MR Image Reconstruction from Sparse Radial Samples Using Bregman Iteration and Inverse Scale Space Methods

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Practical Motivation

- Imaging speed has been the major drawback of MRI $\Rightarrow$ sample the frequency plane sparsely;
- Sparse sampling $\Rightarrow$ in image artifacts and/or low signal to noise ratio (SNR)
  - Backprojection method
  - Gridding method
  - ...

A theory by Candes, Romberg and Tao'04

Consider

- a discrete complex signal f of length N
- a randomly chosen set of frequencies Ω of mean size τN, where $\tau < 1$
- $\#\{t,\ f(t) \neq 0\} \leq \alpha(M) \cdot (\log N)^{-1} \cdot \#\Omega$, $\forall M > 0$

Then

- with the probability at least $1 - O(N^{-M})$,
- f can be reconstructed exactly from

$$\min_g \sum_{t=0}^{N-1} |g(t)|, \text{ s.t. } \hat{g}(\omega) = \hat{f}(\omega) \text{ for all } \omega \in \Omega.$$

An Extension to Signals that have Sparse Representation

$$\hat{g} = \arg\min_{g} ||\psi(g)||_1 \text{ s.t. } FFT(g)|_\Omega = y.$$

- g is the reconstructed image,
- ψ transforms g into a sparse representation,
- y represents the sampled data on Ω,
- solved with an iterative scheme with a projection on the constrained set Ω.

Candes, Romberg & Tao '05; Candes & Romberg '05

Sparse representation for piecewise constant image

- TV regularization

$$\hat{g} = \arg\min_g ||g||_{BV} \text{ s.t. } FFT(g)|_\Omega = y,$$

where $||g||_{BV} := \int |\nabla g| dx.$

- The wavelet transform

$$\hat{g} = \arg\min_g ||g||_W \text{ s.t. } FFT(g)|_\Omega = y,$$

where $||g||_W$ is the summation of the wavelet coefficients after the wavelet transformation.

Numerical phantom results from
Candes et al.

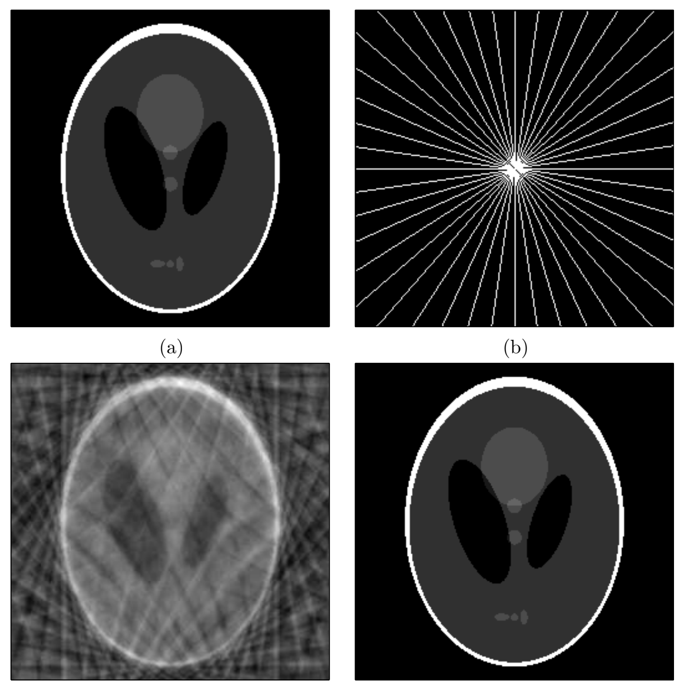


Figure 1: Example of a simple recovery problem. (a) The Logan-Shepp phantom test image. (b) Sampling 'domain' in the frequency plane; Fourier coefficients are sampled along 22 approximately radial lines. (c) Minimum energy reconstruction obtained by setting unobserved Fourier coefficients to zero. (d) Reconstruction obtained by minimizing the total-variation, as in (1.1). The reconstruction is an exact replica of the image in (a).

Previous work:

- Minimizing l^1 norms: *Dobson & Santosa '96; Donoho & Elad '03 ...*
- Sparse representations: *Feuer & Nemirovsky '02; Gribonval & Nielson '04 ...*
- Uncertainty principles: *Donoho & Huo '01; Elad & Brukstein '02 ...*

Challenges

- The raw data is not on uniform grids $\Rightarrow$ FFT is not a good operator to enforce the constraint;
- The target image is sometimes a piecewise smooth image $\Rightarrow$ TV or the wavelet transform alone is not a good sparse representation.

Solutions

- A non-uniform fast Fourier transform applied;
  - NFFT in C++ by *Kunis & Potts '04*
  - NUFFT in Matlab by *Fessler & Sutton '03*
- Wavelet + TV as an approximate sparse representation of a piecewise smooth image.

Lustig, Lee, Donoho & Pauly '05 ISMRM; Lustig, Donoho & Pauly '06 ISMRM; Plett, Guarini, & Irarrazaval '06

Our Model

$$\min_{m} \left\{ \mu ||m||_{BV} + \nu ||m||_{W} + \lambda ||Am - y||_2^2 \right\}, \tag{1}$$

- y is the known undersample data in k-space;
- A is the non-uniform FFT operator;
- μ, ν and λ are non-negative parameters;
  - $\mu = 0 \Rightarrow$ wavelet transformation;
  - $\nu = 0 \Rightarrow$ total variation regularization;
  - Balance μ and ν for piecewise smooth images;
- solved with conjugate gradient descent method and back tracking line search.

More novelties

- Bregman iteration is applied to obtain finer scales; *Bregman ’67; Osher, Burger, Goldfarb, Xu & Yin ’05; He, Marquina & Osher ’05; He, Burger & Osher 06’...*
- Inverse scale space flow method is also experimented; *Burger, Gilboa, Osher & Xu ’06; Xu & Osher ’06...*
- Curvelet + TV is under experimenting to obtain sharper edges. *Candes & Donoho ’02; Candes & Guo ’02; Candes, Demanet, Donoho & Ying ’05 ...*

The Bregman Distance

- Given a convex function φ, the Bregman distance is defined by

$$D_\varphi(u,v) = \varphi(u) - \varphi(v) - < u - v, \partial\varphi(v) >,$$

where $< \cdot, \cdot >$ denotes the inner product in $\mathbb{R}^n$ and $\partial\varphi(y)$ is an element of the sub-gradient of φ at point y.
$\varphi(u) = \int |\nabla u|^2 dx$, $D_\varphi(u,v) = \int |\nabla(u-v)|^2 dx$.

- In general, $D_\varphi(u,v) \neq D_\varphi(v,u)$ and also the triangle inequality does not hold.

- However, $D_\varphi(u,v) \geq 0$ and $D_\varphi(u,v) = 0$ if $u = v$ (if and only if for strictly convex functionals).

The Iterative Refinement Procedure for ROF model

Introduced by Osher et al. for Image Denoising

- ROF model: $\left\{ \min_{u \in BV(\Omega)} ||u||_{BV} + \lambda ||u - f||_{L^2}^2 \right\}$;
- Denote $J(u) = ||u||_{BV}$, $u_0 = 0$ and $v_0 = 0$, for $n > 0$, the nth Bregman iteration is defined as

$$u_n = \arg \min_{u \in BV(\Omega)} \left\{ D_J(u, u_{n-1}) + \lambda ||u - f||_{L^2}^2 \right\},$$

$$\Leftrightarrow u_n = \arg \min_{u \in BV(\Omega)} \left\{ J(u) + \lambda ||u - f - v_{n-1}||_{L^2}^2 \right\},$$

where $v_n = f + v_{n-1} - u_n$.

- It goes on until a stopping criterion is satisfied.

Convergence results and etc.

- Theory proves that $||u_n - f||_{L^2} \leq \sqrt{\frac{J(f)}{n\lambda}}$, if $J(f) < \infty$.
  - noise free image f ;
  - noisy image f.
- Numerical experiments show significant improvement over standard ROF model.
  - Small λ makes the image smooth at earlier iteration;
  - However, image features come back before the noise along the iteration;

Osher, Burger, Goldfarb, Xu & Yin '05; He, Burger & Osher 06'

The Iterative Refinement Procedure for Our Model

- Denote $J(m) = \mu||m||_{BV} + \nu||m||_W$, $m_0 = 0$ and $v_0 = 0$, for $n > 0$, the nth Bregman iteration is defined as

$$m_n = \arg\min_m \left\{ D_J(m, m_{n-1}) + \lambda||Am - y||_{L^2}^2 \right\},$$

$$\Leftrightarrow m_n = \arg\min_m \left\{ J(m) + \lambda||Am - y - v_{n-1}||_{L^2}^2 \right\},$$

where $v_n = y + v_{n-1} - Am_n$.

- Stopping criterion is dependent on the user needs.

Inverse Scale Space Methods

- Divide by $\lambda = \Delta t$, the E-L eqn of (1) becomes

$$\frac{\partial J(m_k^{\Delta t}) - \partial J(m_{k-1}^{\Delta t})}{\Delta t} = -\widehat{A}(Am_k^{\Delta t} - y).$$

- Setting $t_k = k\Delta t$, $p^{\Delta t}(t_k) = \partial J(m_k^{\Delta t})$, $m^{\Delta t}(t_k) = m_k^{\Delta t}$

$$\frac{p^{\Delta t}(t_k) - p^{\Delta t}(t_k - \Delta t)}{\Delta t} = -\widehat{A}(Am^{\Delta t}(t_k) - y).$$

- Letting $\Delta t \to 0$, with initial values $m(0) = p(0) = 0$

$$\partial_t p(t) = -\widehat{A}(Am(t) - y), \;\; p(t) \in \partial J(m(t)),$$

Burger, Gilboa, Osher & Xu'06

Why Bregman Iteration (1)?

A Simple Example by Meyer for ROF model Assume $f = \alpha\chi_R$ where

$$\chi_R(x,y) = \begin{cases} 1 & \text{if } x^2 + y^2 \le R^2; \\ 0 & \text{else.} \end{cases} \tag{2}$$

then the solution of ROF model

$$u = \begin{cases} 0 & \text{if } \alpha R \le \frac{1}{\lambda}, \\ (\alpha - \frac{1}{\lambda R})\chi_R & \text{else.} \end{cases}$$

An Intuitive Explanation

- If $\lambda\alpha R > 1$, optimal decomposition at the iteration
  - 1st: $f + v_0 = \alpha\chi_R = (\alpha - \frac{1}{\lambda R})\chi_R + \frac{1}{\lambda R}\chi_R := u_1 + v_1$;
  - 2nd: $f + v_1 = (\alpha + \frac{1}{\lambda R})\chi_R = \alpha\chi_R + \frac{1}{\lambda R}\chi_R := u_2 + v_2$.
- Denote $\tilde{n} = \min\{n \in \mathbb{N} | n\lambda\alpha R > 1\}$,
  - 1st: $f + v_0 = \alpha\chi_R = 0 + \alpha\chi_R := u_1 + v_1$;
  - ...
  - nth: $f + v_{n-1} = n\alpha\chi_R = (n\alpha - \frac{1}{\lambda R})\chi_R + \frac{1}{\lambda R}\chi_R := u_n + v_n$;
  - (n+1)th: $f + v_n = (\alpha + \frac{1}{\lambda R})\chi_R = \alpha\chi_R + \frac{1}{\lambda R}\chi_R := u_{n+1} + v_{n+1}$.

Why Bregman Iteration (3)?

When $f = \alpha_1 \chi^{R_1}_{(x_1,y_1)} + \alpha_2 \chi^{R_2}_{(x_2,y_2)}$ Assume that the distance between two circles is large enough that $||f||_* = \max(\frac{\alpha_1 R_1}{2}, \frac{\alpha_2 R_2}{2}) =: \frac{\alpha_1 R_1}{2}$, *Kinderman, Osher & Xu '06*.

- If $\frac{1}{2\lambda} \geq \frac{\alpha_1 R_1}{2} > \frac{\alpha_2 R_2}{2}$, f is decomposed as

$$\begin{cases} u &= 0, \\ v &= \alpha_1 \chi^{R_1}_{(x_1,y_1)} + \alpha_2 \chi^{R_2}_{(x_2,y_2)}; \end{cases} \quad (3)$$

Why Bregman Iteration (3): Continue?

- else if $\frac{\alpha_1 R_1}{2} > \frac{1}{2\lambda} \geq \frac{\alpha_2 R_2}{2}$, then the extreme pair (u, v) is as follows,

$$\begin{cases} u &= (\alpha_1 - \frac{1}{\lambda R_1})\chi^{R_1}_{(x_1,y_1)}, \\ v &= \frac{1}{\lambda R_1}\chi^{R_1}_{(x_1,y_1)} + \alpha_2 \chi^{R_2}_{(x_2,y_2)}; \end{cases} \quad (4)$$

- otherwise $\frac{\alpha_1 R_1}{2} \geq \frac{\alpha_2 R_2}{2} > \frac{1}{2\lambda}$, the optimal decomposition of f is

$$\begin{cases} u &= (\alpha_1 - \frac{1}{\lambda R_1})\chi^{R_1}_{(x_1,y_1)} + (\alpha_2 - \frac{1}{\lambda R_2})\chi^{R_2}_{(x_2,y_2)}, \\ v &= \frac{1}{\lambda R_1}\chi^{R_1}_{(x_1,y_1)} + \frac{1}{\lambda R_2}\chi^{R_2}_{(x_2,y_2)}. \end{cases} \quad (5)$$

Why Bregman Iteration (3)? : Continue

- If we denote n_1 and n_2, $n_1 = \min\{n \in \mathbb{N} | n\lambda\alpha_1 R_1 > 1\}$ and $n_2 = \min\{n \in \mathbb{N} | n\lambda\alpha_2 R_2 > 1\}$;
- At the $(n_1 + 1)$th iteration we recover circle of radius R_1 exactly and at the $n_2 + 1$th iteration we recover circle of radius R_2 exactly.
- Particularly, if we choose λ small enough that $n_1 + 1 < n_2$, then at the $(n_1 + 1)$th iteration, the circle of radius R_1 has been recovered exactly while the circle of R_2 is still missing.

A Modification of Meyer's Example

- Assume $y = F\alpha\chi_R$, where F is the FFT operator;
- $||Fm - F\alpha\chi_R||_2^2 = ||m - \alpha\chi_R||_2^2$ for any m;
- The solution of $\min_m \{||m||_{BV} + \lambda||Fm - y||_2^2\}$ is

$$m = \begin{cases} 0 & \text{if } \alpha R \leq \frac{1}{\lambda}, \\ (\alpha - \frac{1}{\lambda R})\chi_R & \text{else.} \end{cases}$$

- The Bregman iteration will have an exact recovery;

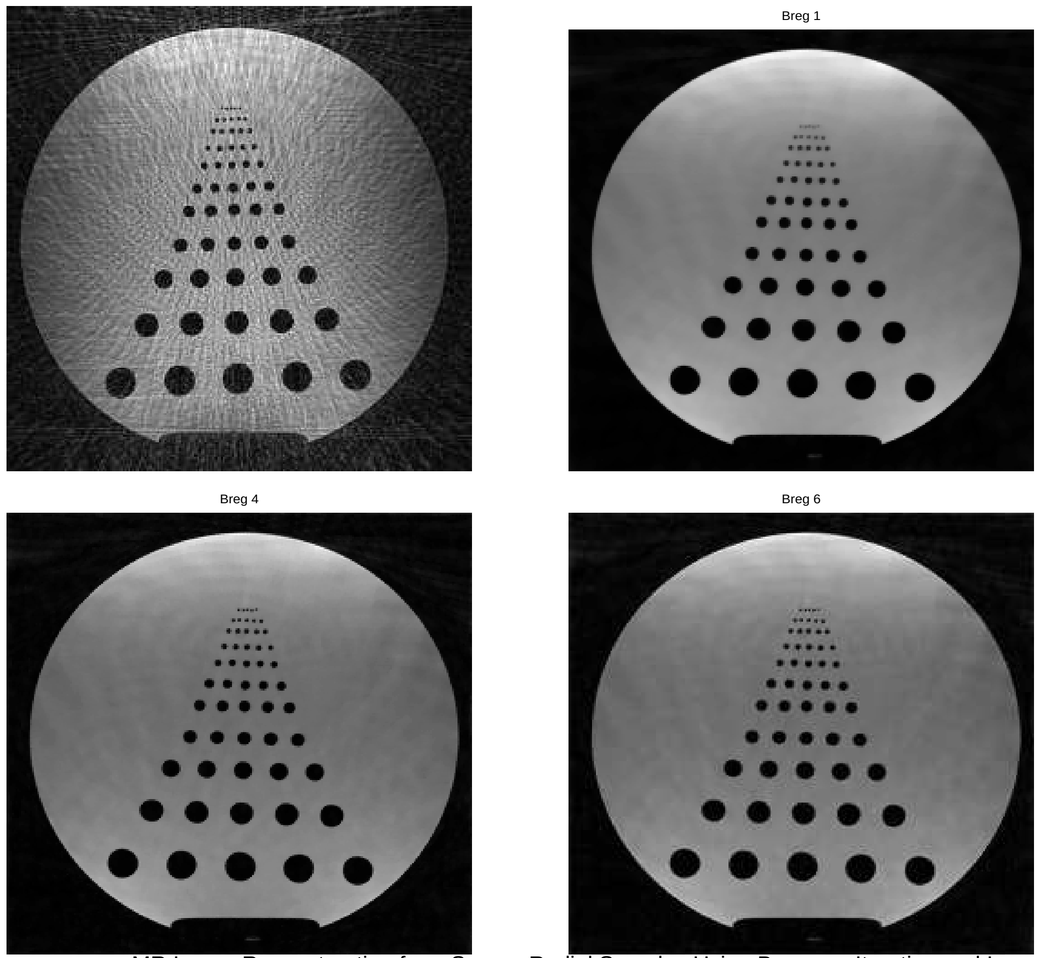
Numerical Results for Phantom (1)

Data Details

- Magnetom Avanto 1.5T scanner
- 63 radial lines with 512 samples each
- Three coils/channels
- Computation time is around 5 minutes per channel per Bregman iteration (Matlab)
- Scanning parameters are TR=4.8ms, TE=2.4ms, flip angle a=60o, FOV=206mm with a resolution of 256 pixels
- $\mu = 1$, $\nu = 0$, and $\lambda = 100$

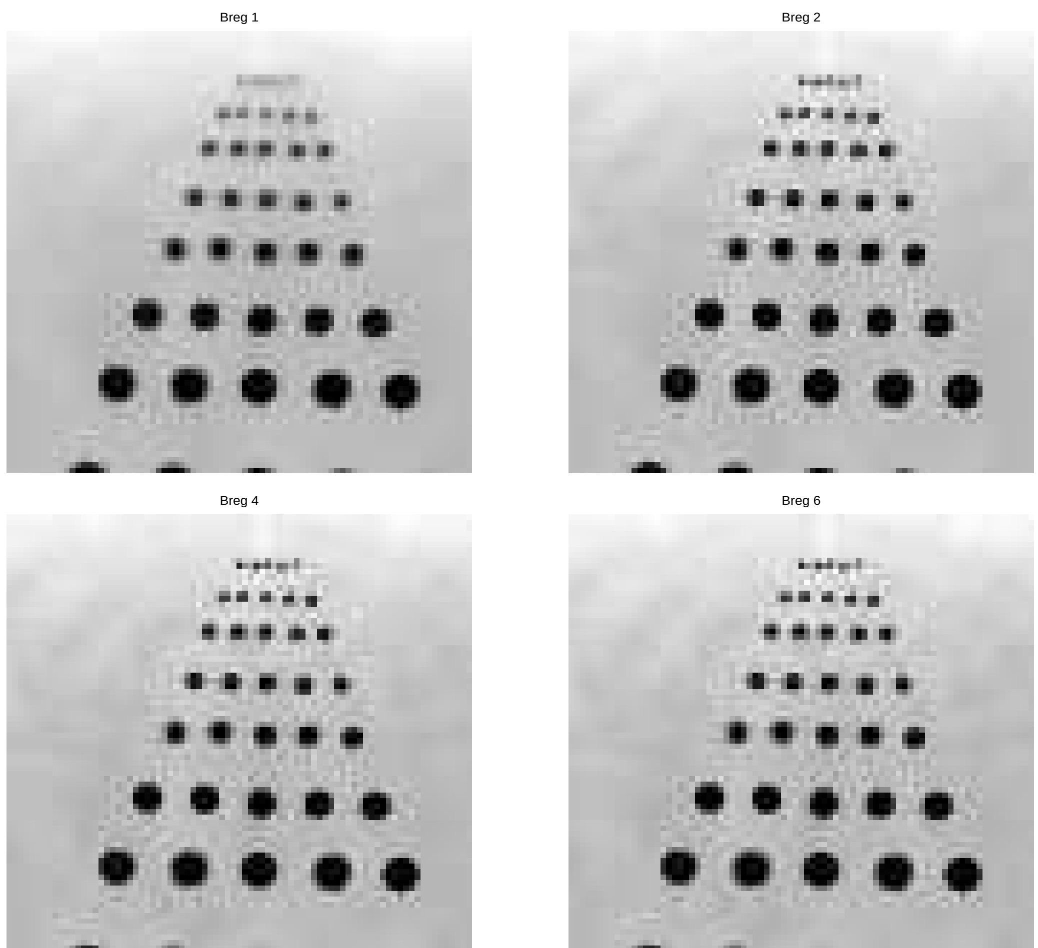
Numerical Results for Phantom (2)

Bregman iteration



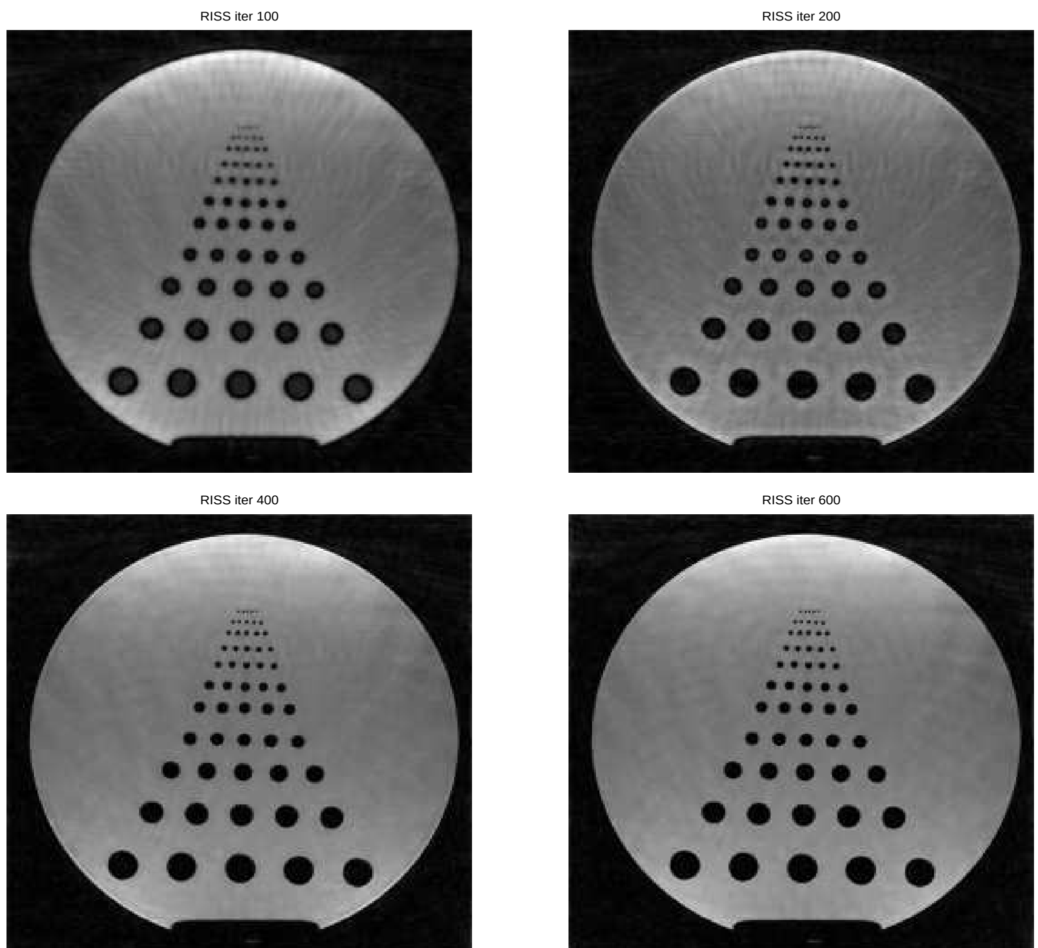
Numerical Results for Phantom (3)

Zoom in Results of Bregman iteration



Numerical Results for Phantom (4)

Inverse Scale Space Methods



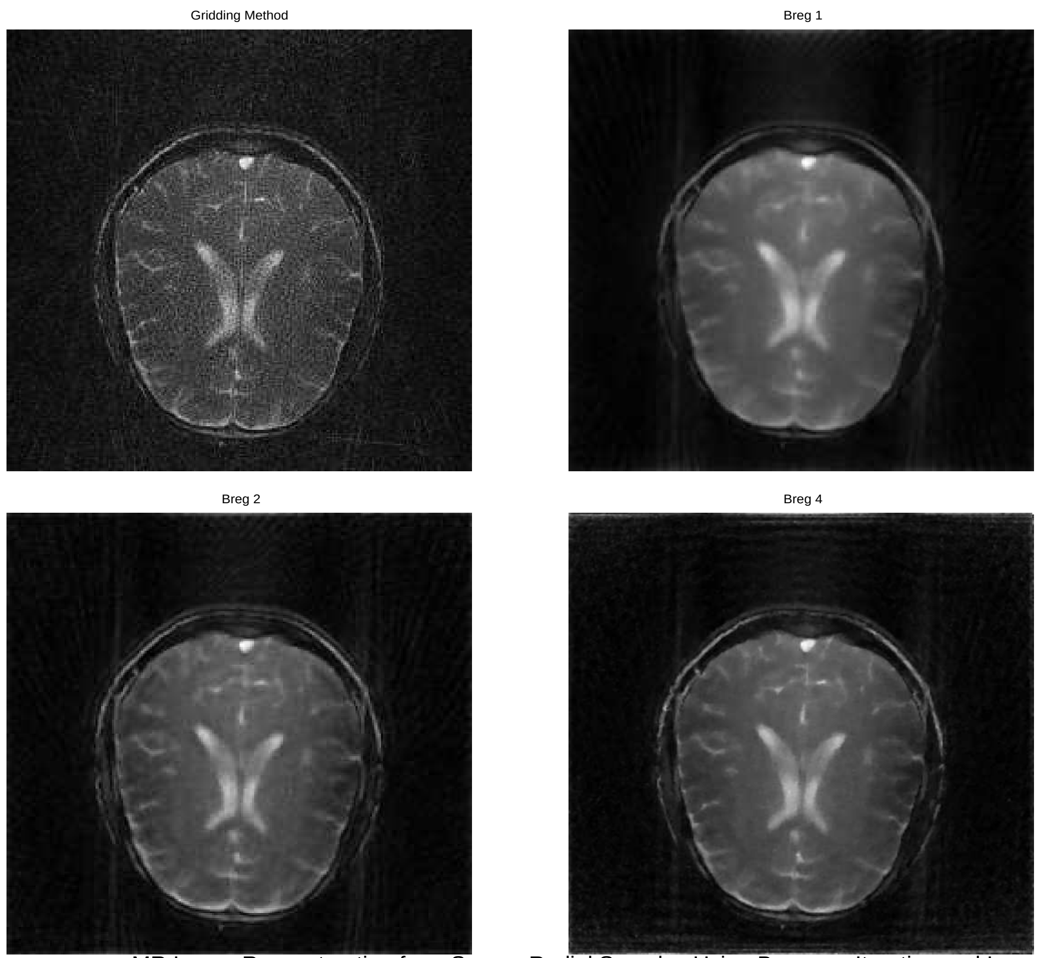
Numerical Results for Head (1)

Data Details

- 62 radial lines with 512 samples each
- Four coils/channels
- Scanning parameters are TR=4.46ms, TE=2.23ms, flip angle a=50o, FOV=250mm with a resolution of 256 pixels
- $\mu = 1$, $\nu = 0.1$, and $\lambda = 200$;

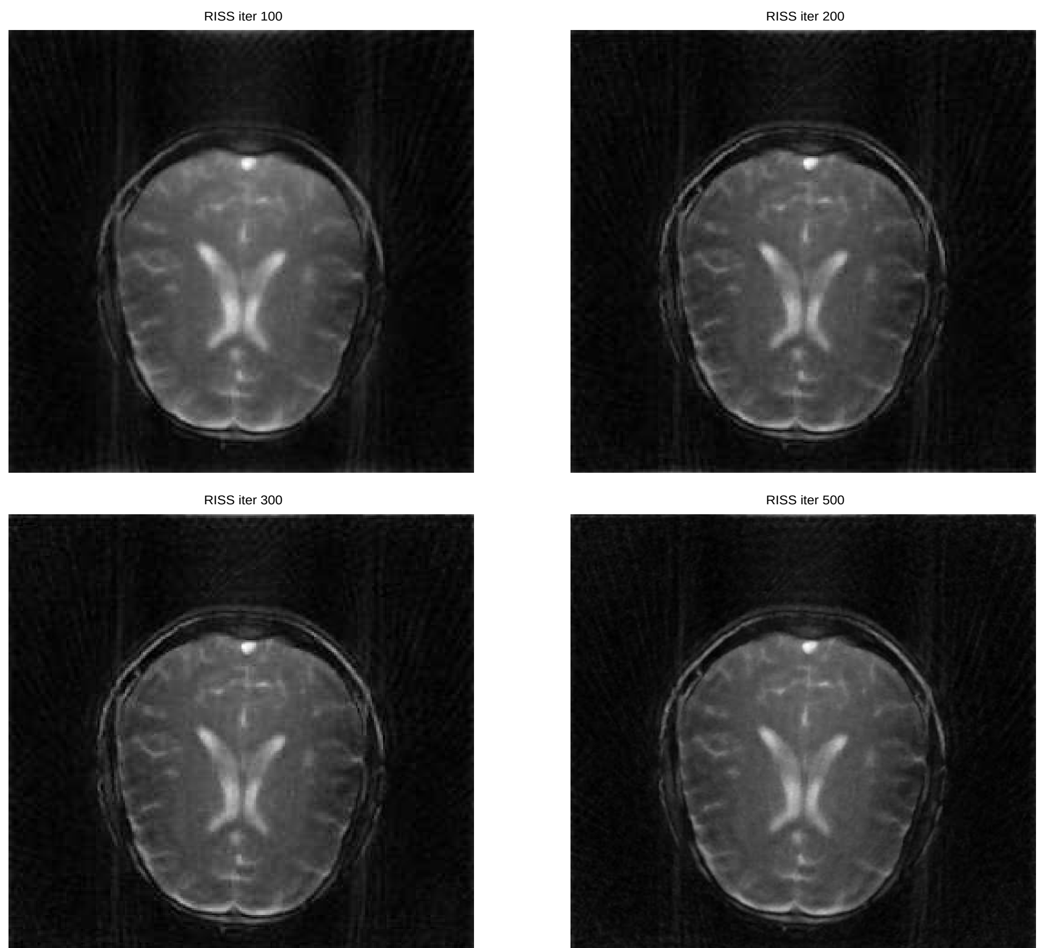
Numerical Results for Head (2)

Bregman iteration



Numerical Results for Head (3)

Inverse Scale Space Methods



MR Image Reconstruction from Sparse Samples

- Propose a new model with wavelet + TV
- Bregman iteration
- Inverse scale space method

The Potential of Curvelet, 5.3% Fourier Domain data

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Thank you for your attention!

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