Medial Representations

Mathematics, Algorithms and Applications

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with contributions from:

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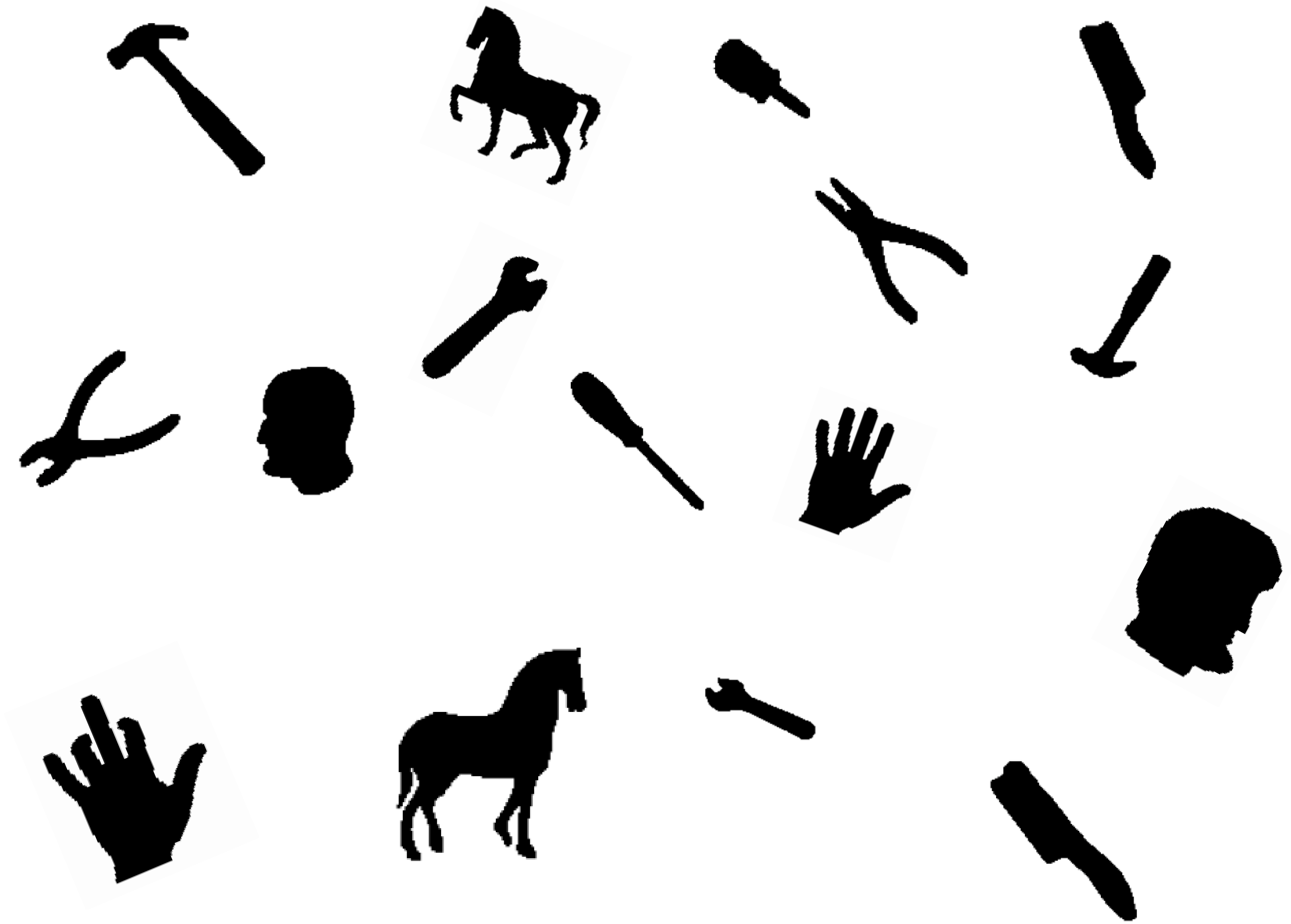
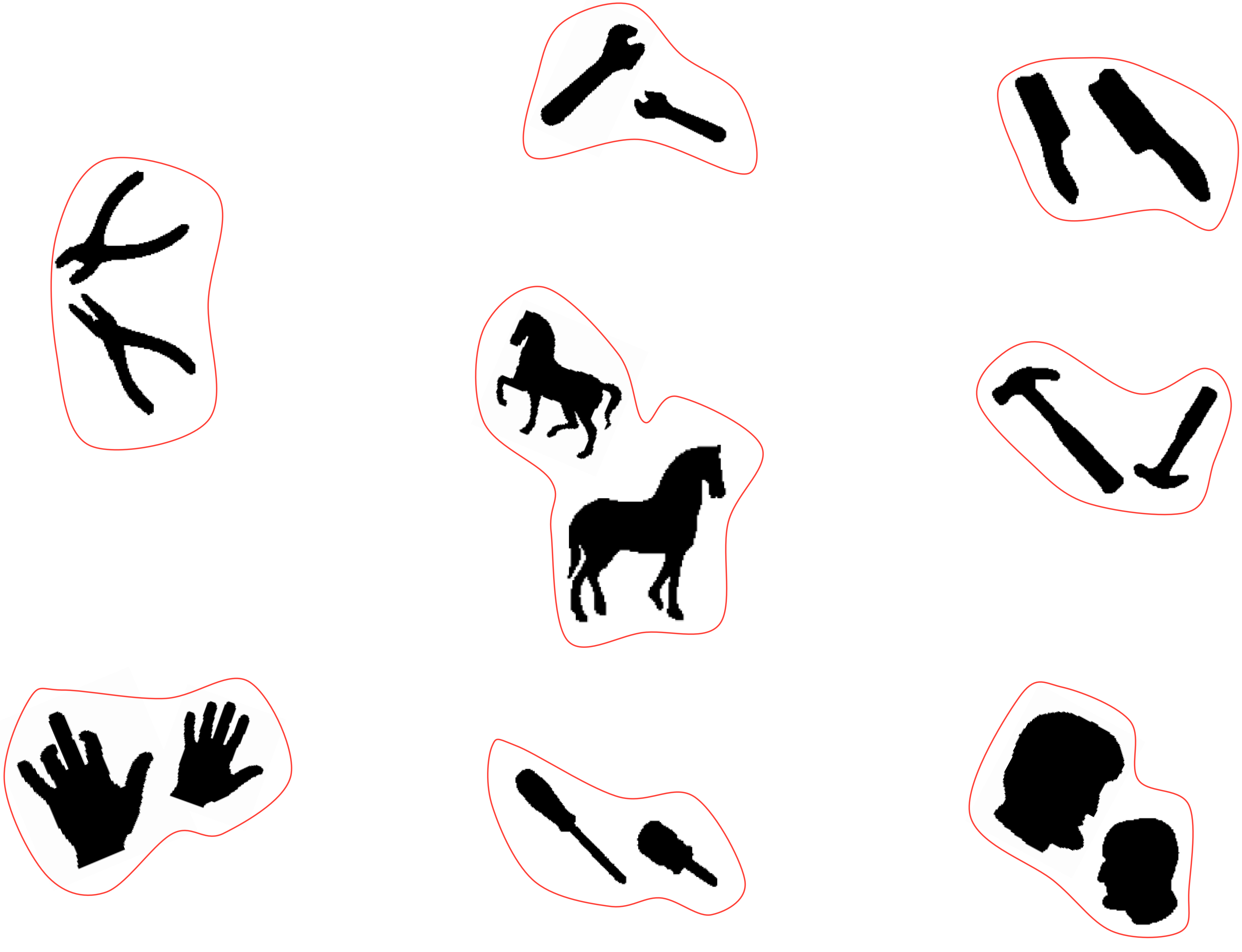
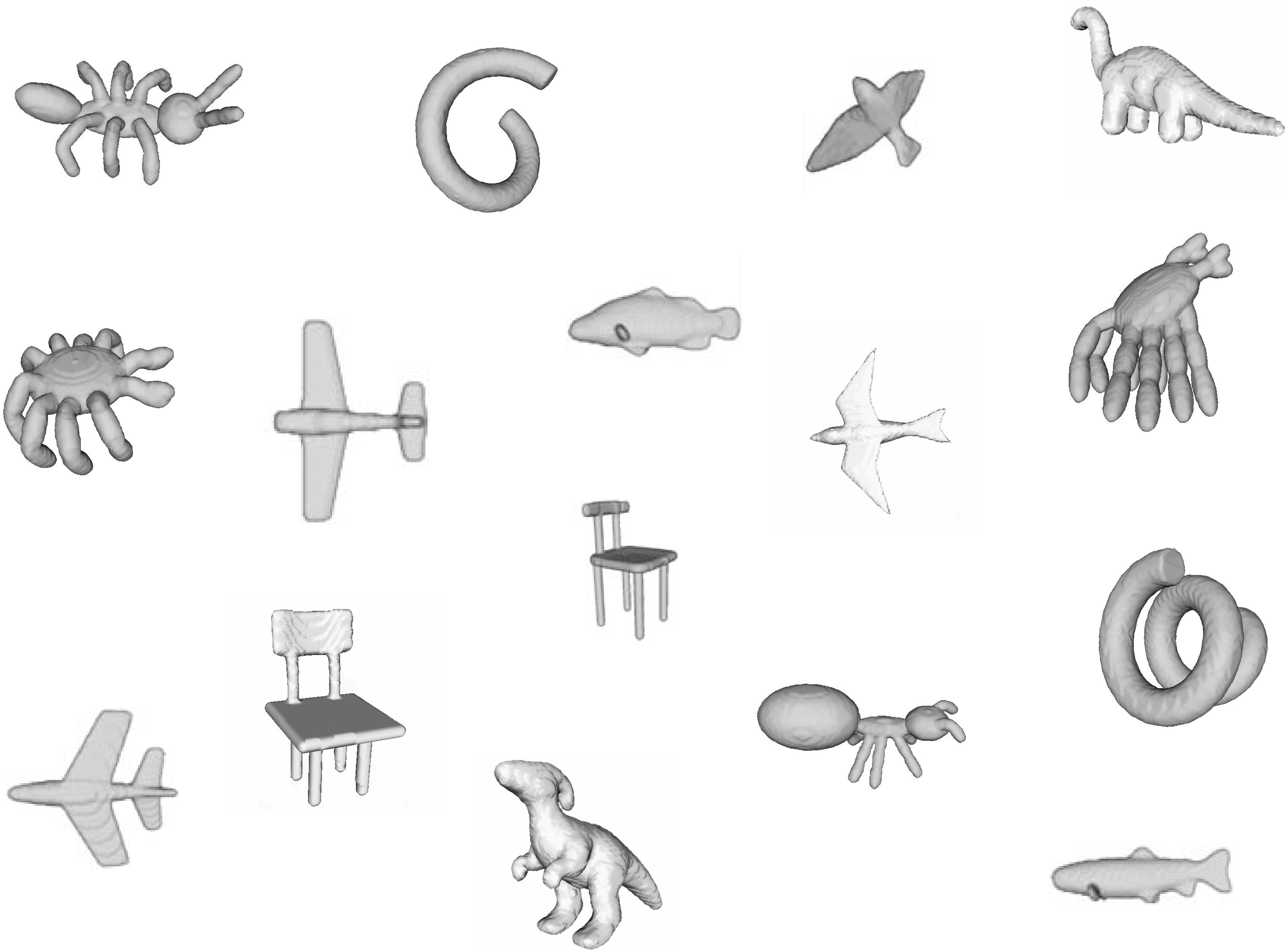
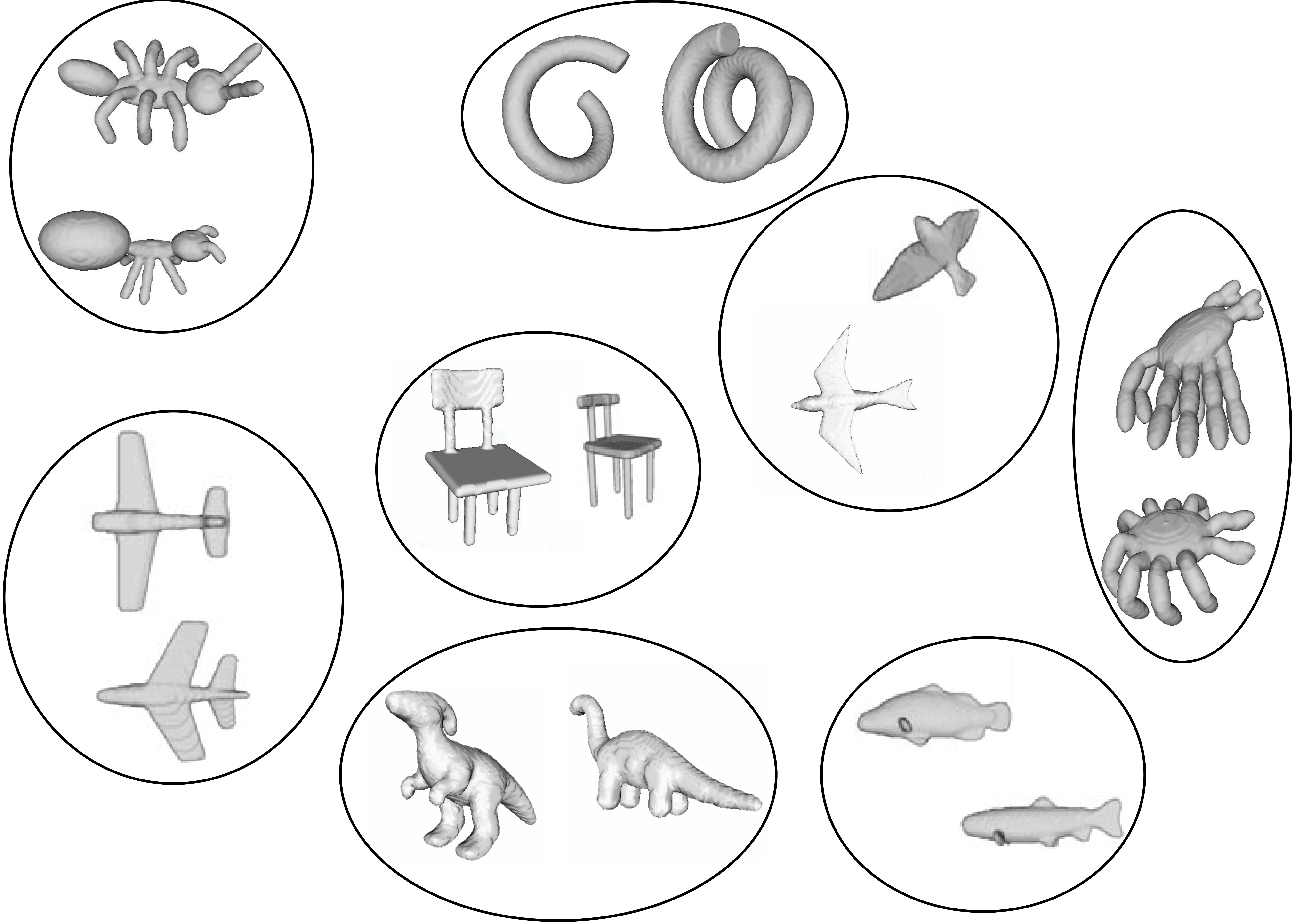
Marcello Pelillo, Carlos Phillips, Ali Shokoufandeh, Svetlana Stolpner, Steve Zucker,

Motivation









Blum's A-Morphologies: 2D

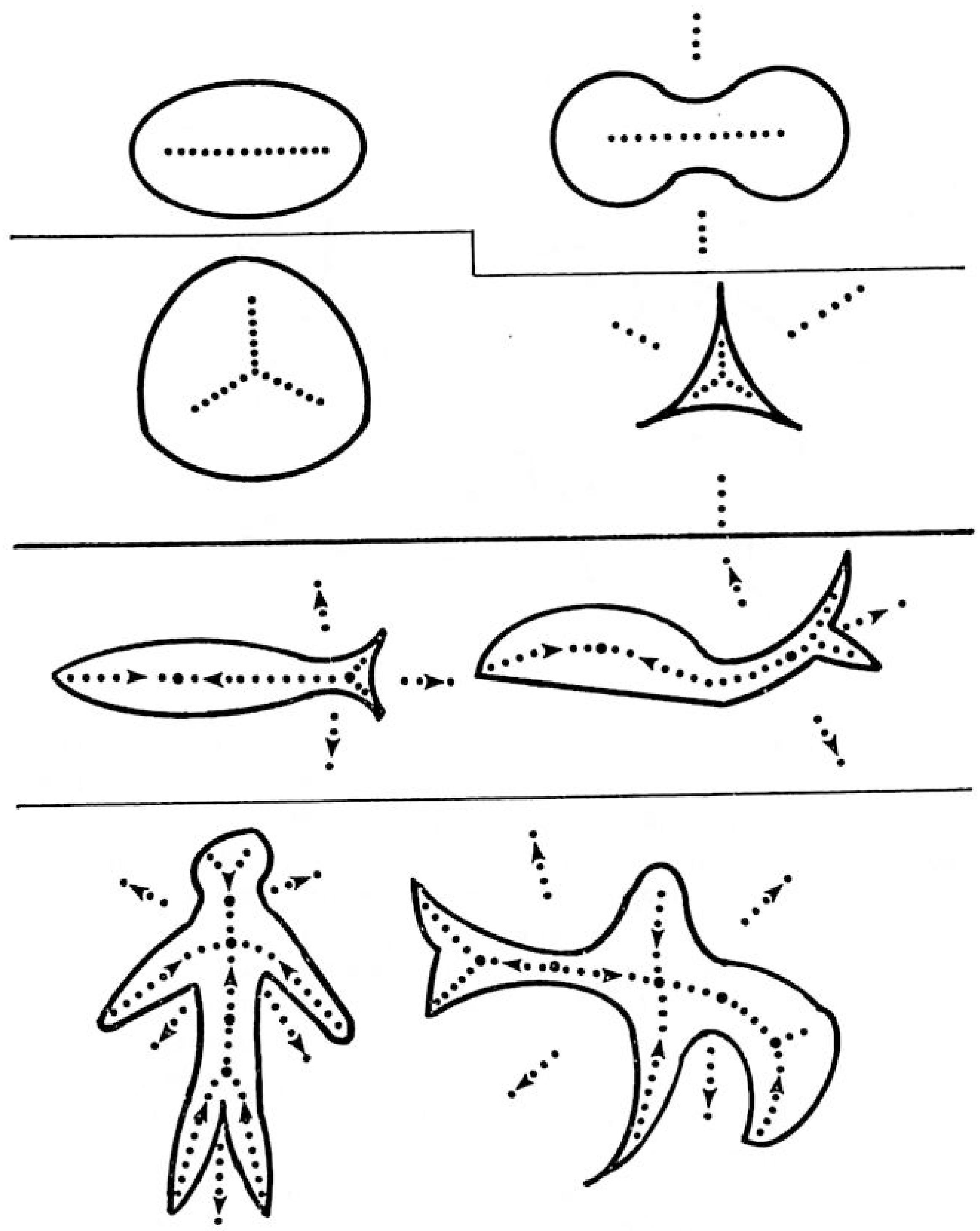


FIG. 29. Equivalent objects in some simple A-morphologies. In the upper half, the object sym-axes have the same topology. In the lower half, the object and ground have the same directed graph.

Blum's A-Morphologies: 3D

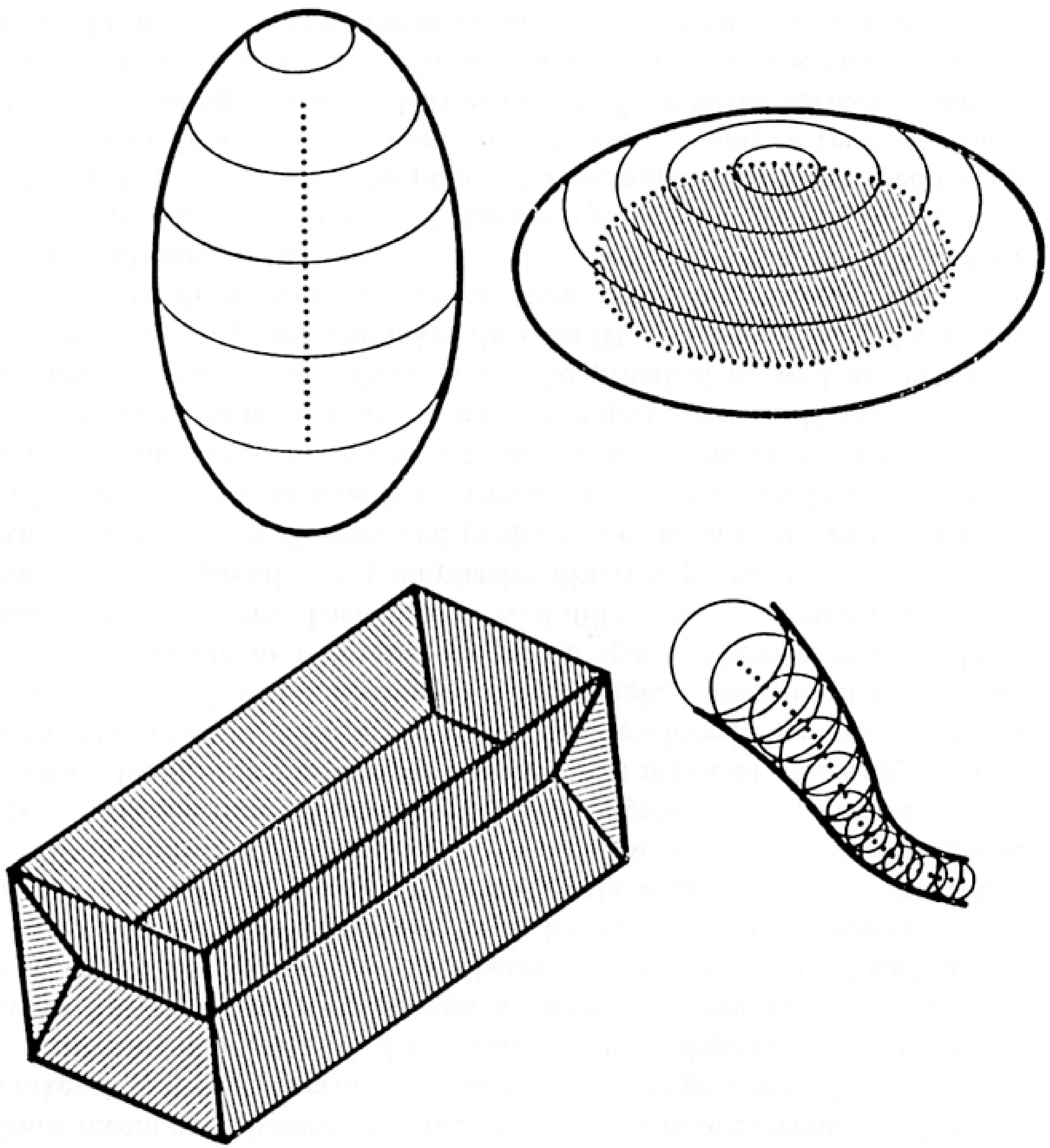
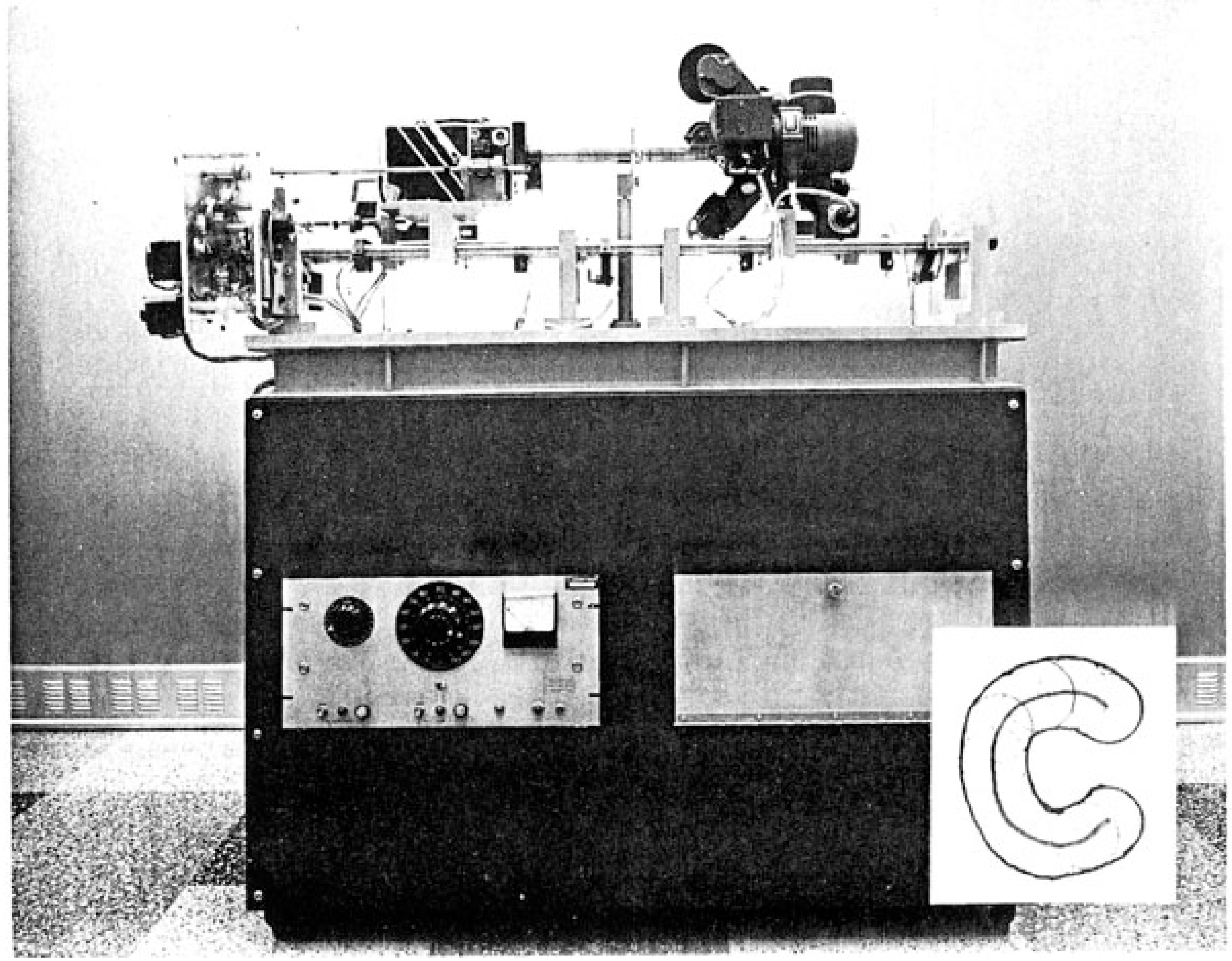


FIG. 42. Sym-axes of some three-dimensional objects. The ellipsoids at the top show that the sym-ax can now be both arcs and surfaces. The bottom shows the sym-ax for a rectangular solid and for a general spherical envelope.

Blum's Grassfire Machine



"Figure 19 shows my first physical embodiment of the process. It uses a movie projector and camera with high contrast film. These are symmetrically driven apart from the lens in such a way as to keep a one to one magnification, but to increase the circle of confusion (defocussing). Corner detection is done by a separate process. I am presently building a closed loop electronic system to do both the wave generation and corner detection."

[A transformation for Extracting New Descriptors of Shape, 1967.]

Mathematics

The Rowboat Analogy

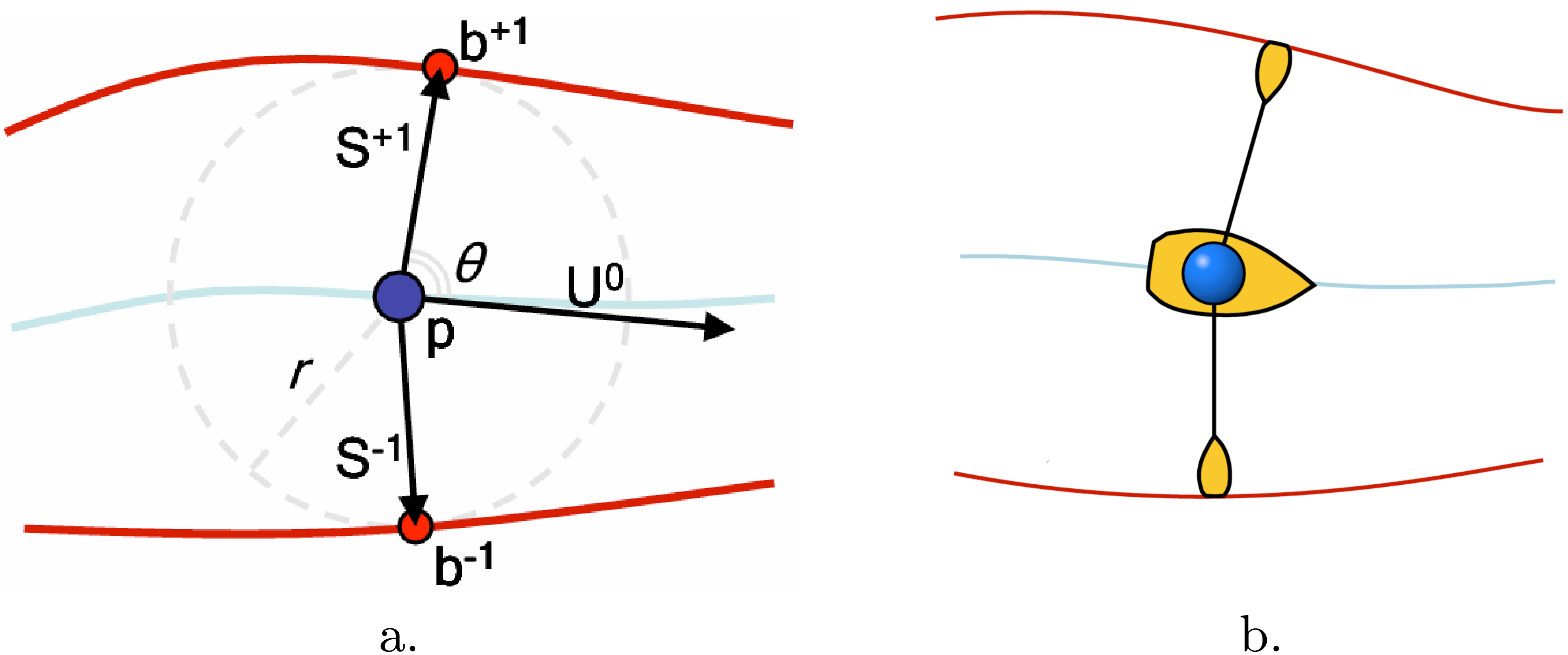


Figure 1.7. Local medial geometry. a. Local geometric properties of a medial point and its boundary pre-image. b. The rowboat analogy for medial points.

Contact Classification

Theorem 1 (Giblin and Kimia) *The internal medial locus of a three-dimensional object Ω generically consists of*

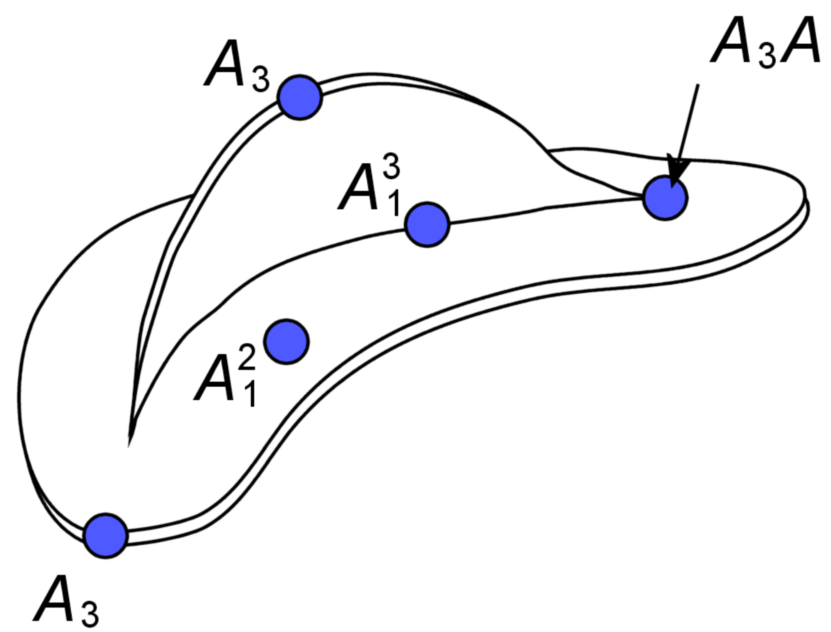
1 *sheets (manifolds with boundary) of A_1^2 medial points;*

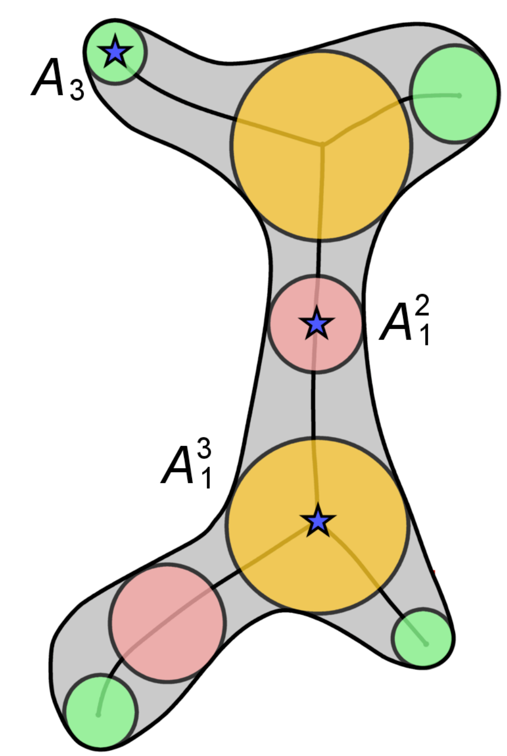
2 *curves of A_1^3 points, along which these sheets join, three at a time;*

3 *curves of A_3 points, which bound the free (unconnected) edges of the sheets and for which the corresponding boundary points fall on a crest;*

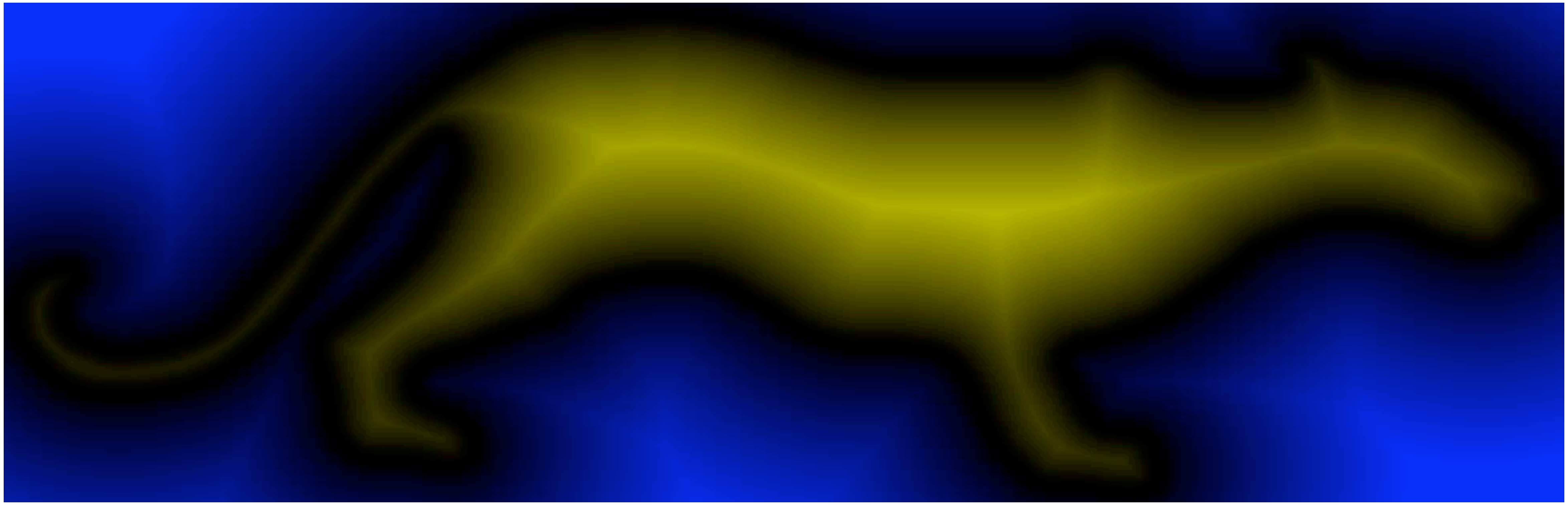
4 *points of type A_1^4, which occur when four A_1^3 curves meet;*

5 *points of type A_1A_3 (i.e., A_1 contact and A_3 contact at a distinct pair of points) which occur when an A_3 curve meets an A_1^3 curve.*

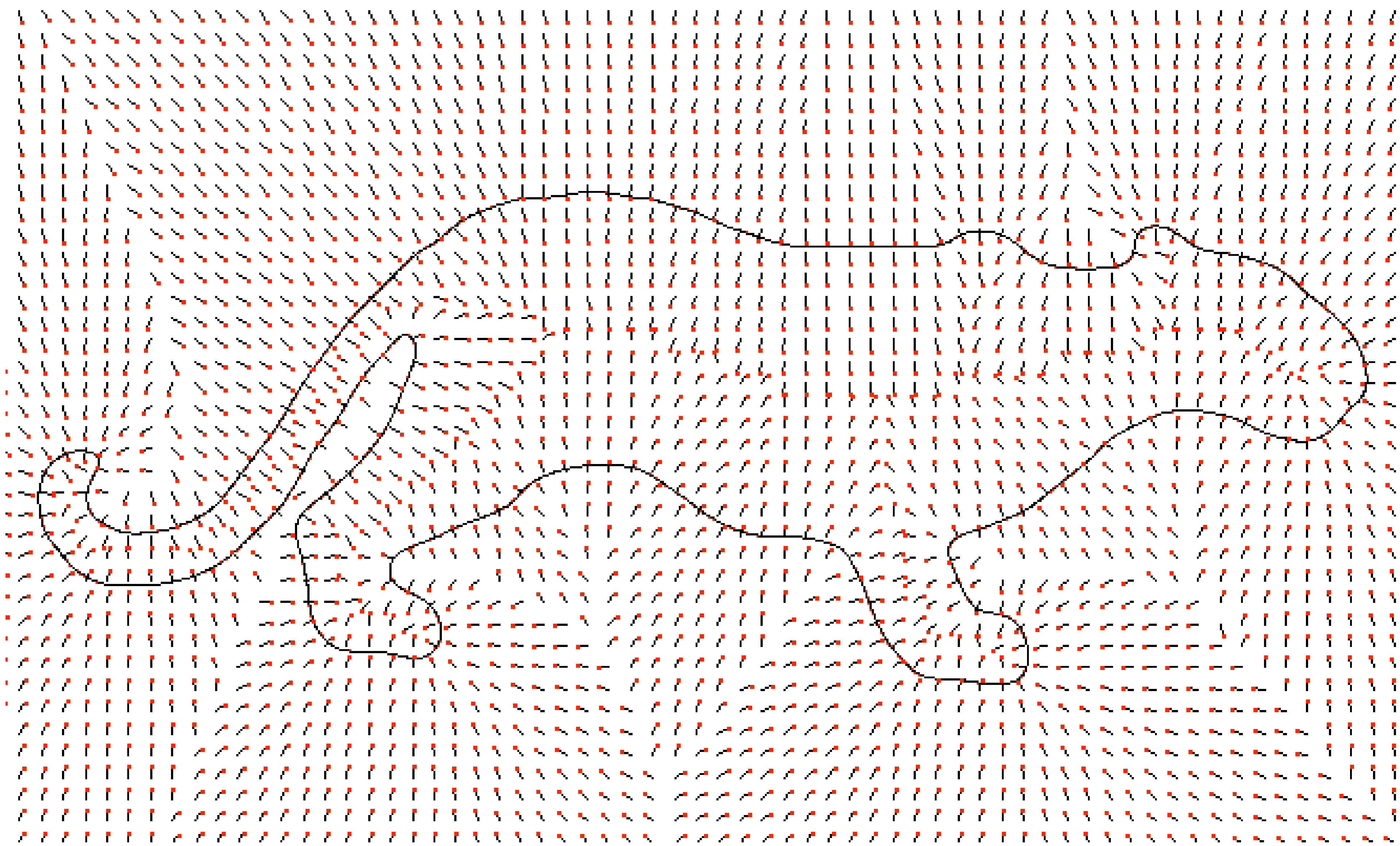




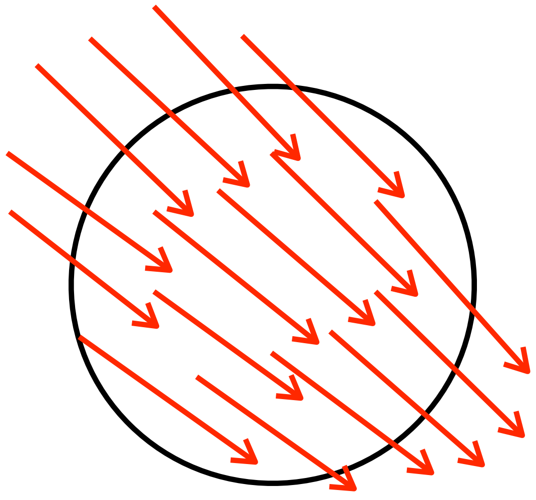
Euclidean Distance Function



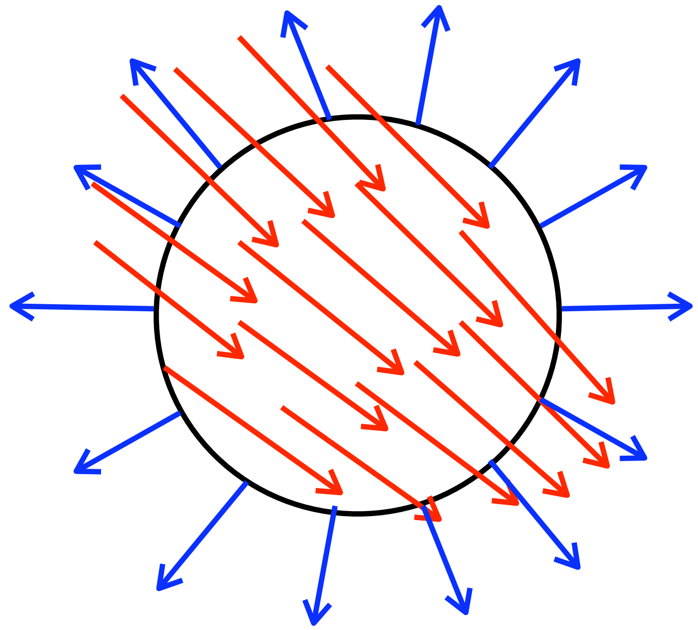
Gradient Vector Field



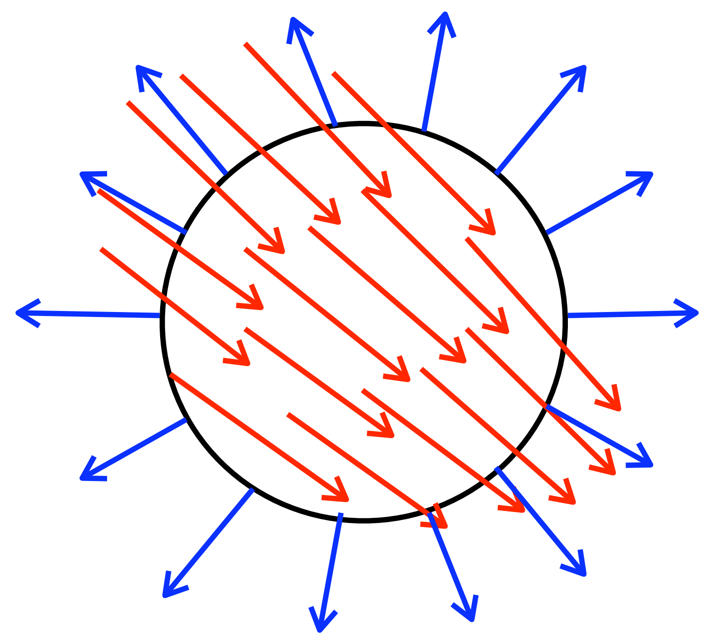
Outward Flux



Outward Flux



Outward Flux

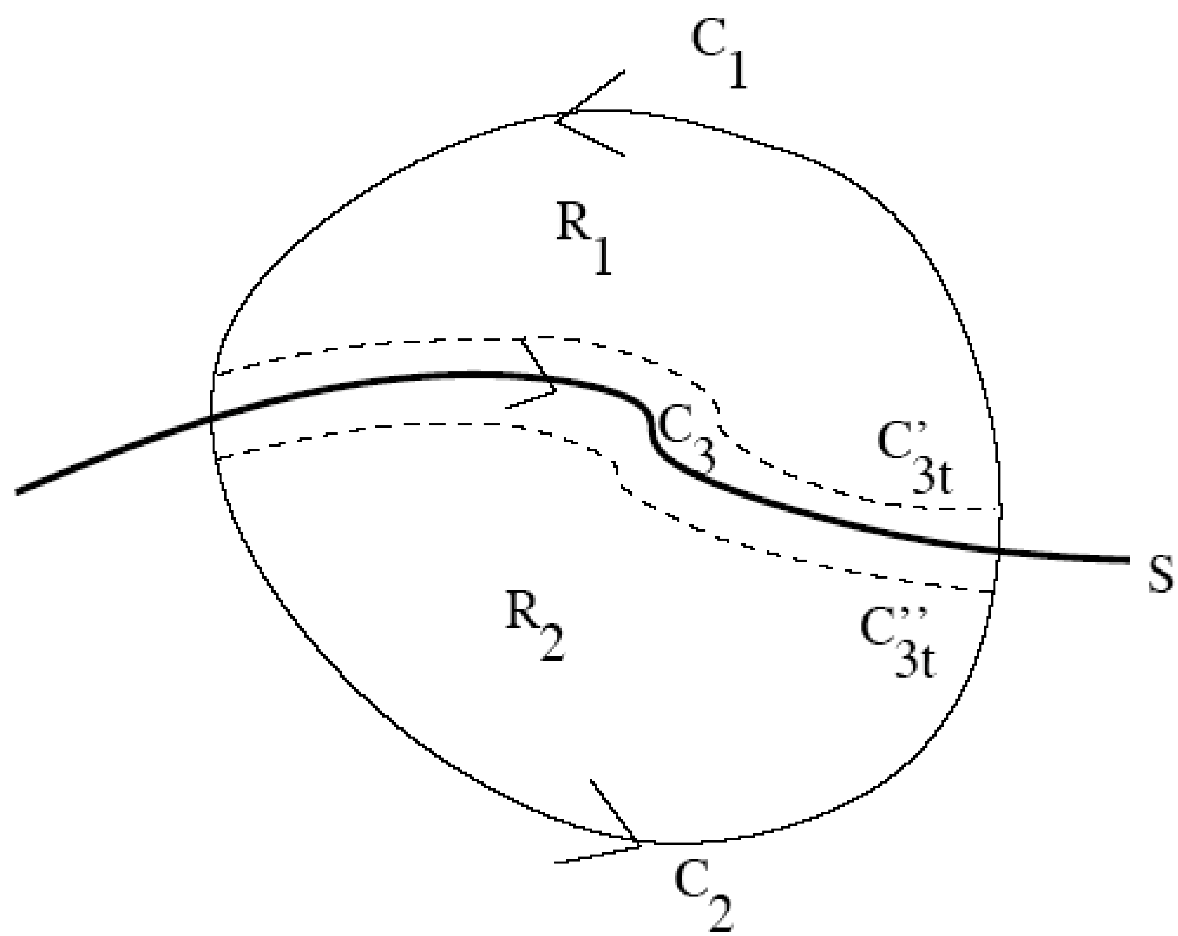


Definition 1 *The outward flux of $\dot{\mathbf{q}}$ through ∂R is given by*

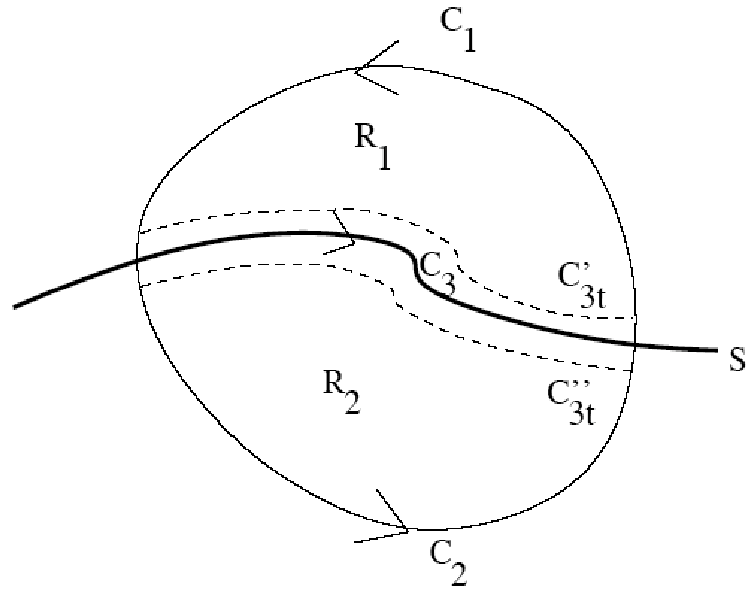
$$\int_{\partial R} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \;\, \mathrm{d}s$$

Definition 2 *The average outward flux of $\dot{\mathbf{q}}$ through ∂R is given by*

$$\frac{\int_{\partial R} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \;\, \mathrm{d}s}{\int_{\partial R} \;\mathrm{d}s}$$



Let S be a branch of the skeleton and let $R = R_1 \cup R_2$ be a path connected region which intersects it. Let $\partial R = C_1 \cup C_2$ and $C_3 = S \cap R$. Let C'_{3t}, C''_{3t} be parallel curves to C_3 which approach C_3 as $t \to 0$. Let R_{1t} and R_{2t} be the regions obtained from R_1 and R_2 by removing the region between the curves C'_{3t} and C''_{3t} Finally, let $\dot{\mathbf{q}}_+$ denote $\dot{\mathbf{q}}$ above S and $\dot{\mathbf{q}}_-$ denote $\dot{\mathbf{q}}$ below S.



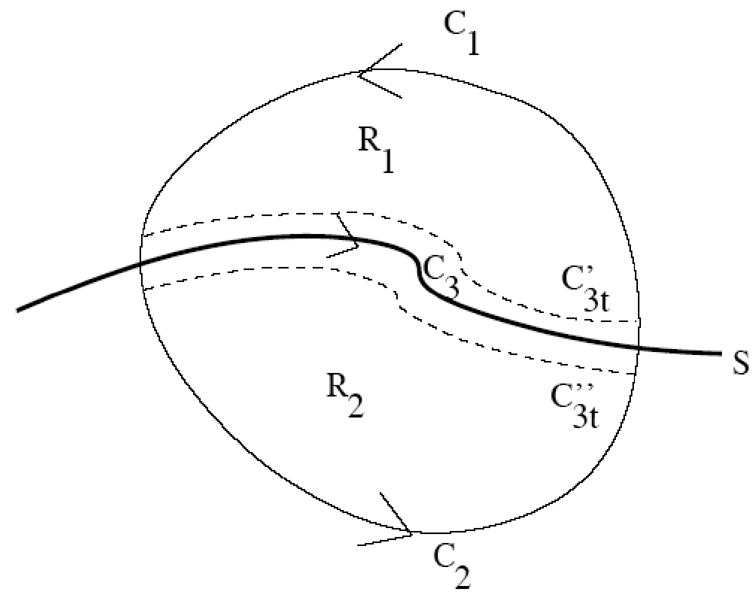
The outward flux of $\dot{\mathbf{q}}$ through ∂R is given by

$$\int_{\partial R} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \; \mathrm{d}s = \int_{C_1} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \; \mathrm{d}s + \int_{C_2} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \; \mathrm{d}s.$$

Applying the divergence theorem to R_{1t} and R_{2t}

$$\int_{R_{1t}} \mathrm{div}(\dot{\mathbf{q}})\, \mathrm{d}v = \int_{C_{1t}} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \; \mathrm{d}s + \int_{C'_{3t}} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \; \mathrm{d}s,$$

$$\int_{R_{2t}} \mathrm{div}(\dot{\mathbf{q}})\, \mathrm{d}v = \int_{C_{2t}} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \; \mathrm{d}s + \int_{-C''_{3t}} \langle\, \dot{\mathbf{q}}, \mathcal{N} \,\rangle \; \mathrm{d}s.$$

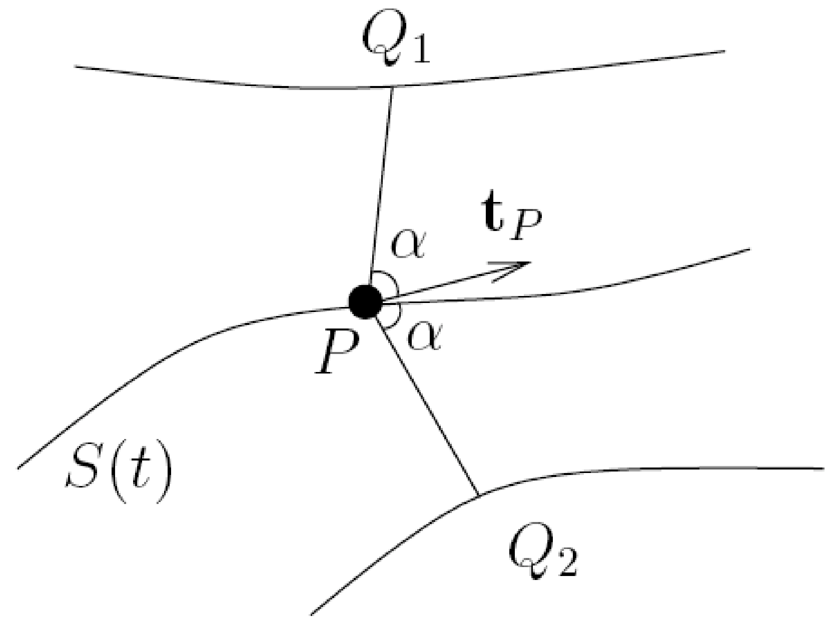


Adding the above two equations we have

$$\int_{R_{1t}} \operatorname{div}(\dot{\mathbf{q}})\, \mathrm{d}v + \int_{R_{2t}} \operatorname{div}(\dot{\mathbf{q}})\, \mathrm{d}v =$$

$$\int_{C_{1t}} \langle\, \dot{\mathbf{q}}, \mathcal{N}\,\rangle\ \mathrm{d}s\ + \int_{C_{2t}} \langle\, \dot{\mathbf{q}}, \mathcal{N}\,\rangle\ \mathrm{d}s\ +$$

$$\int_{C'_{3t}} \langle\, \dot{\mathbf{q}}, \mathcal{N}\,\rangle\ \mathrm{d}s\ + \int_{-C''_{3t}} \langle\, \dot{\mathbf{q}}, \mathcal{N}\,\rangle\ \mathrm{d}s.$$



It is a standard property that the tangent to the skeleton bisects the the angle between $\dot{\mathbf{q}}_+$ and $\dot{\mathbf{q}}_-$ at a skeletal point (see Figure 2). Thus, on C_3 we have

$$\langle\, \dot{\mathbf{q}}_+, \mathcal{N}_+ \,\rangle = \langle\, \dot{\mathbf{q}}_-, \mathcal{N}_- \,\rangle\,, \tag{2}$$

where $\mathcal{N}_+, \mathcal{N}_-$ denote the normals to C_3 from above and from below, respectively. Thus, one can take the limit as $t \rightarrow 0$ of both sides of the above equation to obtain the following extension of the divergence theorem

(extended) Divergence Theorem

Theorem 1 *For a path connected region R which contains part of a skeletal curve S, the divergence of the vector field $\dot{\mathbf{q}}$ is related to its flux through ∂R by the following equation*

$$\int_{R=R_1\cup R_2} \mathrm{div}(\dot{\mathbf{q}})\,\mathrm{d}v =$$

$$\int_{\partial R} \langle\, \dot{\mathbf{q}}, \mathcal{N}\,\rangle\; \mathrm{d}s + 2\int_{C_3} \langle\, \dot{\mathbf{q}}, \mathcal{N}_{C_3}\,\rangle\; \mathrm{d}s.$$

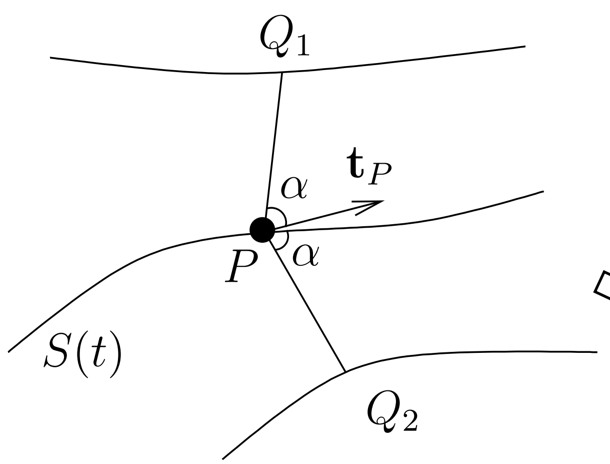
Corollary. *The OF for a region shrinking to a skeleton point satisfies:*

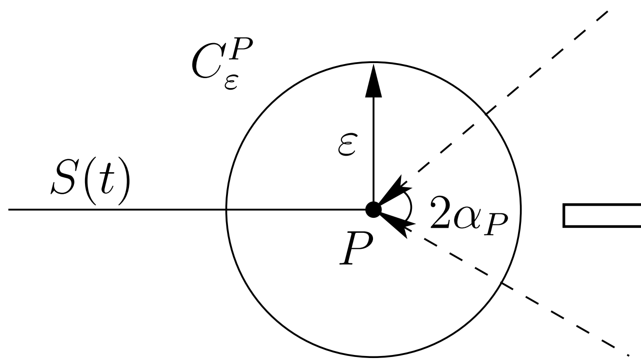
$$\lim_{R\to P} \mathrm{OF}_R \longrightarrow 2\left(\langle \nabla D(P),\ \mathcal{N}\rangle\right) \mathrm{length}(C_3)$$

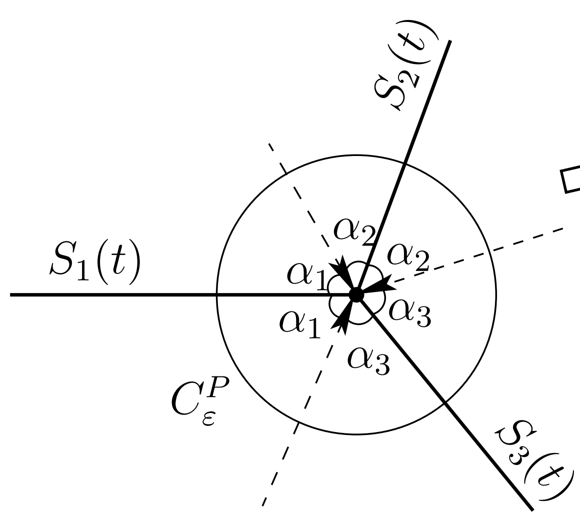
Circular Neighborhoods

(Dimitrov, Damon, Siddiqi, CVPR'03)

The AOF for shrinking circular regions:

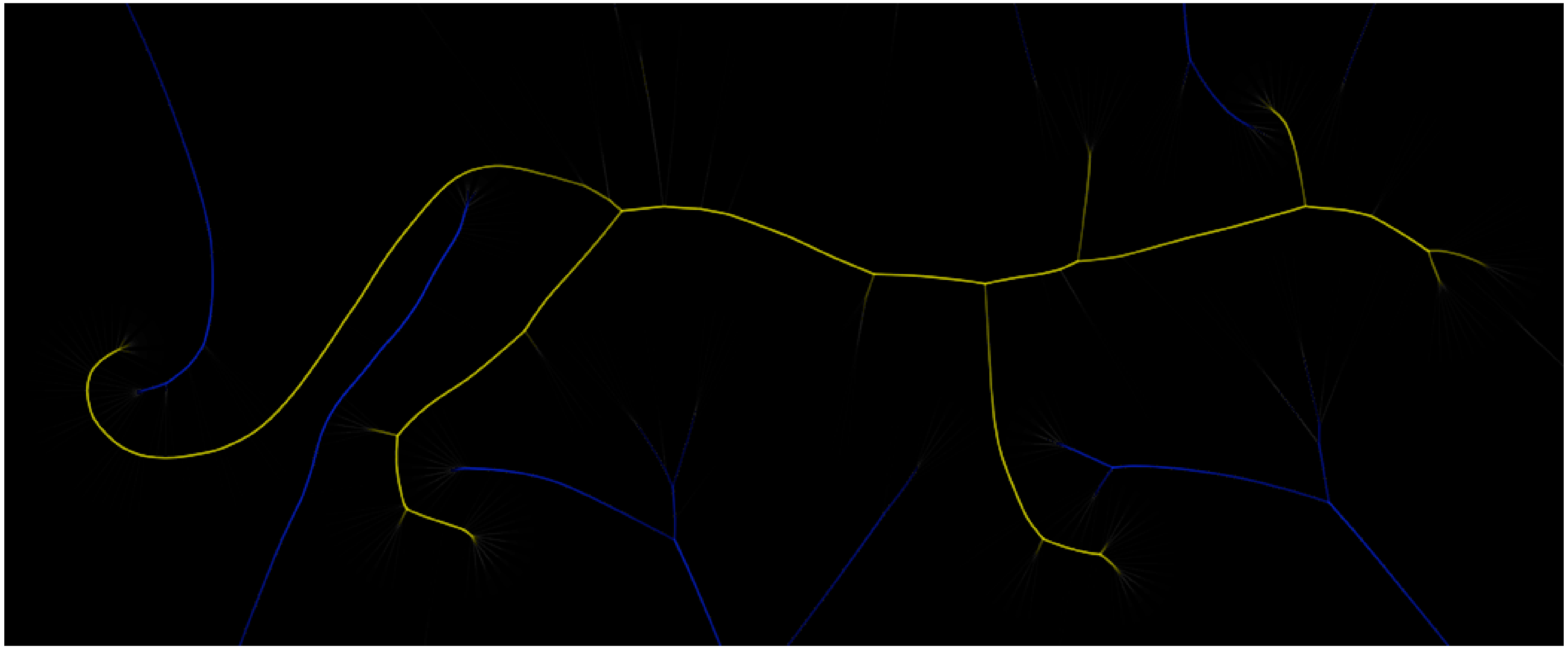






$R \to P$	$\mathrm{AOF}_R \to x$
Regular Points	$-\frac{2}{\pi}\sin\alpha$
End-Points	$-\frac{1}{\pi}(\sin\alpha + \alpha)$
Junction Points	$-\frac{1}{\pi}\sum_{i=1}^{n}\sin\alpha_i$
Non-Skeletal Points	0

Average Outward Flux



Damon: Skeletal Structures

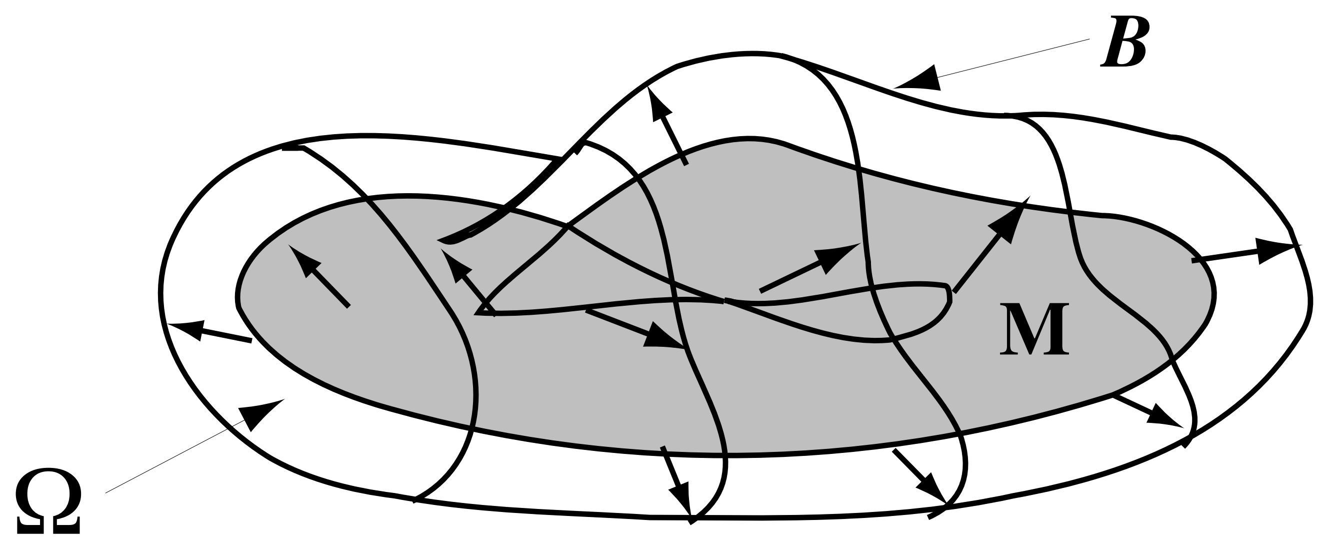


FIGURE 2. A Skeletal Structure (M, U) defining a region Ω with smooth boundary $\mathcal{B}$

radial shape operator

$$S_{rad}(v) \quad = \quad -\mathrm{proj}_U(\frac{\partial U_1}{\partial v})$$

radial curvature

$$K_{rad} = \det(S_{rad})$$

Damon: Radial Flow

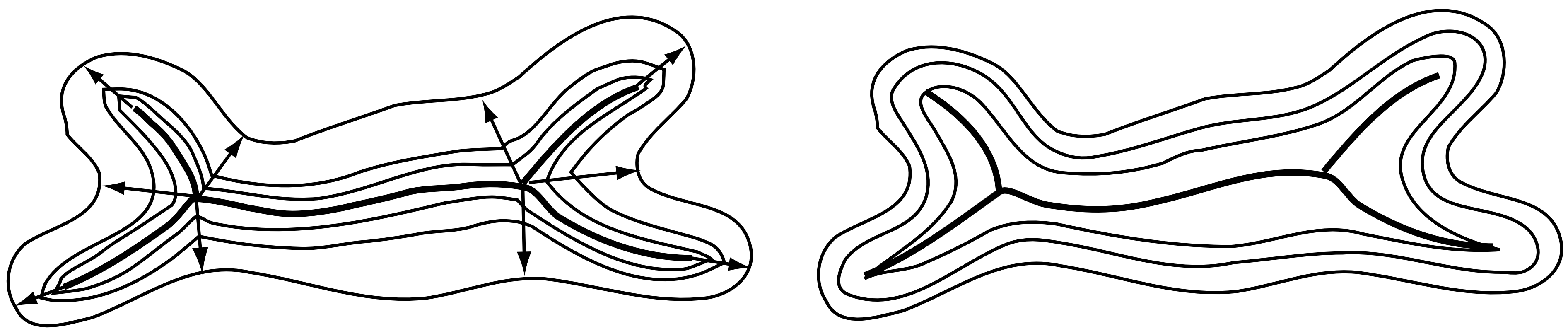


Figure 3. a) Radial Flow and b) Grassfire Flow

- radial curvature condition + edge condition + compatibility condition ensure smoothness of boundary
- complete characterization of local and relative differential geometry of boundary in terms of radial shape operator on skeletal structure

(Rigorous) Divergence Theorem

Theorem 9 (Modified Divergence Theorem)**.** *Let* Ω *be a region with smooth boundary* $\mathcal{B}$ *defined by the skeletal structure. Also, let* Γ *be a region in* Ω *with regular piecewise smooth boundary. Suppose* F *is a smooth vector field with discontinuities across* M*, then*

$$\int_{\Gamma} div\, F\, dV \;=\; \int_{\partial\Gamma} F \cdot \mathbf{n}_{\Gamma}\, dS \;-\; \int_{\tilde{\Gamma}} c_F\, dM \tag{7.1}$$

where $\tilde{\Gamma} = \tilde{M} \cap \pi^{-1}(M \cap \Gamma)$*.*

$\mathrm{proj}_{TM}(F) = c_F \cdot U_1$, where proj_{TM} denotes projection onto U along TM

Boundary Integrals as Medial Integrals

Theorem 1. *Suppose (M, U) is a skeletal structure defining a region with smooth boundary $\mathcal{B}$ and satisfying the partial Blum condition. Let $g : \mathcal{B} \to \mathbb{R}$ be Borel measurable and integrable with respect to the Riemannian volume measure. Then,*

$$\int_{\mathcal{B}} g \, dV \quad = \quad \int_{\tilde{M}} \tilde{g} \cdot \det(I - rS_{rad}) \, dM \tag{3.1}$$

where $\tilde{g} = g \circ \psi_1$.

Algorithms

Algorithm

Algorithm 2: Topology Preserving Thinning.

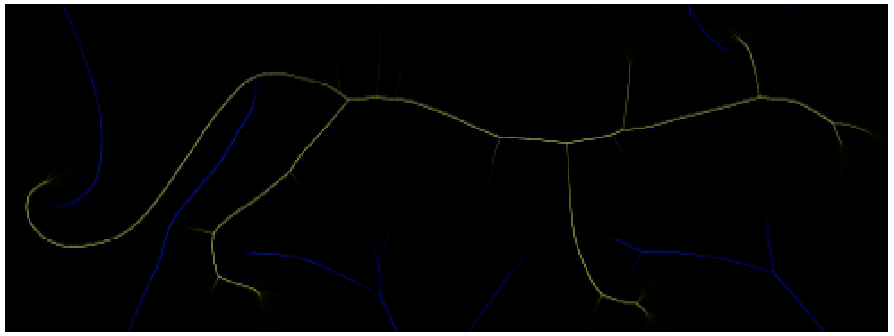
Data : Object, Average Outward Flux Map.
Result : (2D or 3D) Skeleton.

```
for (each point x on the boundary of the object) do
    if (x is simple) then
        insert(x, maxHeap) with AOF(x) as the sorting key for insertion;
while (maxHeap.size > 0) do
    x = HeapExtractMax(maxHeap);
    if (x is simple) then
        if (x is an end point) and (AOF(x) < Thresh) then
            mark x as a medial surface (end) point;
        else
            Remove x;
            for (all neighbors y of x) do
                if (y is simple) then
                    insert(y, maxHeap) with AOF(y) as the sorting key for in-
                    sertion;
```

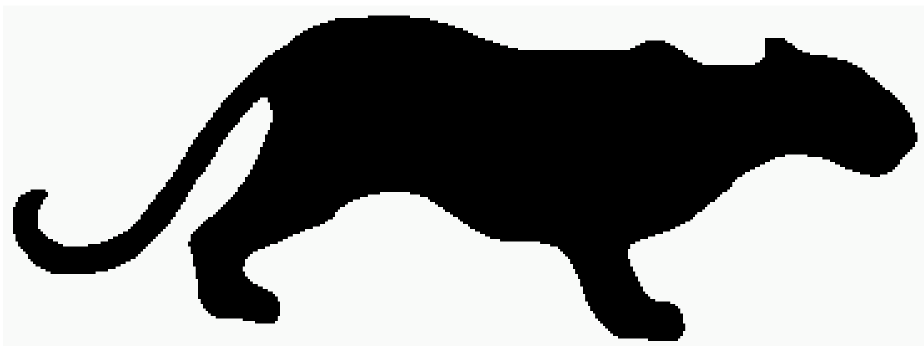
Validation

To verify the theoretical results, boundary points corresponding to regular skeletal points are reconstructed according to: $Q_{1,2} = P + rR(\pm\alpha)\mathbf{t}_P$

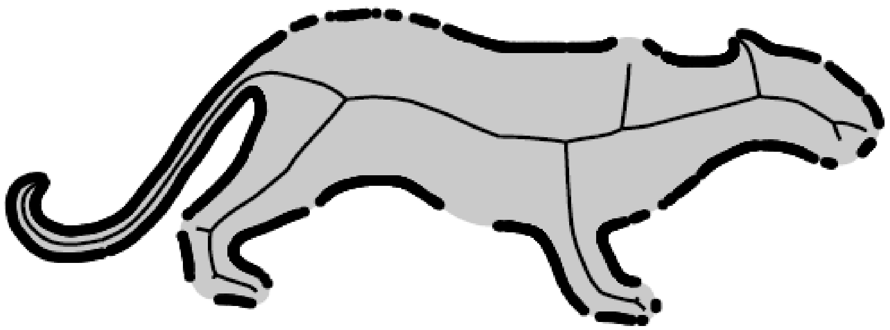
STEP 1. Start with a binary shape.



STEP 2. Compute AOF of shape using circular regions.

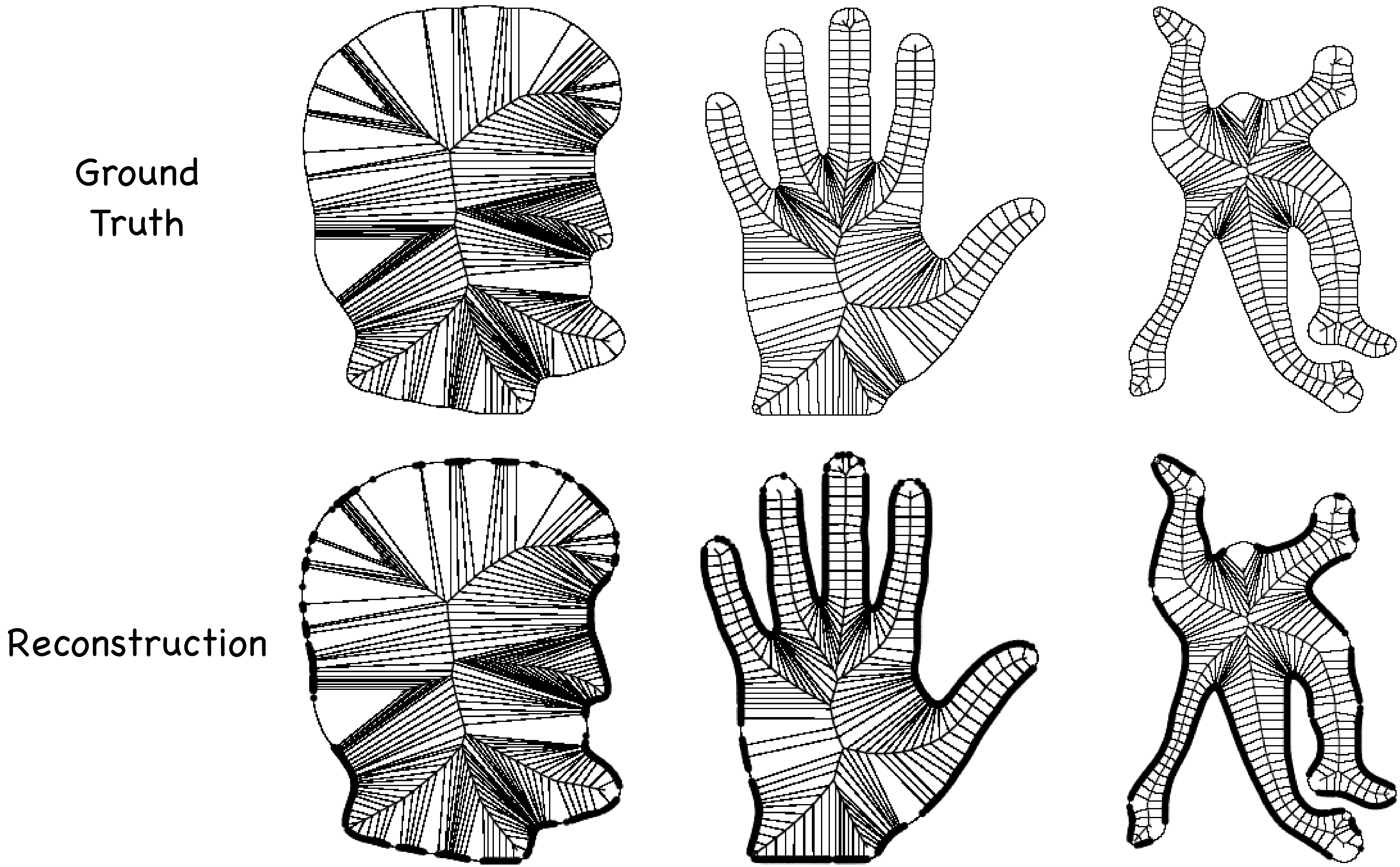


STEP 3. Compute skeleton with algorithm presented in [3].



STEP 4. Using our results for shrinking circular regions, reconstruct boundary points from regular skeletal points.

Validation



The limiting average outward flux value determines the object angle, which in turn is used to recover the associated bi-tangent points, shown as filled circles.

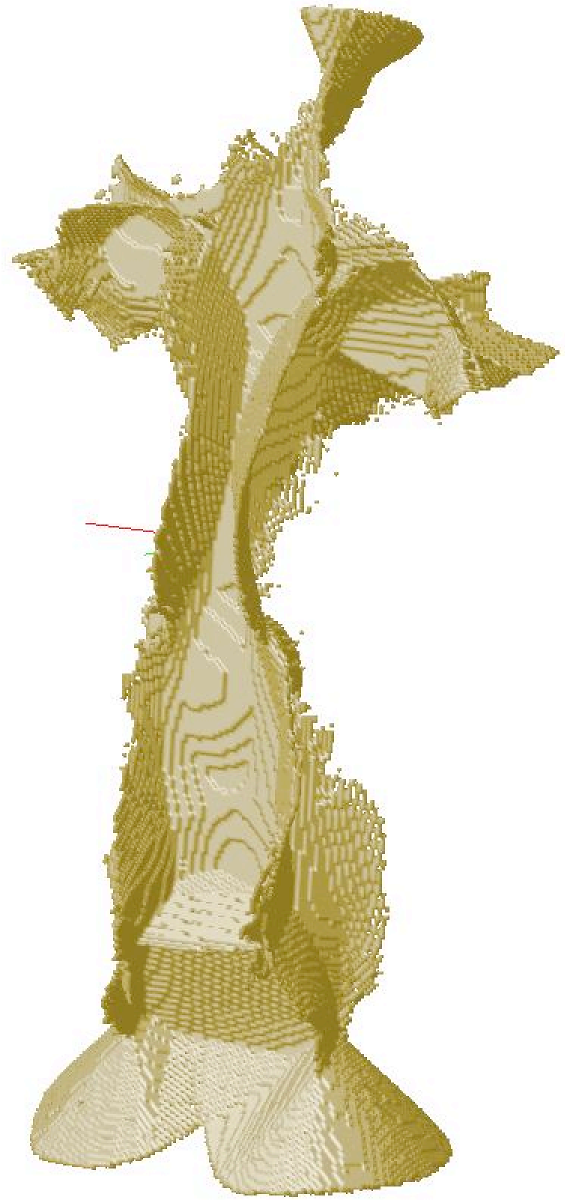
Brain Ventricles

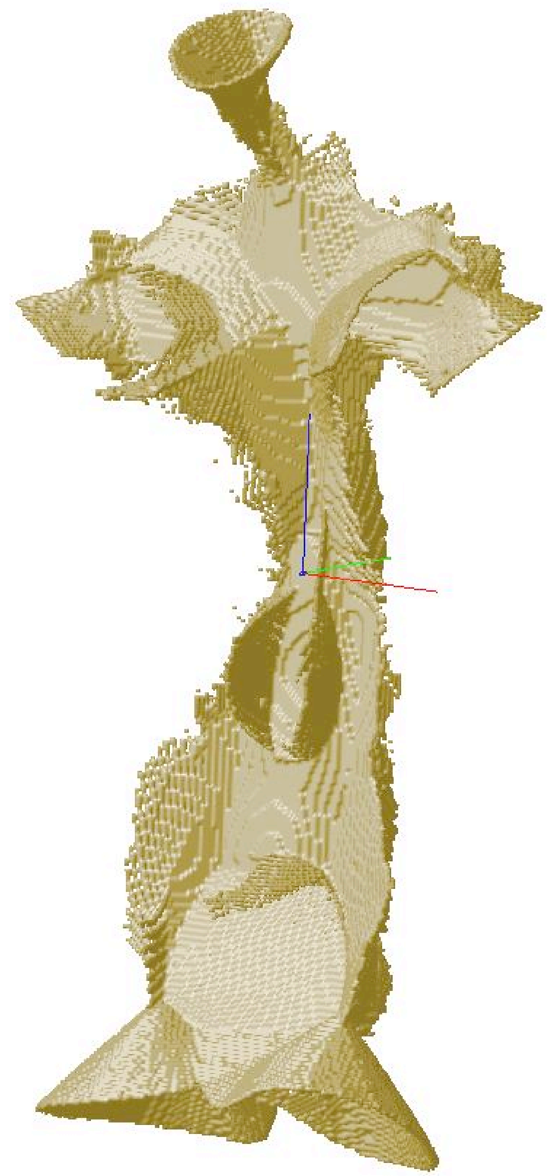
Original

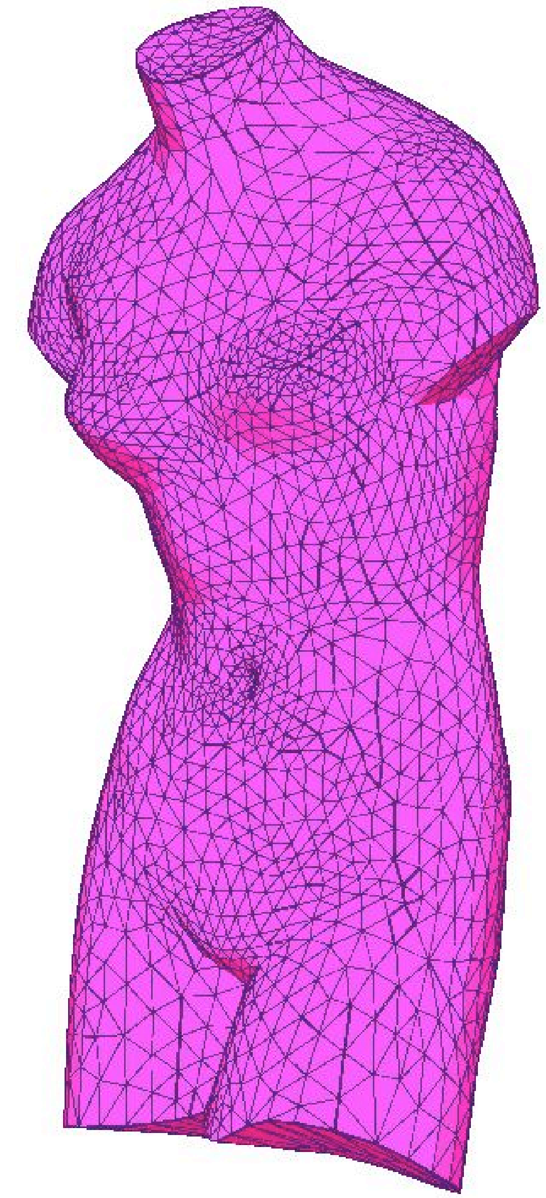
Medial Surface

Venus de Milo

Circa 100 BC

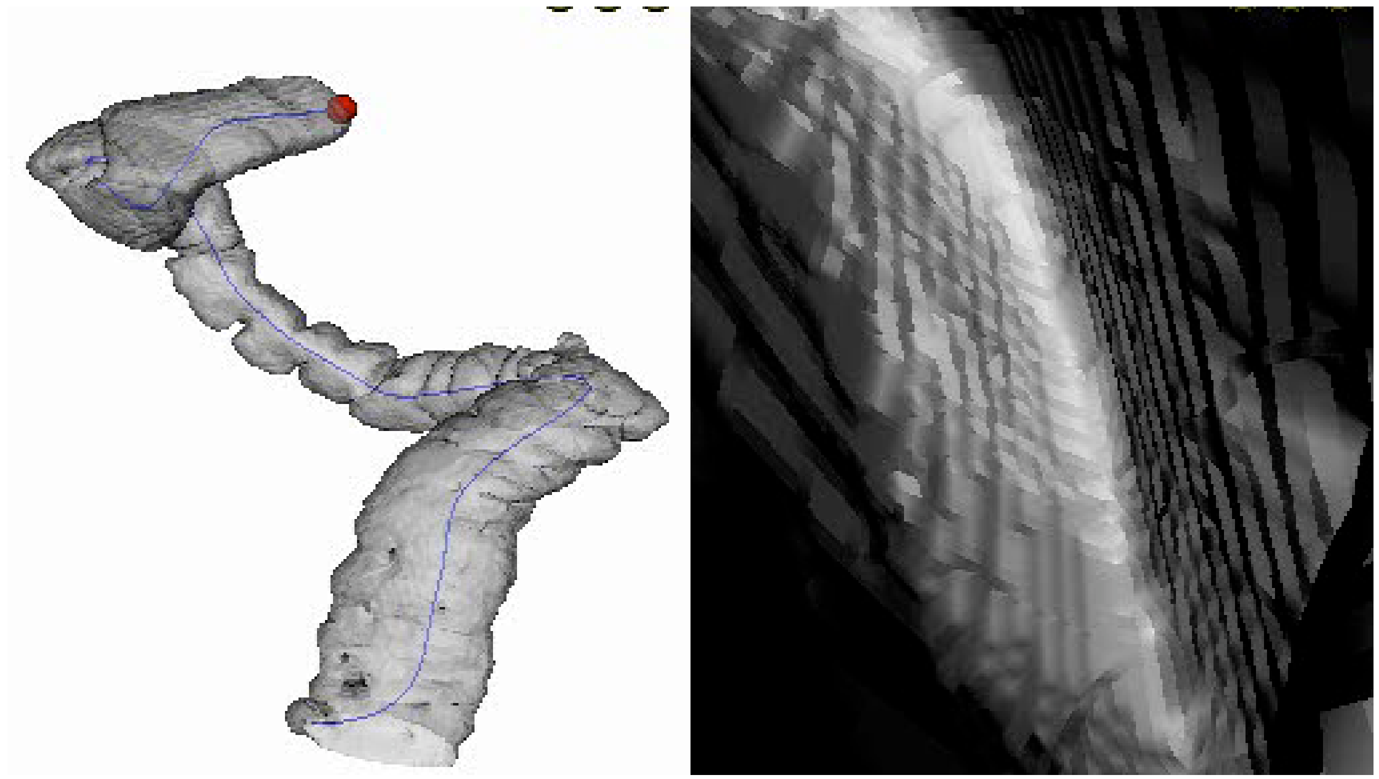




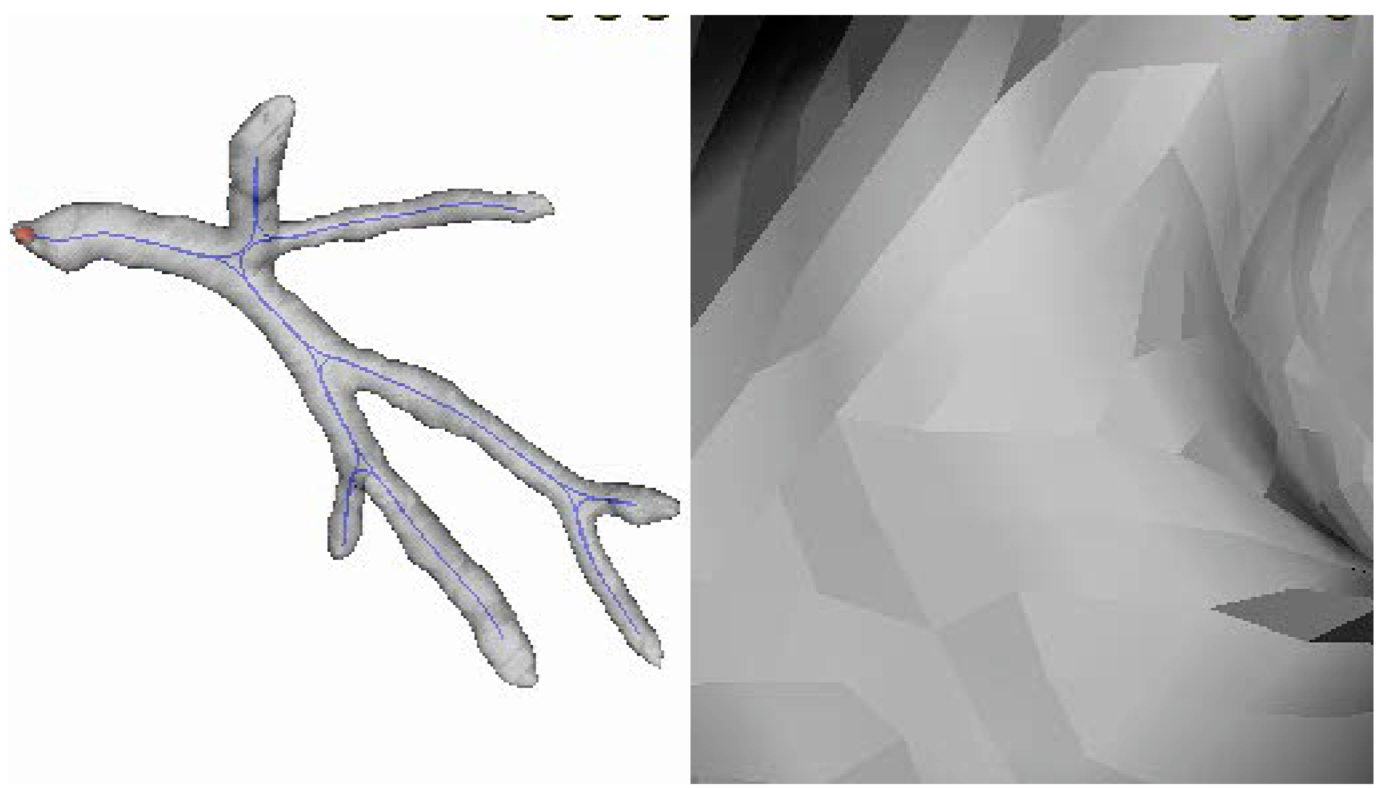


Applications

Virtual Endoscopy

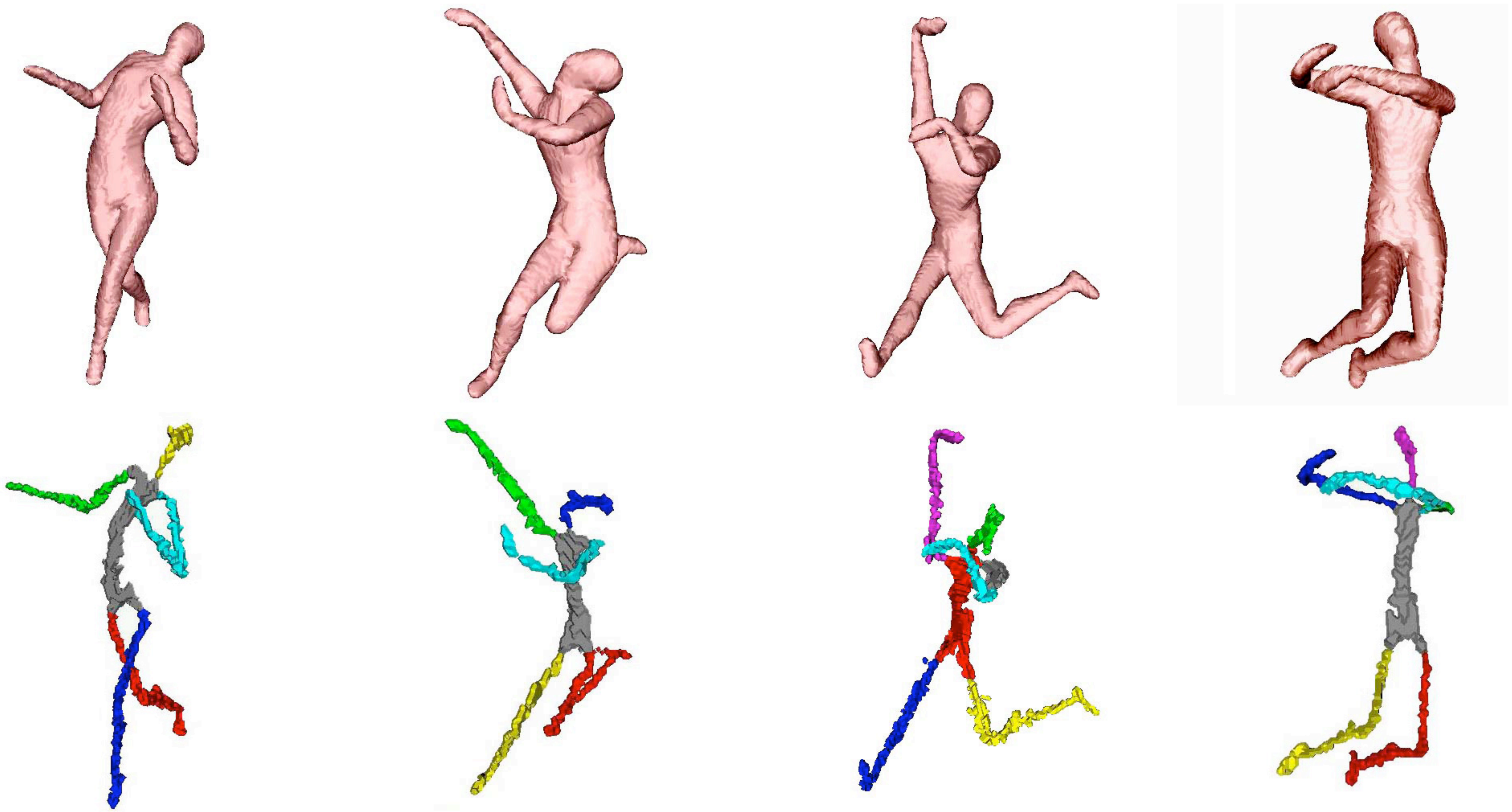


Colon



Arteries

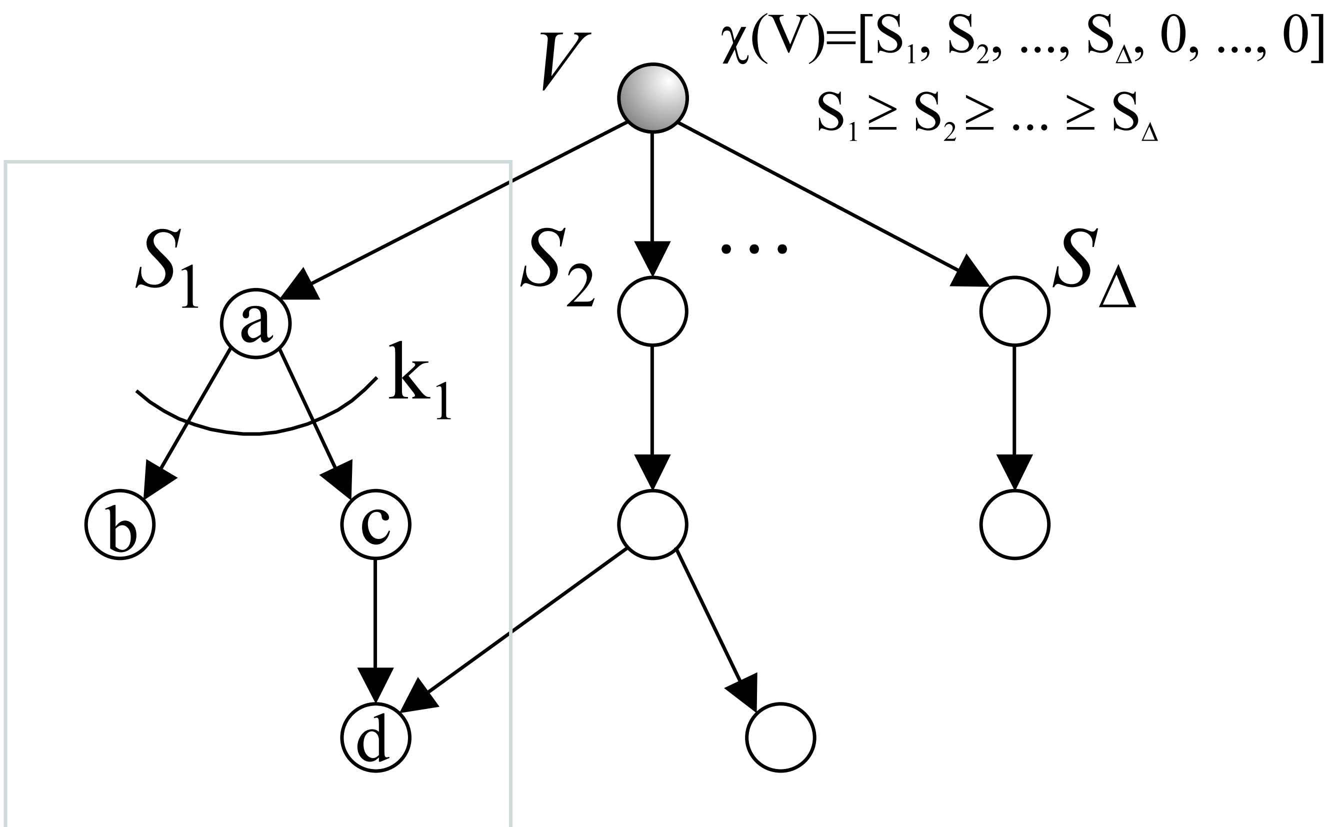
3D Medial Graph Matching



Medial Graph Matching

- Edit Distance Based Approaches
  (Sebastian, Kline, Kimia; Hancock, Torsello)
  - motivated by string edit distances
  - polynomial time algorithm for trees, (but need to define edit costs)
- Maximum Clique Approaches
  (Pelillo et al.)
  - subgraph isomorphism -> maximum clique in an association graph
  - discrete combinatorial problem -> continuous optimization
- Graph Spectra-Based Approaches
  (Shokoufandeh et al.)
  - eigenvalue analysis of adjacency matrix for DAGs
  - separation of "topology" and "geometry"
  - extension to handle indexing

A Topological Signature Vector



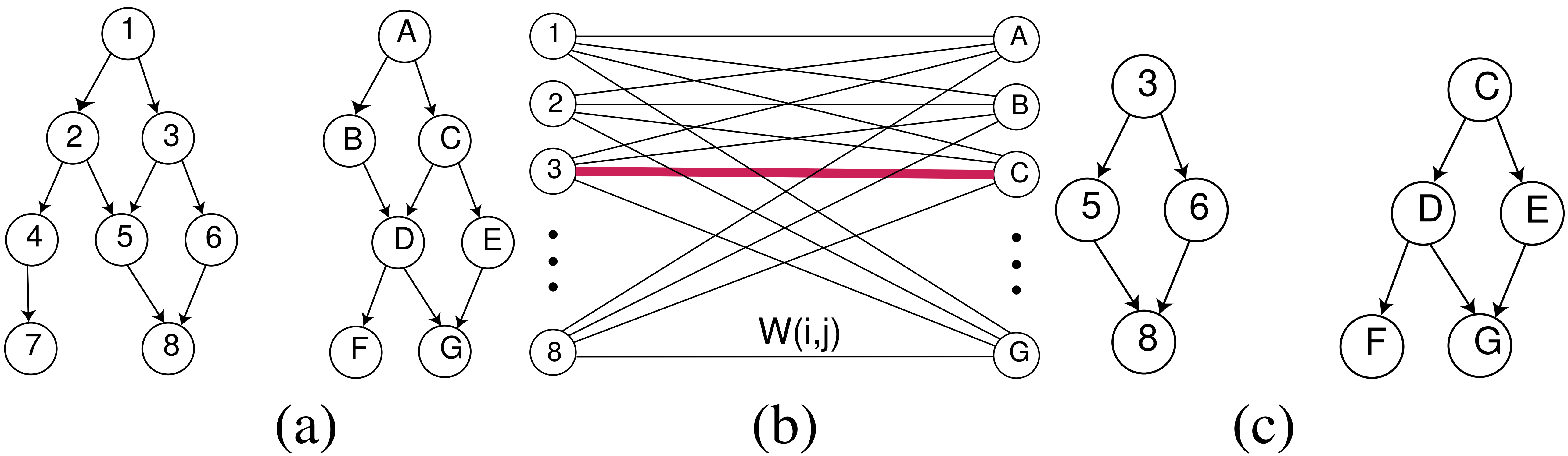
$$
\begin{array}{c|cccc}
 & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\hline
\mathbf{a} & 0 & 1 & 1 & 0 \\
\mathbf{b} & -1 & 0 & 0 & 0 \\
\mathbf{c} & -1 & 0 & 0 & 1 \\
\mathbf{d} & 0 & 0 & -1 & 0
\end{array}
$$

S_1 Antisymmetric Adjacency Matrix

$$S_i = |\lambda_1| + |\lambda_2| + \ldots + |\lambda_{k_i}|$$

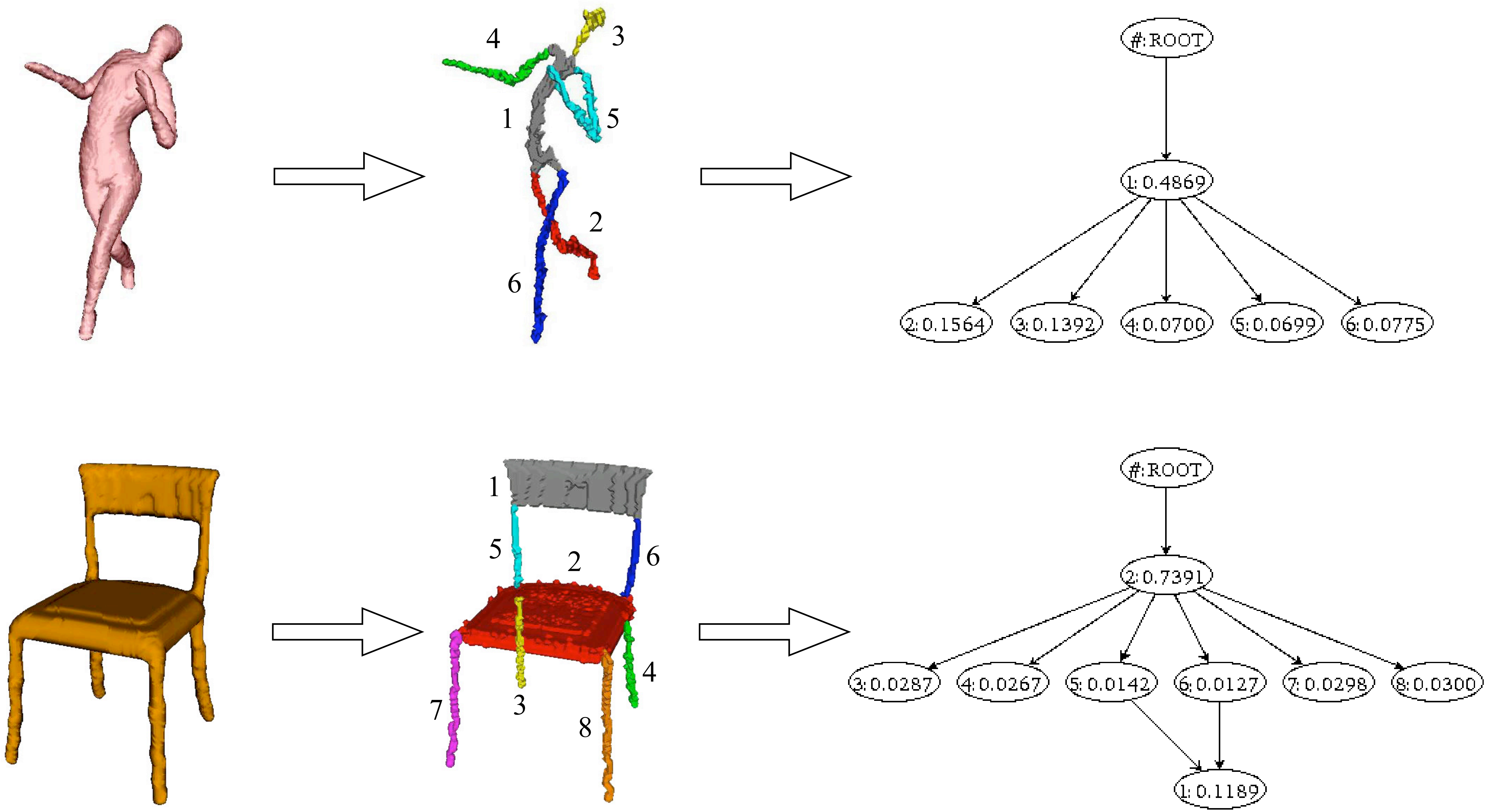
- At node "a" compute the sum of the magnitudes of the "k" largest eigenvalues of the adjacency matrix of the subgraph rooted at "a".
- Carry out this process recursively at all nodes.
- The sorted sums become the components of the "TSV" assigned to node V.

Matching Algorithm



- (a) Two DAGs to be matched.
- (b) A bi-partite graph is formed, spanning their nodes but excluding their edges. The edge weights W(i,j) in the bi-partite graph encode node similarity as well as TSV similarity. The two most similar nodes are found, and are added to the solution set of correspondences.
- (c) This process is applied, recursively, to the subgraphs of the two most similar nodes. This ensures that the search for corresponding nodes is focused in corresponding subgraphs, in a top-down manner.

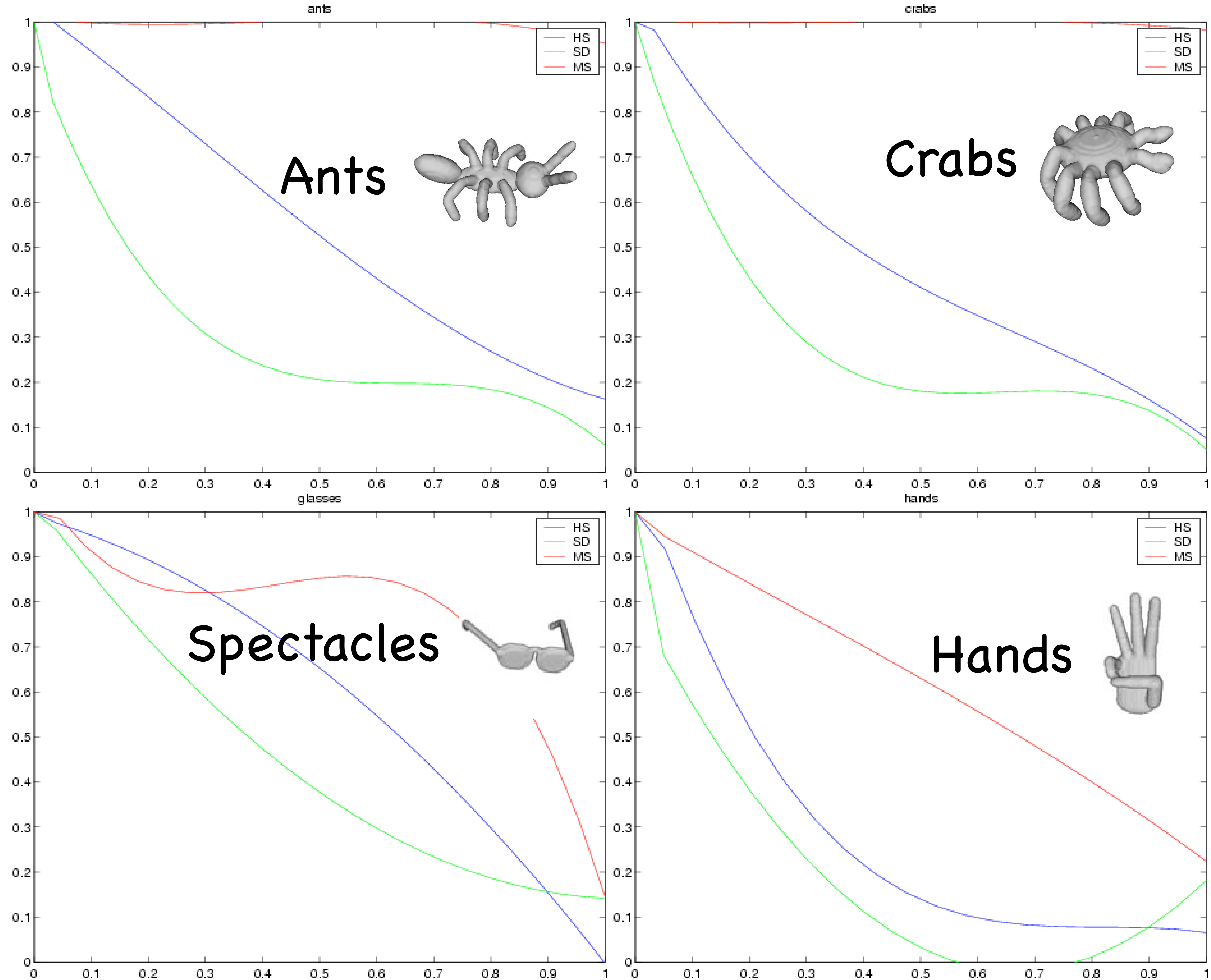
Medial Surfaces to DAGs

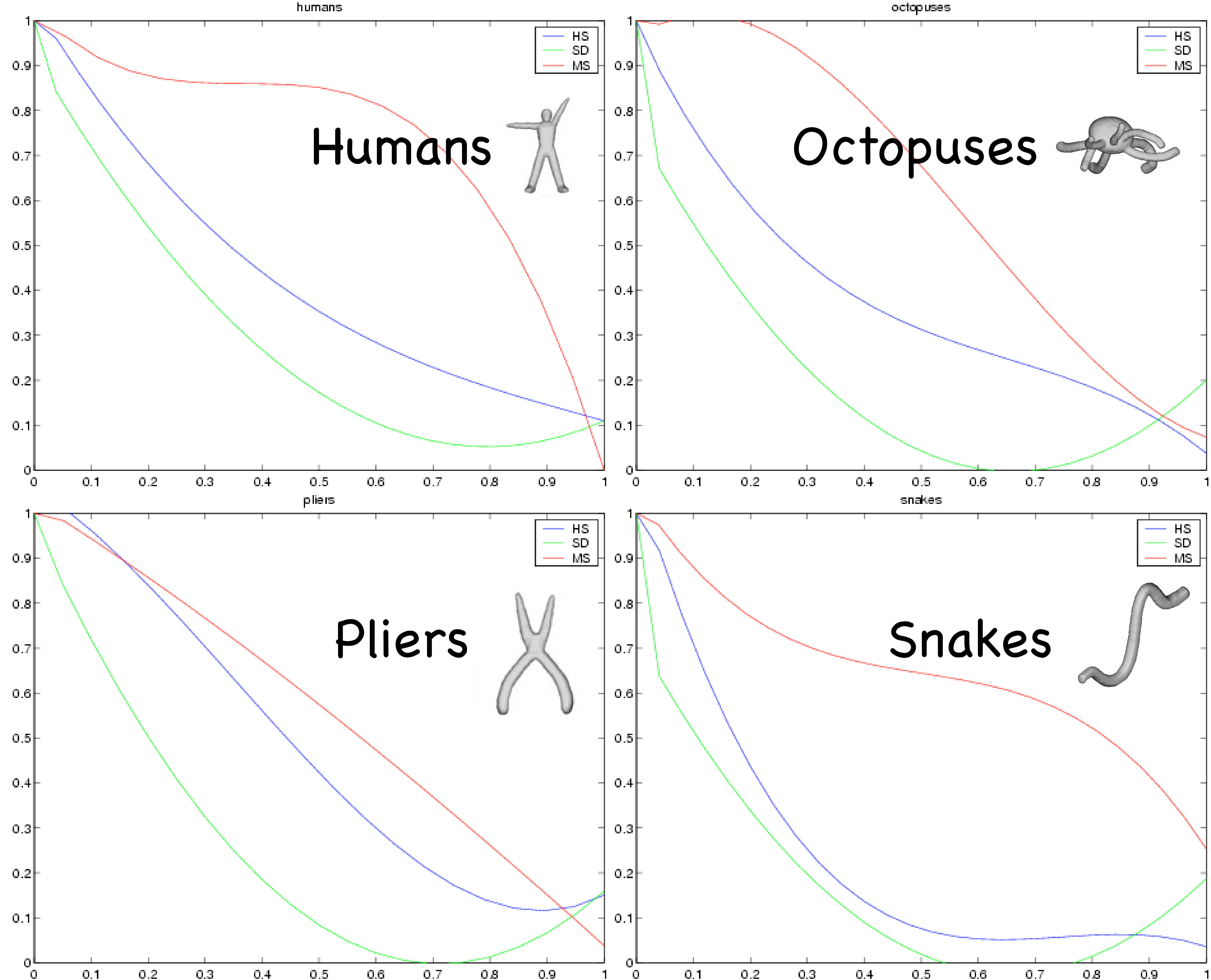


(Malandain, Bertrand, Ayache, IJCV'03)

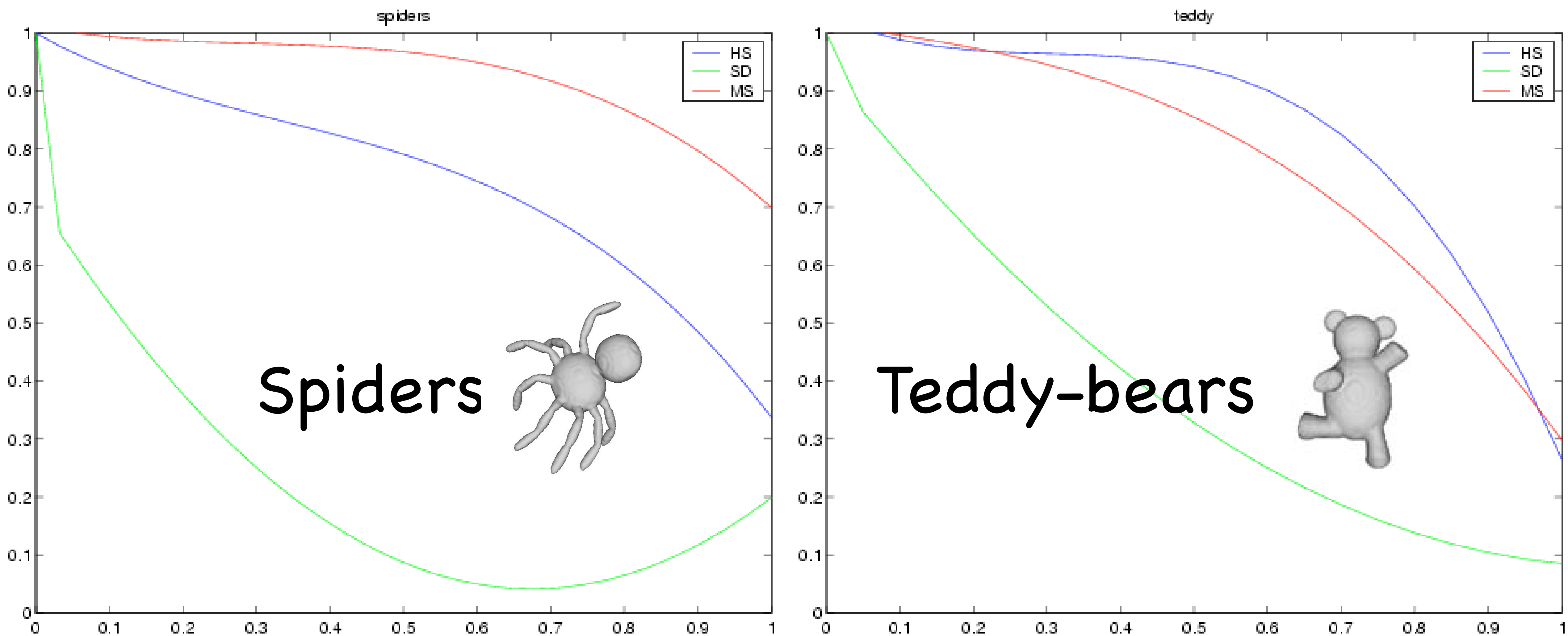
3D Object Models: The McGill Shape Benchmark

- 420 models reflecting 18 object classes
- Severe Articulation: hands, humans, teddy-bears, eyeglasses, pliers, snakes, crabs, ants, spiders, octopuses
- Moderate or No Articulation: planes, birds, chairs, tables, cups, dolphins, four-limbed animals, fish
- Precision Vs Recall Experiments: shape distributions of Osada et al. (SD), harmonic spheres of Kazhdhan et al. (HS) and medial surfaces (MS).

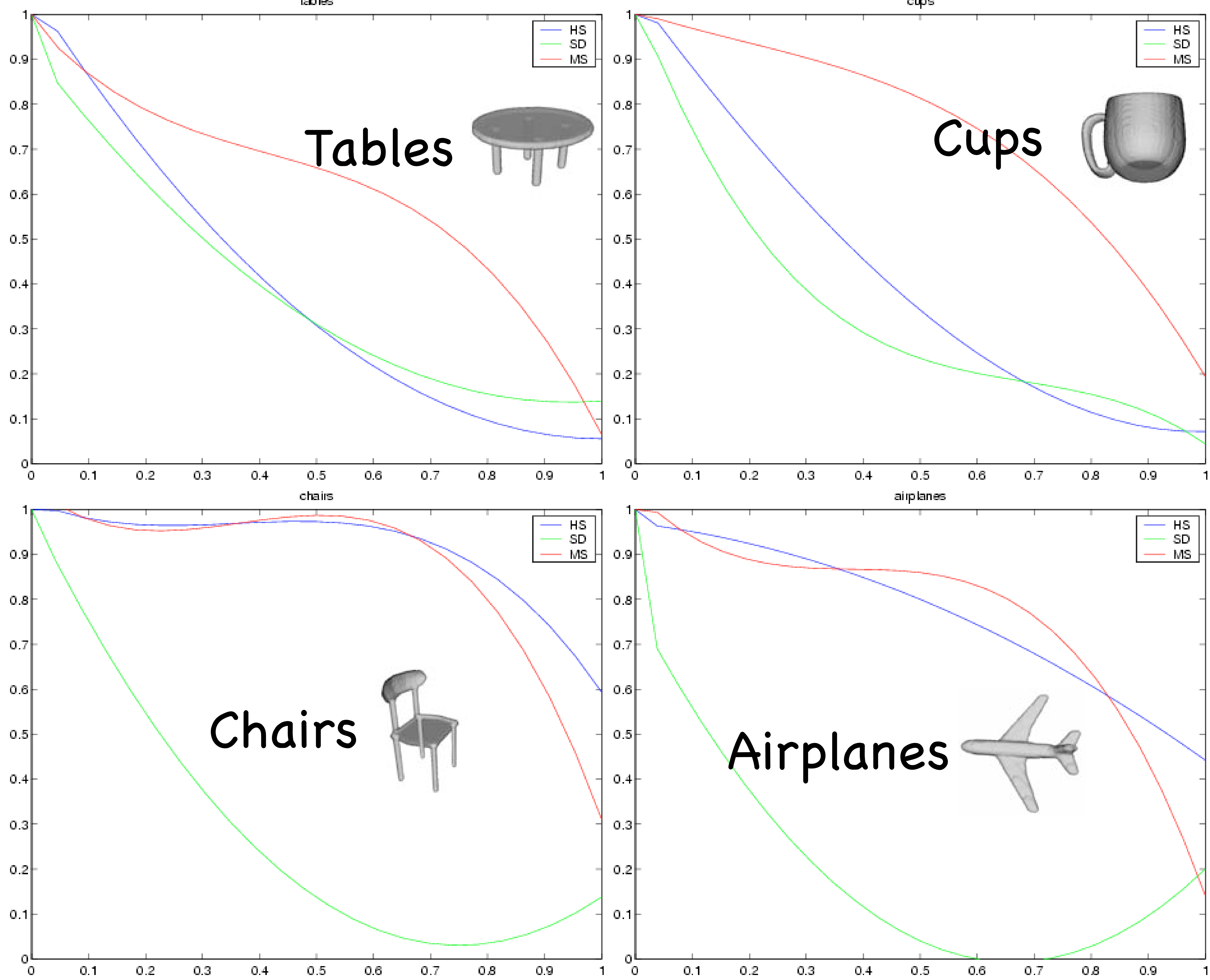

ants
HS
SD
MS
Ants
crabs
HS
SD
MS
Crabs
glasses
HS
SD
MS
Spectacles
hands
HS
SD
MS
Hands
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



humans
HS
SD
MS
Humans
octopuses
HS
SD
MS
Octopuses
pliers
HS
SD
MS
Pliers
snakes
HS
SD
MS
Snakes
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1



spiders
HS
SD
MS
Spiders
teddy
HS
SD
MS
Teddy-bears


tables
cups
chairs
airplanes
HS
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MS
Tables
Cups
Chairs
Airplanes
0
0.1
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0.8
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1

Summary

Medial Representations

Mathematics, Algorithms and Applications

Kaleem Siddiqi and Stephen M. Pizer
Springer (in press, 2006)

- Chapter 1: Pizer, Siddiqi, Yushkevich: “Introduction”
- PART 1 - MATHEMATICS
- Chapter 2: Giblin, Kimia: “Local Forms and Transitions of the Medial Axis”
- Chapter 3: Damon: “Geometry and Medial Structure”
- PART 2 - ALGORITHMS
- Chapter 4: Siddiqi, Bouix, Shah: “Skeletons Via Shocks of Boundary Evolution”
- Chapter 5: Borgefors, Nystrom, Sanniti di Baja: “Discrete Skeletons from Distance Transforms.”

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- Chapter 6: Szekely: "Voronoi Skeletons"
- Chapter 7: Amenta and Choi: "Voronoi Methods for 3D Medial Axis Approximation"
- Chapter 8: Pizer et al.: "Synthesis, Deformation and Statistics of 3D Objects Via M-reps"
- PART 3 - APPLICATIONS
- Chapter 9: Pizer et al.: "Statistical Applications with Deformable M-Reps"
- Chapter 10: Siddiqi et al: "3D Model Retrieval Using Medial Surfaces"
- Chapter 11: Leymarie, Kimia: "From the Infinitely Large to the Infinitely Small"

Selected References

- Bouix, Siddiqi, "Optics, Mechanics and Hamilton-Jacobi Skeletons" [Advances in Imaging and Electron Physics, 2005]
- Damon, "Global Geometry of Regions and Boundaries via Skeletal and Medial Integrals" [preprint, 2003]
- Dimitrov, "Flux Invariants for Shape" [M.Sc. thesis, McGill, 2003]
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- Pelillo, Siddiqi, Zucker, "Matching Hierarchical Structures Using Association Graphs" [ECCV'98, PAMI'99]

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- Shokoufandeh, Macrini, Dickinson, Siddiqi, Zucker, "Indexing Hierarchical Structures Using Graph Spectra" [CVPR'99, PAMI'05]
- Siddiqi, Bouix, Tannenbaum, Zucker, "Hamilton-Jacobi Skeletons" [ICCV'99, IJCV'02]
- Siddiqi, Shokoufandeh, Dickinson, Zucker, "Shock Graphs and Shape Matching" [ICCV'98, IJCV'99]
- Stolpner, Siddiqi "Revealing Significant Medial Structure in Polyhedral Messhes" [3DPVT'06]