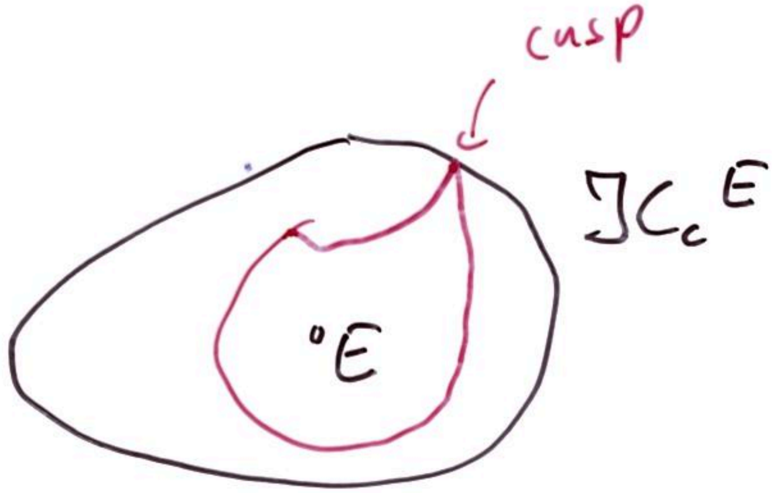
If Moon hit $\partial \mathfrak{K}_c^E$

$\Uparrow$

zero velocity

curve

$\Longrightarrow$ cusp



Lunarity : Period of Moon

$\equiv$ Month

Hill : Studies family of periodic orbits varying period

Cusp occurs

Hill : End of family "Moon of maximal lunarity"

Poincaré, Adams:

Family can be continued

$\rightsquigarrow$ occurence of exterior loop