

Morphology for Matrix-Fields: Ordering vs PDE

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in collaboration with

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Matrix-Valued Images

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- ◆ **Description: Matrix-valued image (or matrix field):**
function with values in $\text{Sym}_n(\mathbb{R})$, the set of real, symmetric $n \times n$ -matrices

$$F : \Omega \subset \mathbb{R}^3 \longrightarrow \text{Sym}_n(\mathbb{R})$$

- ◆ **Sources:**

- in civil engineering and solid mechanics: **diffusion** and **permittivity tensors** and stress-strain relationships describe anisotropic behaviour
- in image analysis: **structure tensor** (also called Förstner interest operator)
- **diffusion tensor magnetic resonance imaging (DT-MRI)**

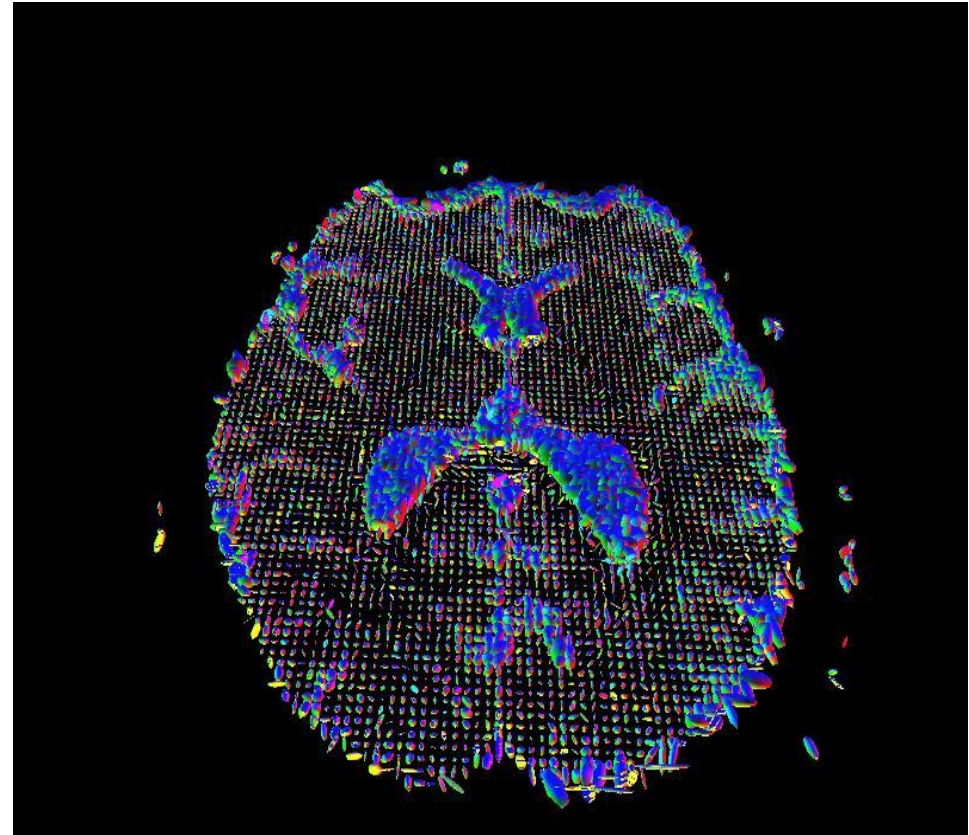
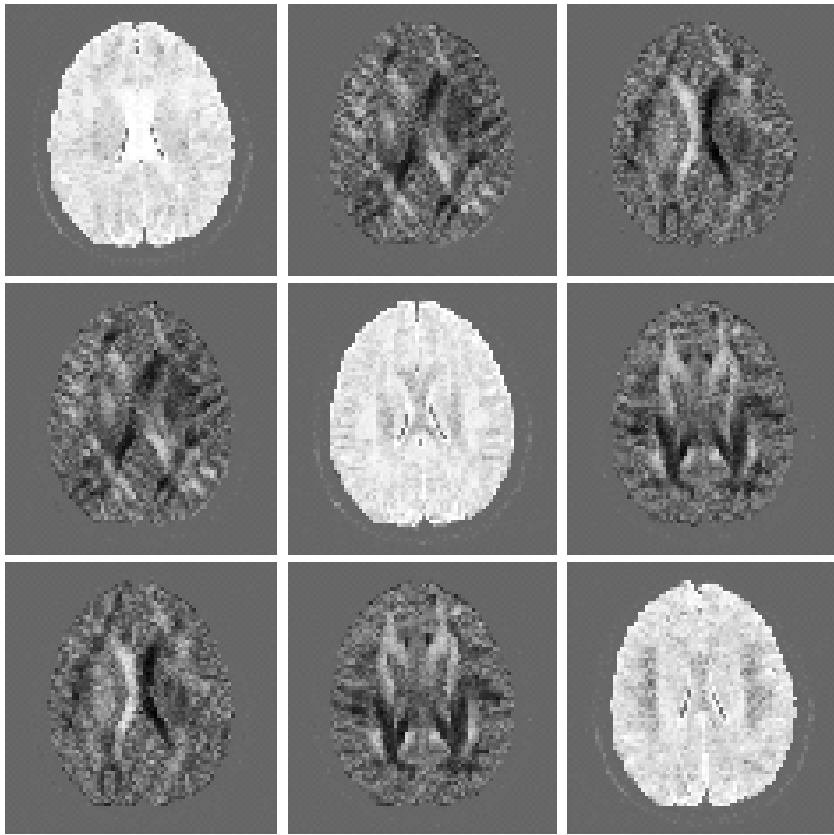
- ◆ **Properties:**

- $A \in \text{Sym}_n^+(\mathbb{R})$ are **positive (semi-)definite:**

$$q_A(x) := x^\top A x \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

- quadratic form $q_A(x)$ describes **isoprobability surface**, $q_A(x) = 1$
- reflects the diffusive property of water molecules in tissue

Visualisation



Slice of 3D DT-MRI data of a human head.

Left: Channelwise, tiled view. **Right:** Visualisation by ellipsoids via quadratic form

DT-MRI data: Courtesy of Anna Villanova, TU Eindhoven

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Content

- ◆ Matrix-valued data
- ◆ Morphology for matrix-fields via Loewner ordering
 - Basic idea in the 2×2 -matrix case
 - Extensions to 3×3 - and larger matrices
 - Experiments
- ◆ Mathematical Morphology via PDEs
 - Matrix-valued morphological PDEs
 - Matrix-valued solution schemes
 - Experiments
- ◆ Concluding Remarks

Basic morphological operations in the scalar case:

- ◆ Greyscale **dilation** \oplus replaces the greyvalue of the image $f(x, y)$ by its supremum within a mask defined by B :

$$(f \oplus B)(x, y) := \sup \{f(x - x', y - y') \mid (x', y') \in B\}$$

- ◆ while **erosion** \ominus is determined by

$$(f \ominus B)(x, y) := \inf \{f(x + x', y + y') \mid (x', y') \in B\}$$

Erosion and dilation of an image with disc-shaped structuring elements

Top: Dilation



Original



radius = 10



radius = 20



Bottom: Erosion



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- ◆ Combinations of erosion and dilation operations lead to
 - opening, closing
 - top hats
 - derivatives
- ◆ Morphological “Laplacian”

$$\Delta_B F := (f \oplus B) - 2 \cdot F + (f \ominus B)$$

- ◆ **Interpretation:** It approximates the second directional derivative $\partial_{\eta\eta} f$ where η denotes the direction of the steepest slope

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Morphological Laplacians are useful for designing so-called **shock filters**

- ◆ **Idea:** Apply dilations around maxima and erosions around minima:

$$S_B f := \begin{cases} f \oplus B & \text{if } \Delta_B f < 0) \\ f & \text{if } \Delta_B f = 0) \\ f \ominus B & \text{if } \Delta_B f > 0) \end{cases}$$

- ◆ experimentally their iterates converge towards a *steady state* given by a *piecewise constant segmented image*
- ◆ discontinuities (“shocks”) between the segments

The basic morphological operations of **dilation** and **erosion** rely on the definition of **infimum** and **supremum**

Problem: What is the right notion of infimum and supremum for matrices, the right matrix-infimum (MI), the right matrix-supremum (MS)?

- ◆ In the **scalar** case: Infimum and supremum are based on an **ordering**
- ◆ In the **vectorial** case: generally **no suitable ordering** on vector spaces!
- ◆ In the **matrix-valued** case:
 - **Plus:** - There is a **partial ordering** on $\text{Sym}(n)$, the so-called **Loewner ordering**
 - **Minus:** - It is **not** a lattice ordering.
 - MI / MS must be **rotationally invariant**
 - MI / MS must **preserve positive definiteness**
 - MI / MS must **depend continuously on input** matrices

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“Loewner approach” for 2×2 -matrices: the basic idea

Definition: (Loewner ordering)

Let $A, B \in \text{Sym}(n)$. Then $A \leq B$ if and only if $B - A$ is positive semidefinite.

How does the corresponding **ordering cone** $\text{Sym}^+(2)$ in $\text{Sym}(2)$ look like ?

The mapping

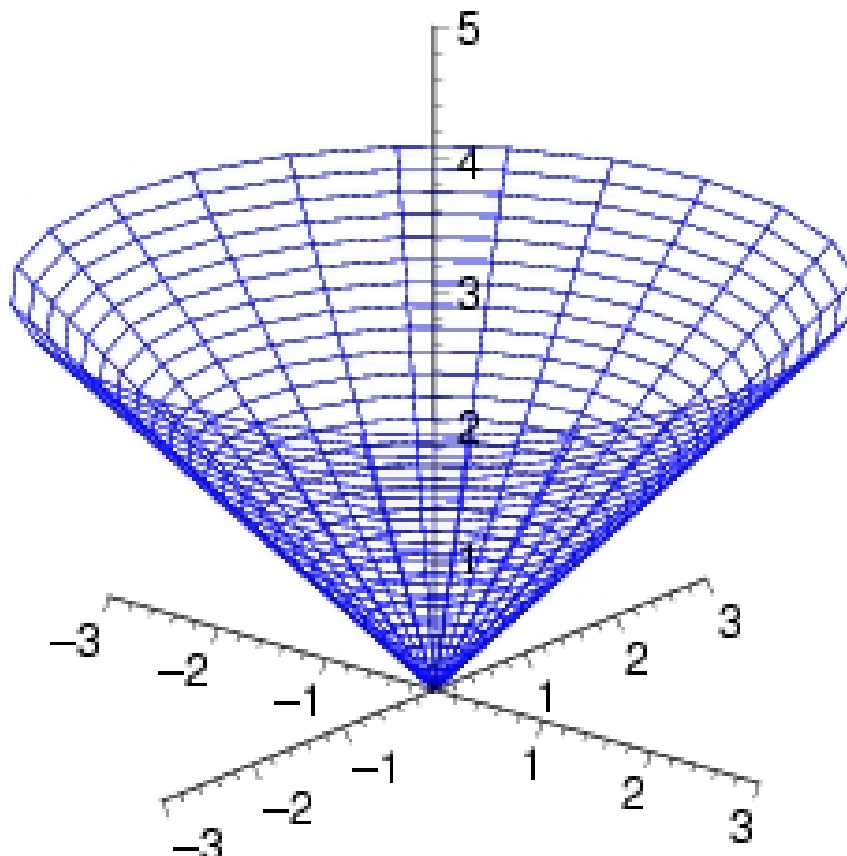
$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(2\beta, \gamma - \alpha, \gamma + \alpha)^\top$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} z - y & x \\ x & z + y \end{pmatrix} \longleftrightarrow (x, y, z)^\top$$

creates an **isomorphic image** of the **cone** $\text{Sym}^+(2)$ in the Euclidean \mathbb{R}^3

Cone of the Loewner Ordering

The convex cone $\text{Sym}^+(2)$ in \mathbb{R}^3 corresponding to the Loewner ordering in $\text{Sym}(2)$:



Loewner ordering cone with 90° angle at its vertex

How can this cone be used to find a matrix-supremum?

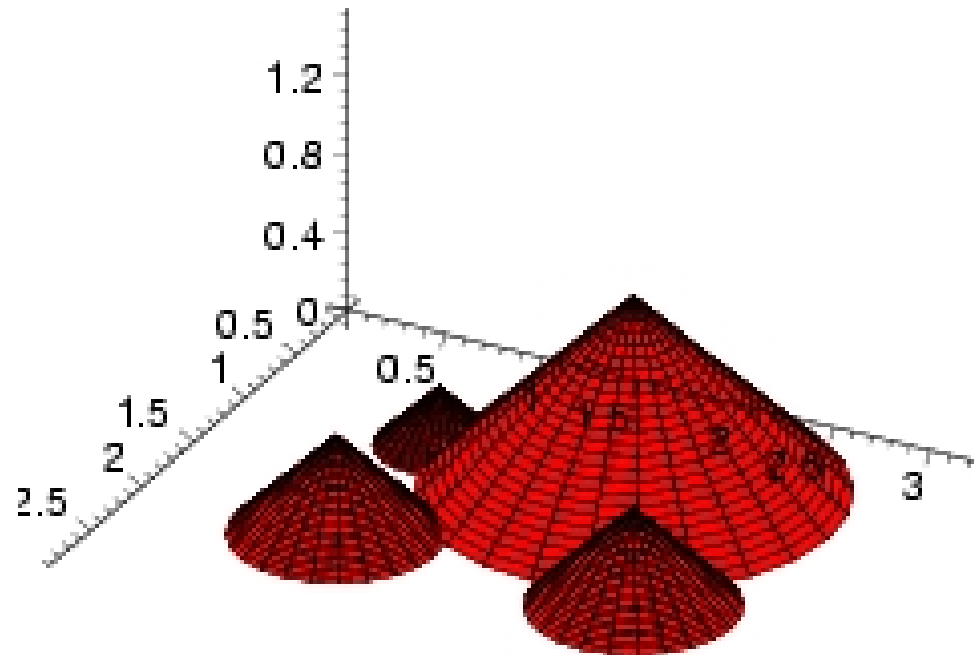
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Matrix-Supremum via Loewner Ordering I

In order to find matrix-supremum $M = MS(A_1, \dots, A_n)$ of a set of matrices $A_1, \dots, A_n \in \text{Sym}(2)$ consider

- ◆ the **penumbra** of each matrix A_i :

$$A_i - \text{Sym}^+(2)$$



Penumbras of the matrices A_i

- ◆ **Note:** The **vertex** of each penubral cone specifies a matrix **uniquely**

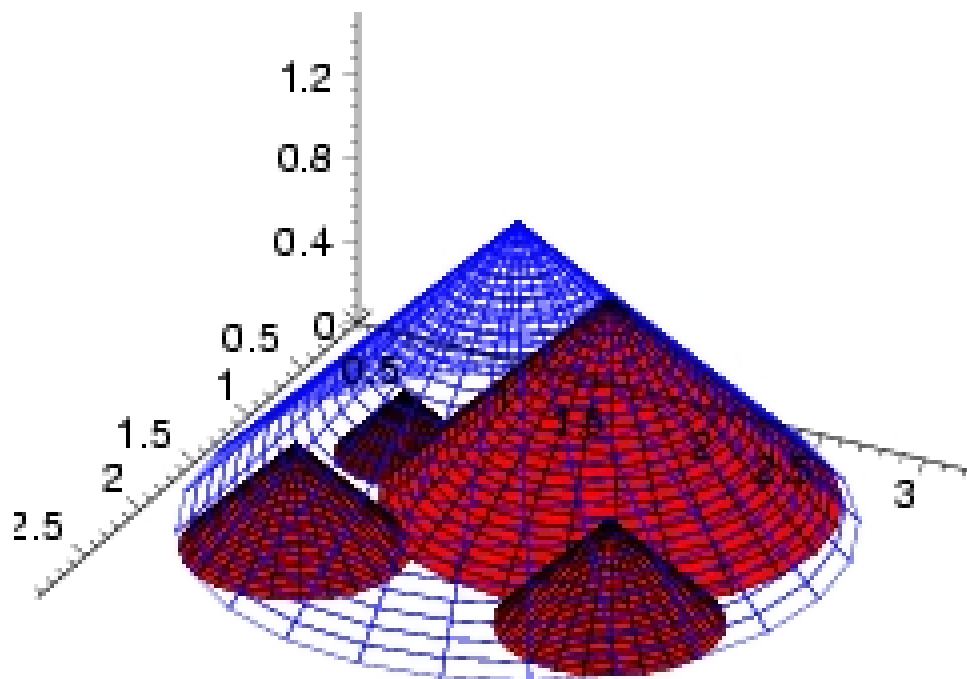
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Matrix-Supremum via Loewner Ordering II

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In order to find the matrix-supremum $M = \max(A_1, \dots, A_n)$ of a set of matrices $A_1, \dots, A_n \in \text{Sym}(2)$ consider

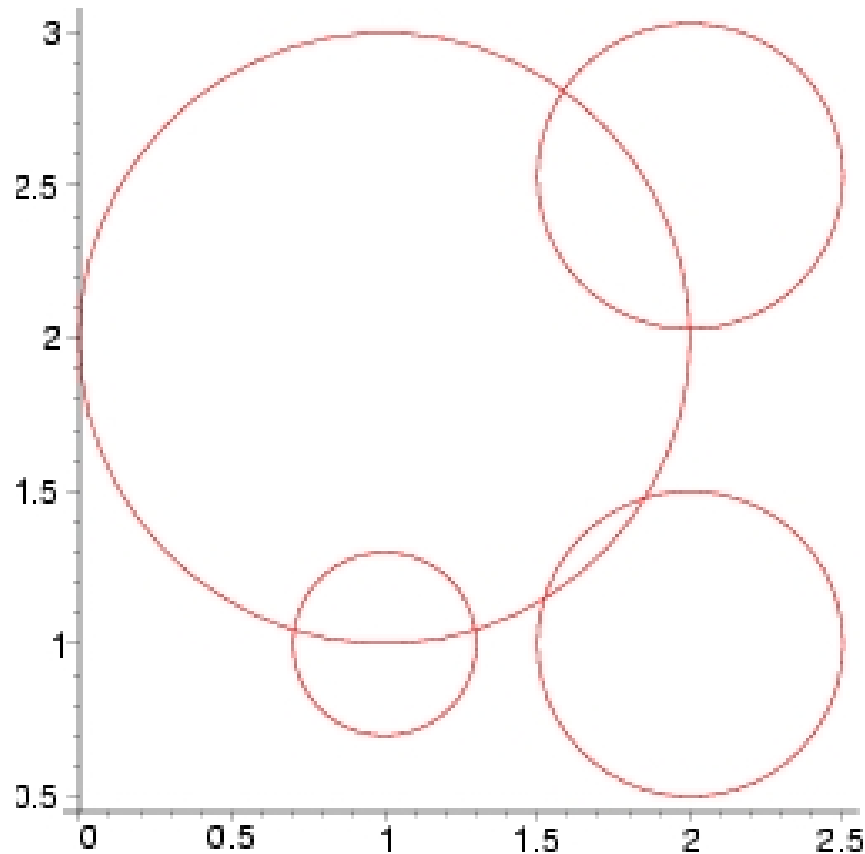
- ◆ the **penumbra** of each matrix A_i
- ◆ and find the “**covering**” cone.



Covering cone encasing the penumbral cones of the A_i 's

How to find this covering cone **computationally** ?

- ◆ The bases of the penumbral cones are circles C_i in the x-y-plane



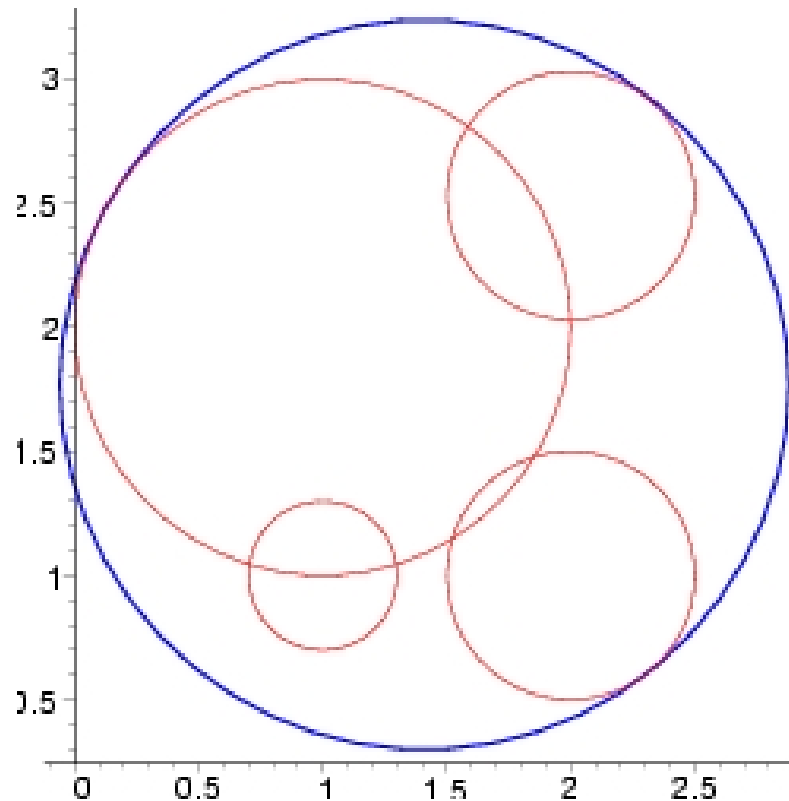
Circles as bases of cones

- ◆ **Note:** A circle (center and radius) determines the penumbra **uniquely**

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Matrix-Supremum via Loewner Ordering IV: Minimal Circle

- ◆ The bases of the penumbral cones are circles C_i in the x-y-plane
- ◆ **Goal:** find the **smallest circle C** enclosing the **circles C_i**



Minimal enclosing circle C

- ◆ An algorithm of B. Gärtner (ETH Zürich, 1999) finds this circle C
- ◆ This enclosing circle C determines the **matrix-supremum**

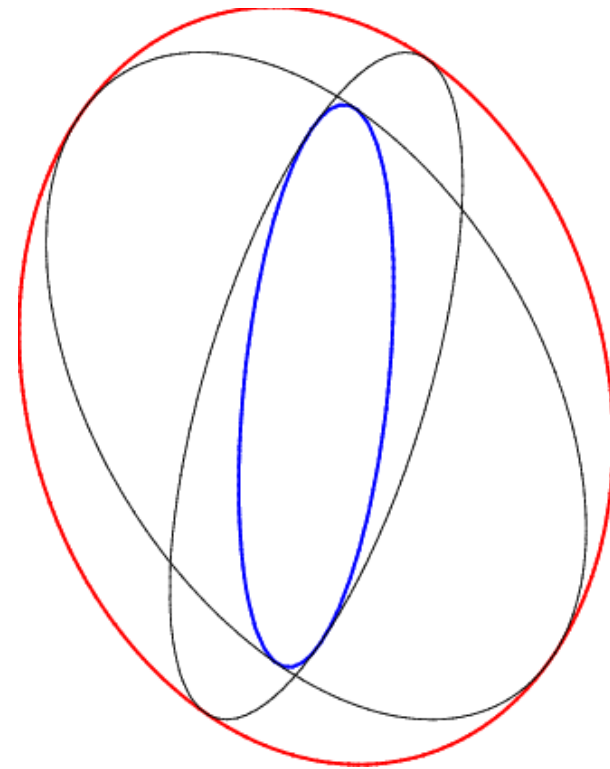
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Matrix-Infimum via Loewner Ordering

- ◆ The matrix-infimum m is obtained via the matrix-supremum of $A_1^{-1}, \dots, A_n^{-1}$:

$$m := \text{MI}(A_1, \dots, A_n) := (\text{MS}(A_1^{-1}, \dots, A_n^{-1}))^{-1}$$

Example: Loewner approach,
maximal and **minimal** ellipses



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How does all this generalise to 3×3 - or larger matrices ?

Answer:

- ◆ The basic idea carries over in spirit to the higher order case.
- ◆ No 'visualising' mapping is known
- ◆ The base of the cone is much more complicated, it is **not** a **strictly convex set**
- ◆ Sample the **extreme points** of the base and find the smallest enclosing (higher dimensional) ball
- ◆ The center and the radius of this ball determine the penumbral cone, that is, the matrix-supremum MS
- ◆ MI via MS

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Loewner Ordering

◆ We obtain simple formulas with I as $n \times n$ -identity matrix:

- the **center** c_M of the circumfering ball associated with M is given by

$$c_M := M - \frac{\text{trace}(M)}{n} I$$

- its **radius** r satisfies ($v \in \mathbb{R}^3, \|v\| = 1$)

$$r := \|M - \text{trace}(M)vv^\top - c_M\| = \text{trace}(M) \sqrt{1 - \frac{1}{n}}$$

- the **vertex** M of the associated penumbra is obtained by

$$M = c_M + \frac{r}{n} \frac{1}{\sqrt{1 - \frac{1}{n}}} I$$

Properties of the approach based on the Loewner ordering:

- ◆ rotationally invariant,
- ◆ preserves positive definiteness,
- ◆ continuous dependence on the input matrices A_i ,
- ◆ extendable to indefinite matrices,
- ◆ extendable to higher order matrices.

For more details on Loewner based matrix morphology:

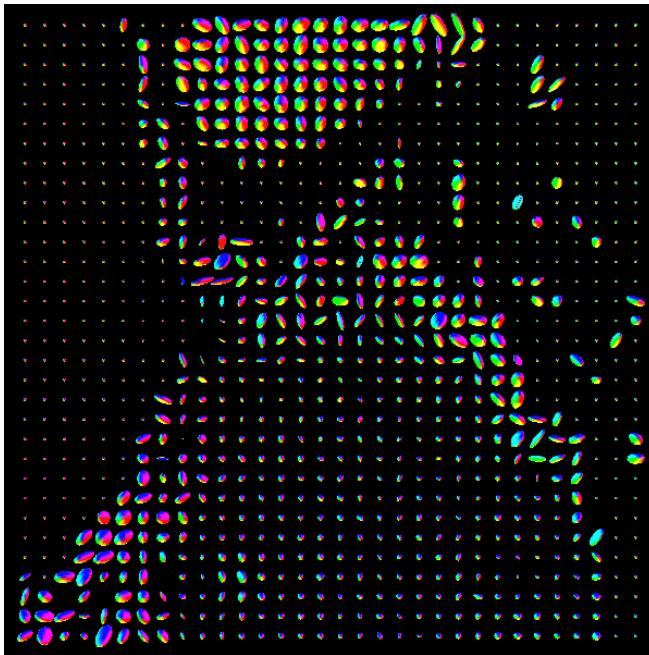
B.B. et al., Mathematical Morphology for Tensor Data Induced by the Loewner Ordering in Higher Dimensions. Preprint 2005 (to be published in IEEE, Sig. Proc.)

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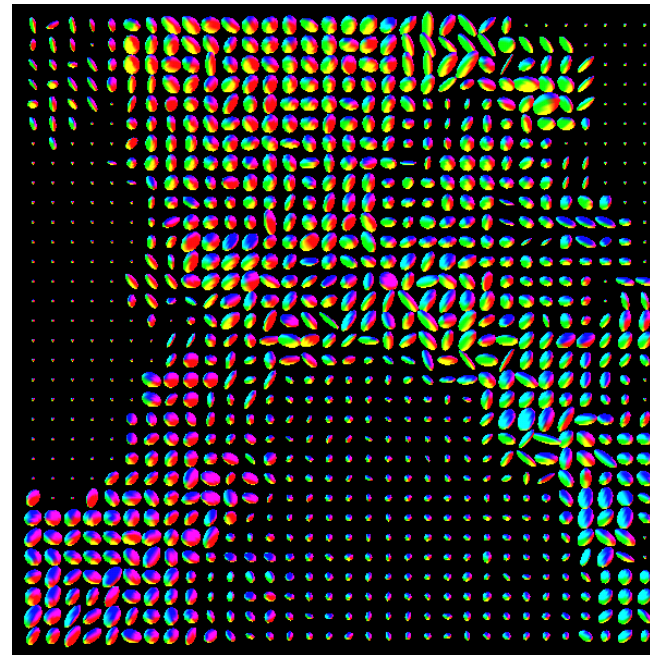
Experiments

Dilation and erosion

of a 3D matrix field F with a ball-shaped structuring element B of radius 2.

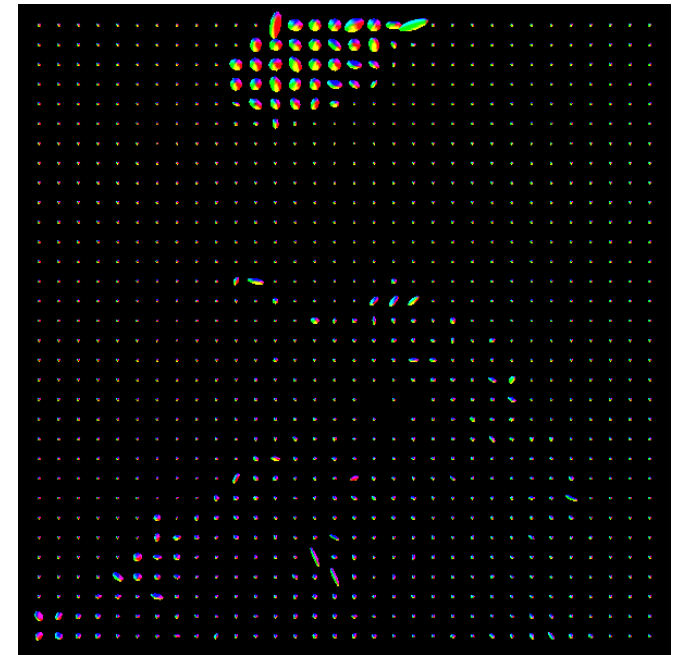


Original



Dilation

$$F \oplus B$$



Erosion

$$F \ominus B$$

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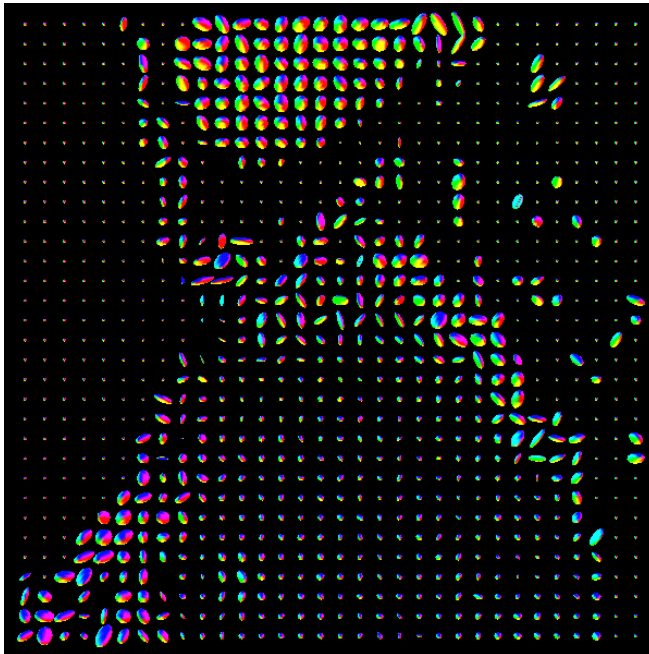
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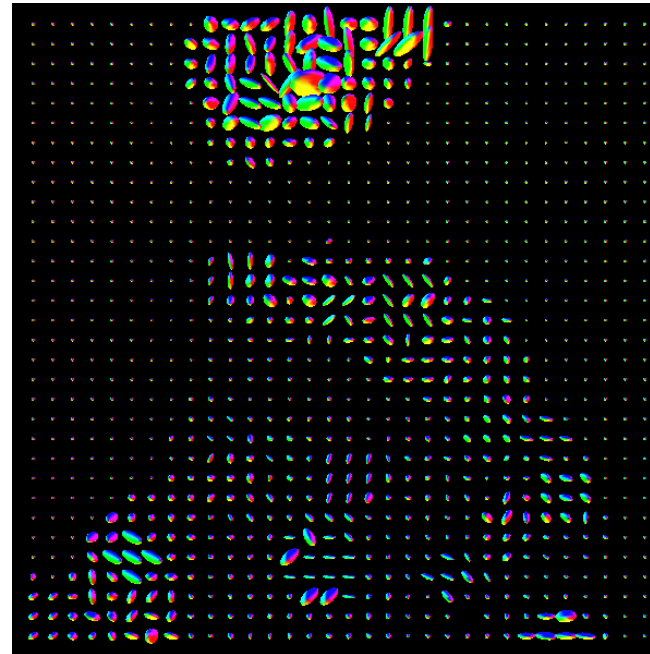
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Opening and closing

of a 3D matrix field F with a ball-shaped structuring element B of radius 2.

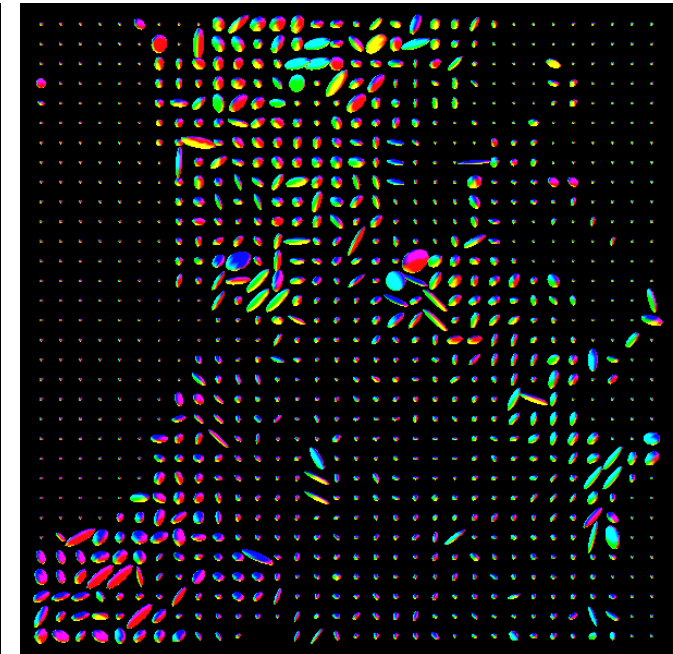


Original



Opening

$$F \circ B = (F \ominus B) \oplus B$$



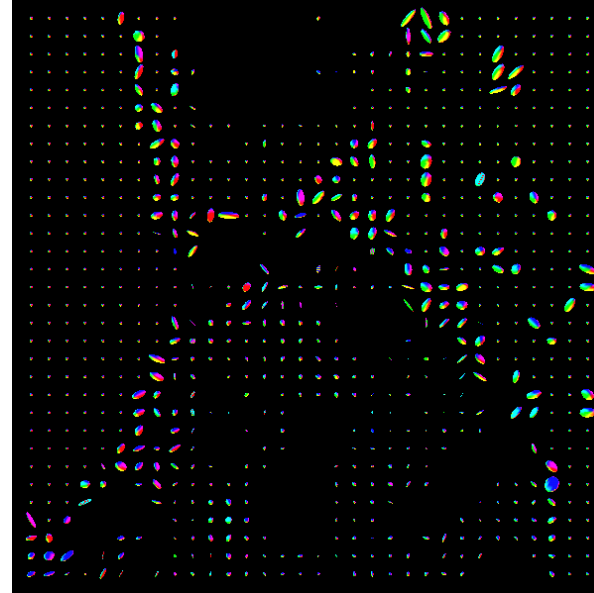
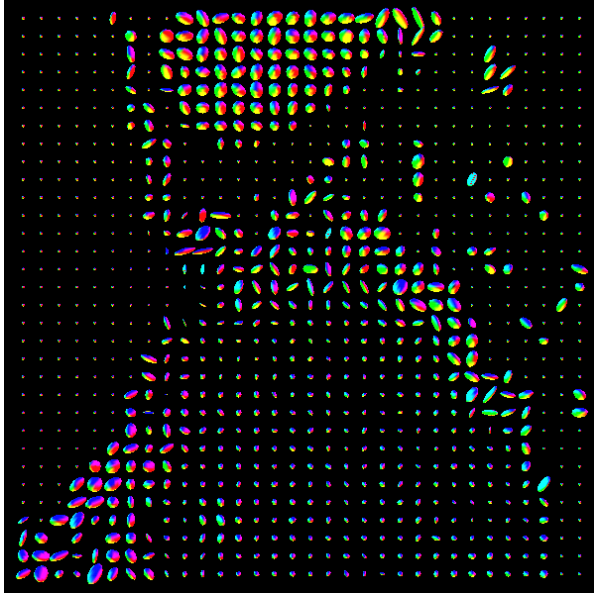
Closing

$$F \bullet B = (F \oplus B) \ominus B$$

Top Hats

of a 3D matrix-field F with a ball-shaped structuring element B of radius 2.

Original

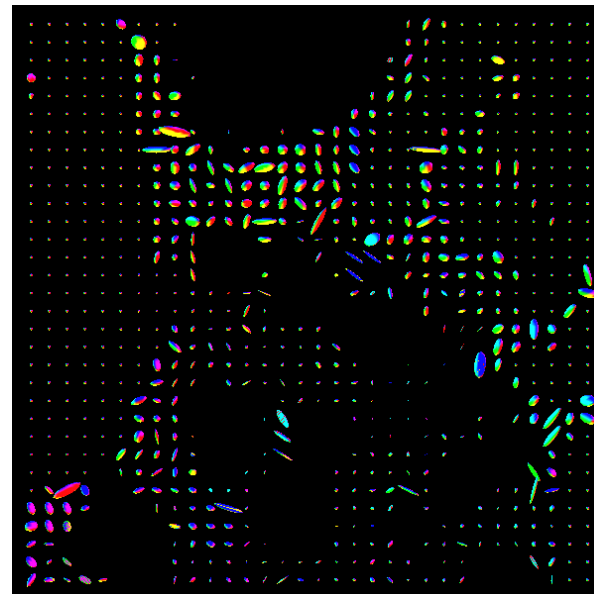
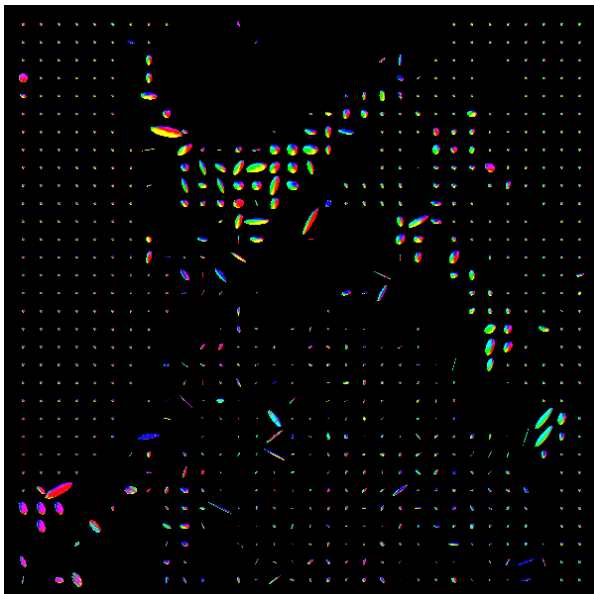


White
top hat

$$F - (F \circ B)$$

Black
top hat

$$(F \bullet B) - F$$



Self-dual
top hat

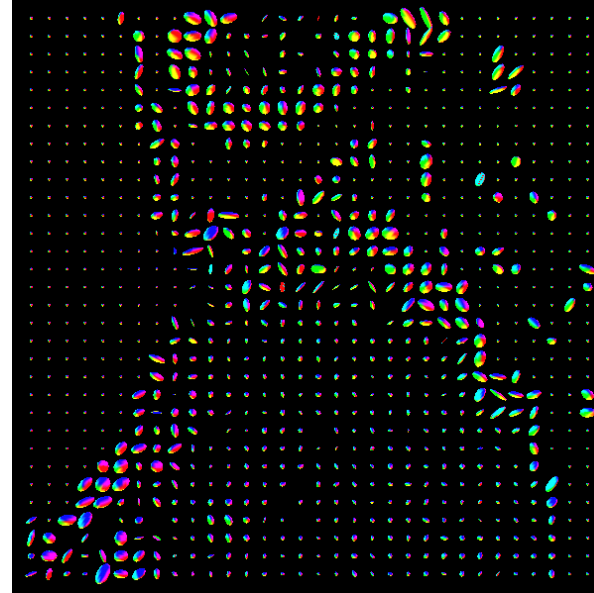
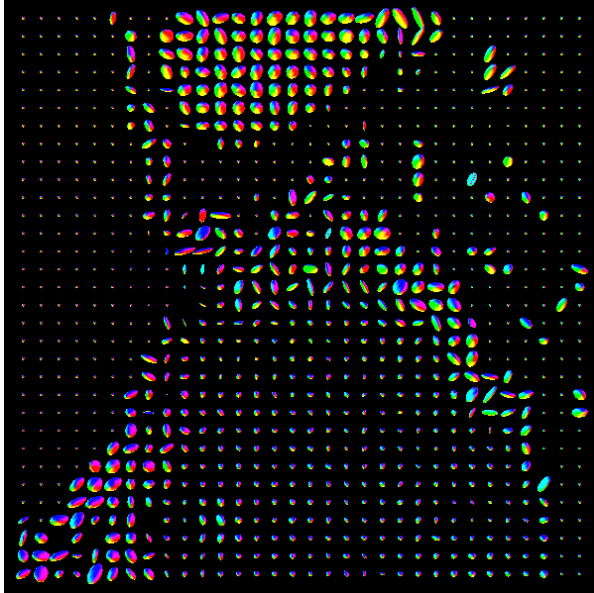
$$(F \bullet B) - (F \circ B)$$

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Morphological derivatives

of a 3D matrix-field F with a ball-shaped structuring element B of radius 2.

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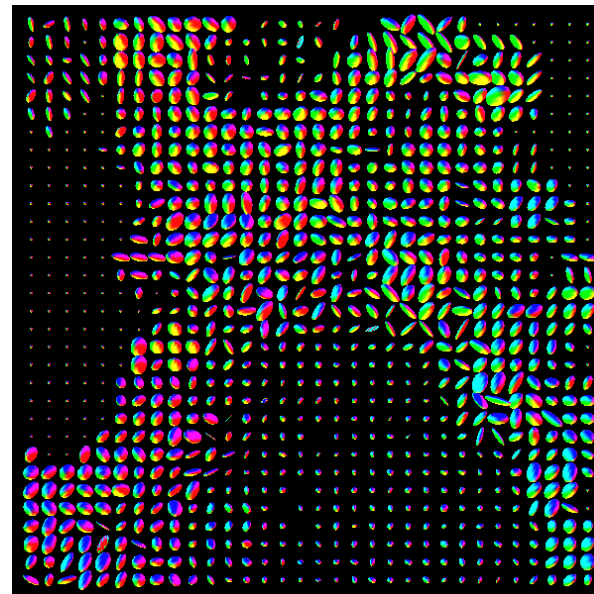
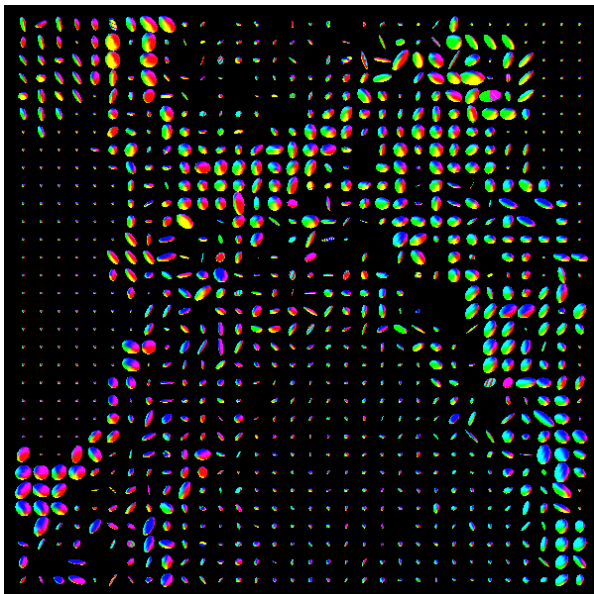


External
Gradient

$$(F \oplus B) - F$$

Internal
Gradient

$$F - (F \ominus B)$$



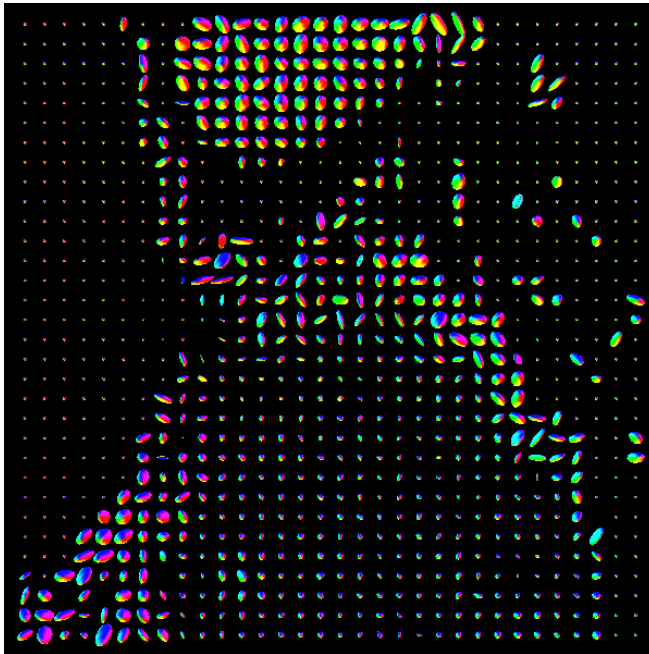
Beucher
Gradient

$$(F \oplus B) - (F \ominus B)$$

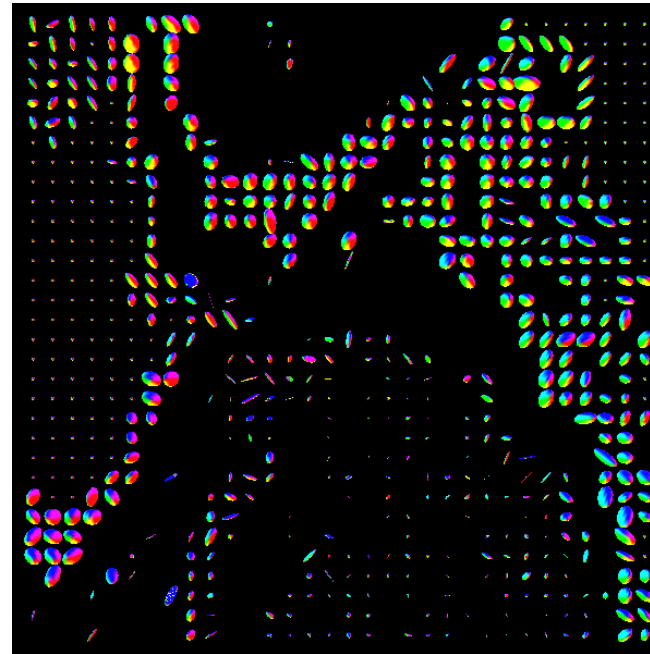
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Morphological Laplacian and shock filtering

of a 3D matrix field F with a ball-shaped structuring element of radius 2.

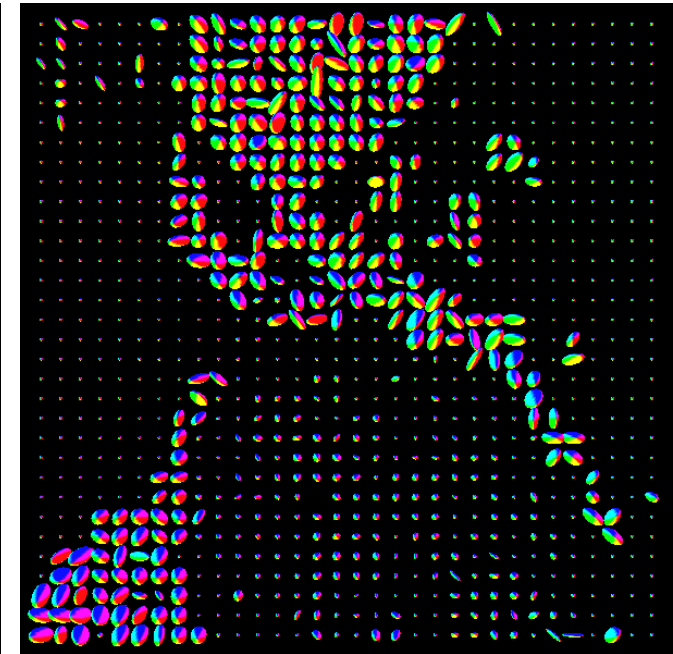


Original



Morphological Laplacian

$$(F \oplus B) - 2 \cdot F + (F \ominus B)$$



Shock filtering

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Continuous Morphology

Basic Approach (Boomgaard/Dorst '92): Nonlinear partial differential equations that mimic the process of dilation and erosion.

- ◆ **Situation:** Original image $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, transformed version u
- ◆ **Dilation** with a ball-shaped structuring element:

$$\partial_t u = \|\nabla u\|$$

- ◆ **Erosion** with a ball-shaped structuring element:

$$\partial_t u = -\|\nabla u\|$$

with initial condition $u(x, y, 0) = f(x, y)$.

Advantages of PDE framework:

- ◆ Sophisticated machinery of numerical solution methods for PDEs is available
- ◆ Continuous approach allows for sub-pixel accuracy

Scalar PDE: $\partial_t u = \|\nabla u\| = \sqrt{(\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2}$

How to find a PDE for matrix-valued data $U = (u_{ij})_{ij} \in \text{Sym}_n(\mathbb{R})$?

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Scalar PDE: $\partial_t u = \|\nabla u\| = \sqrt{(\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2}$

How to find a PDE for matrix-valued data $U = (u_{ij})_{ij} \in \text{Sym}_n(\mathbb{R})$?

- ◆ Define functions on $\text{Sym}_n(\mathbb{R})$: If $U = V^\top \text{diag}(\lambda_1, \dots, \lambda_n) V$ and $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$,

$$h(U) := V^\top \text{diag}(h(\lambda_1), \dots, h(\lambda_n)) V$$

- ◆ Generalise partial derivatives ∂_ω , with $\omega \in \{t, x_1, \dots, x_d\}$:

$$\bar{\partial}_\omega U := (\partial_\omega u_{ij})_{ij}$$

- ◆ Generalise gradient ∇ :

$$\bar{\nabla} U := (\bar{\partial}_{x_1} U, \dots, \bar{\partial}_{x_d} U)^\top \in (\text{Sym}_n(\mathbb{R}))^d$$

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Matrix PDE:

$$\bar{\partial}_t U = |\bar{\nabla} u|_2 = \sqrt{(\bar{\partial}_x U)^2 + (\bar{\partial}_y U)^2 + (\bar{\partial}_z U)^2}$$

How to solve the morphological matrix PDE ?

Numerical solution through the matrix-valued counterparts of the scalar schemes for the scalar PDEs

Example: OS-scheme by Osher & Sethian (1997)

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- ◆ Osher-Sethian scheme, **scalar-valued** numerical approximation in 1D:

$$\begin{aligned} \frac{u(i)^{(n+1)} - u(i)^{(n)}}{\tau} &= \\ &= \left[\left(\min \left(\frac{u(i)^{(n)} - u(i-1)^{(n)}}{h}, 0 \right) \right)^2 + \left(\max \left(\frac{u(i+1)^{(n)} - u(i)^{(n)}}{h}, 0 \right) \right)^2 \right]^{1/2} \end{aligned}$$

- ◆ **matrix-valued** counterpart, numerical approximation in 1D:

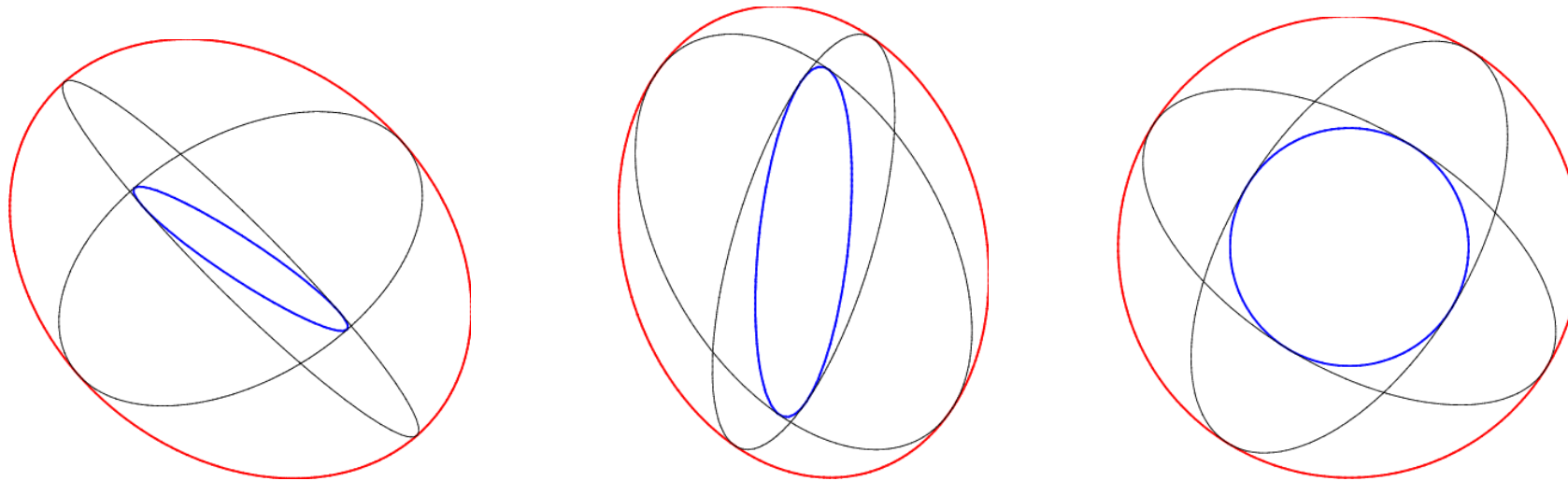
$$\begin{aligned} \frac{U(i)^{(n+1)} - U(i)^{(n)}}{\tau} &= \\ &= \left[\left(\min \left(\frac{U(i)^{(n)} - U(i-1)^{(n)}}{h}, 0 \right) \right)^2 + \left(\max \left(\frac{U(i+1)^{(n)} - U(i)^{(n)}}{h}, 0 \right) \right)^2 \right]^{1/2} \end{aligned}$$

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Definition: Let $A, B \in \text{Sym}_n(\mathbb{R})$ then

$$\max(A, B) := \frac{1}{2}(A + B + |A - B|)$$

$$\min(A, B) := \frac{1}{2}(A + B - |A - B|)$$



Maximal and **minimal** matrices $\in \text{Sym}_2(\mathbb{R})$

Remark: The maximal and minimal matrices are the one induced by the **Loewner ordering:** $A \geq B \iff A - B$ positive semidefinite

For comparison of ordering- and PDE-based matrix-morphology:

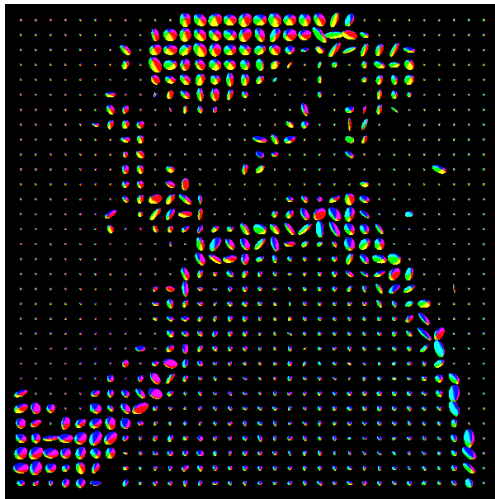
B.B. et al., Morphology for Tensor Data: Ordering versus PDE-Based Approach.

To be published in JMIV 2006.

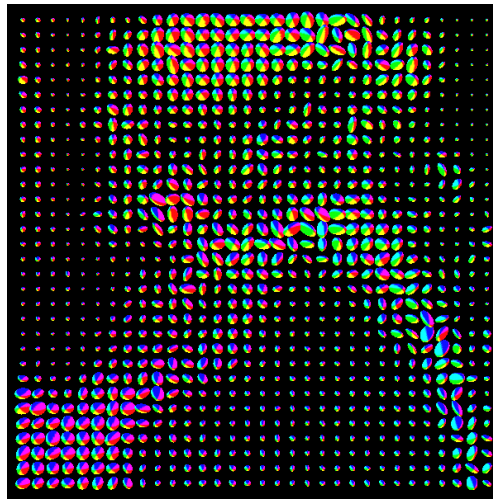
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Erosion and dilation of matrix-valued images by matrix-valued OS-scheme

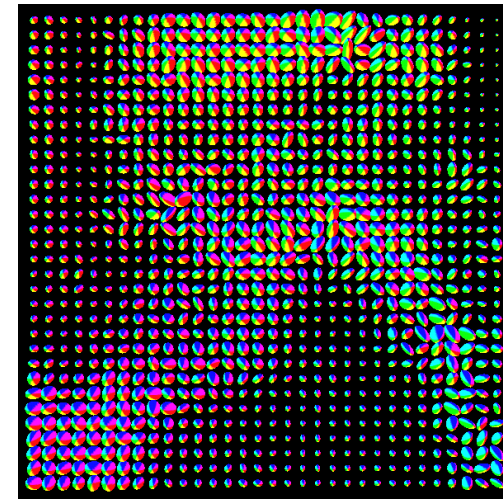
Top: Dilation



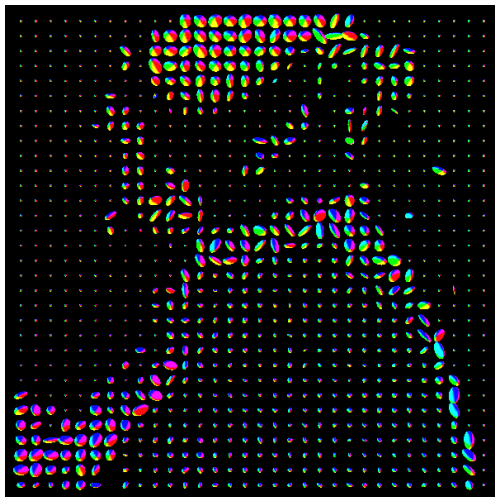
Original



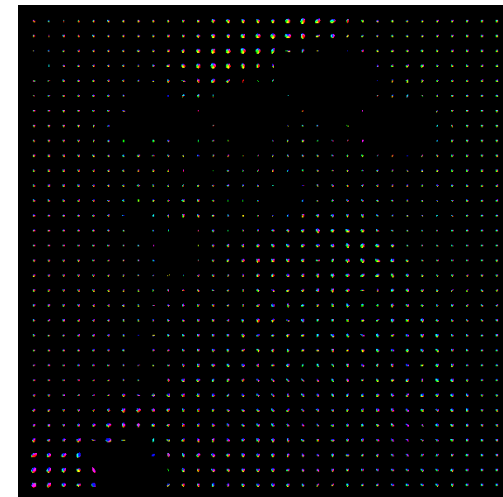
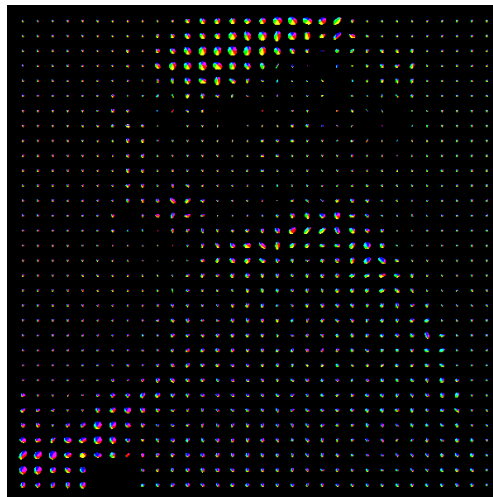
$t = 4$



$t = 10$



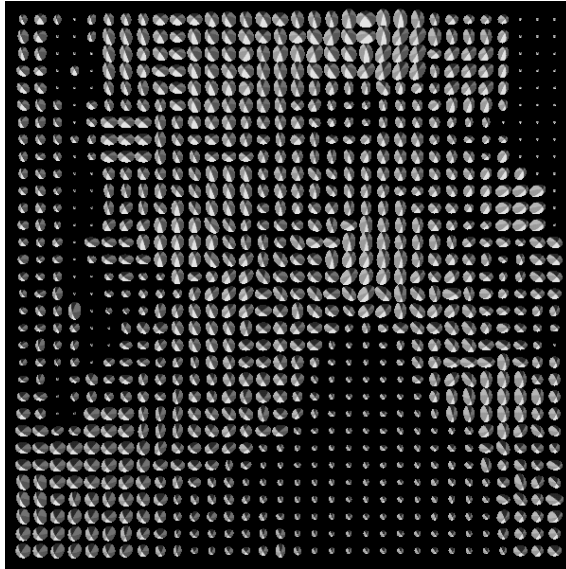
Bottom: Erosion



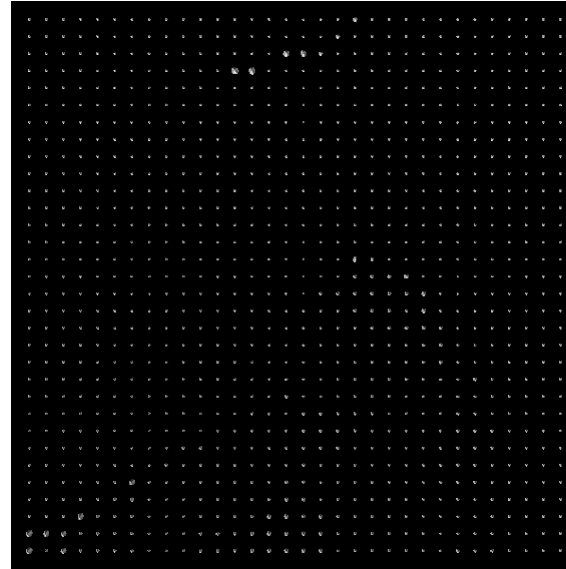
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Experiments: Ordering vs PDE II

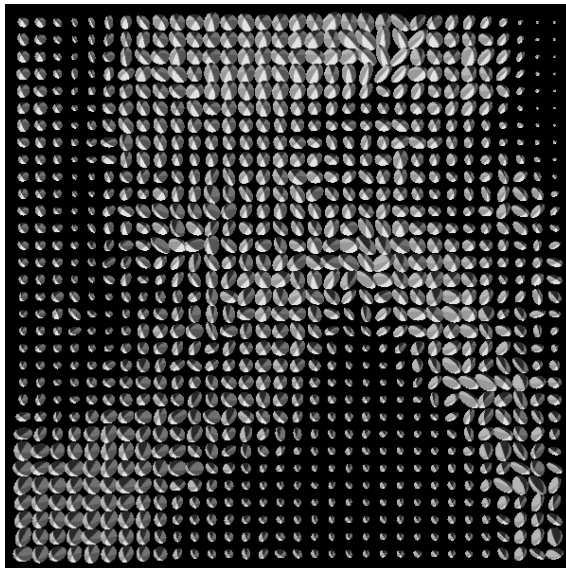
Ordering based approach (ball-shaped structuring element $BSE(\sqrt{2})$)



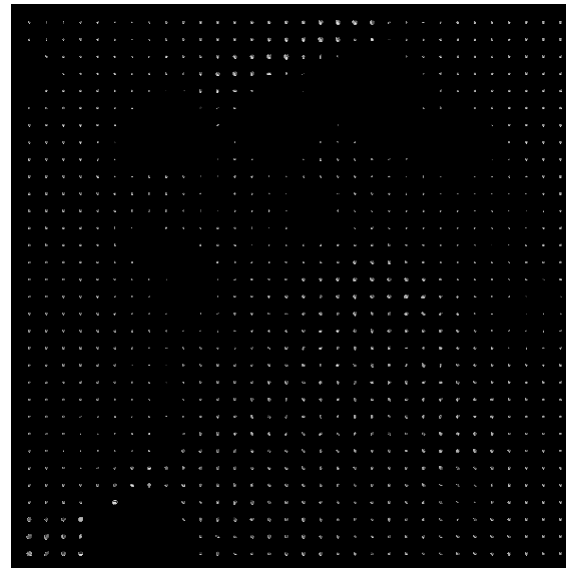
Dilatation



Erosion



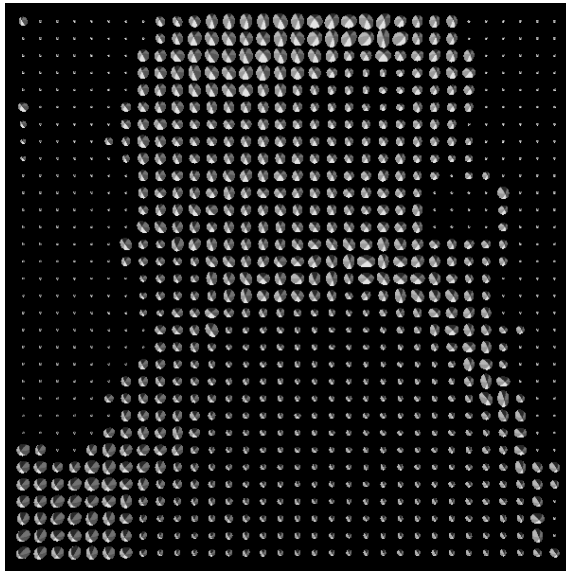
PDE based approach (stopping time 2)



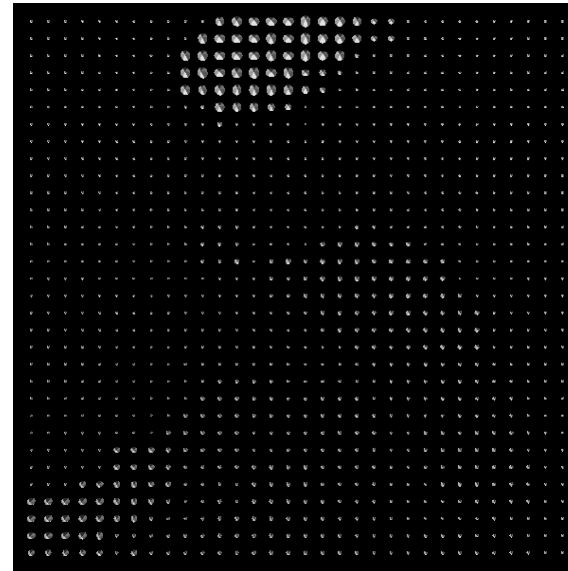
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Experiments: Ordering vs PDE III

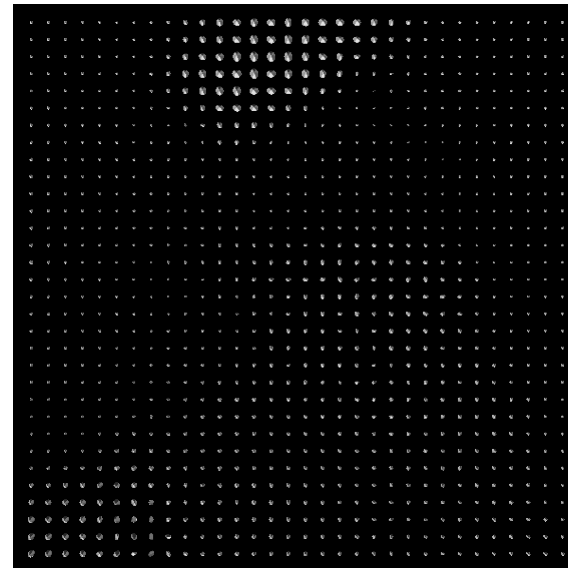
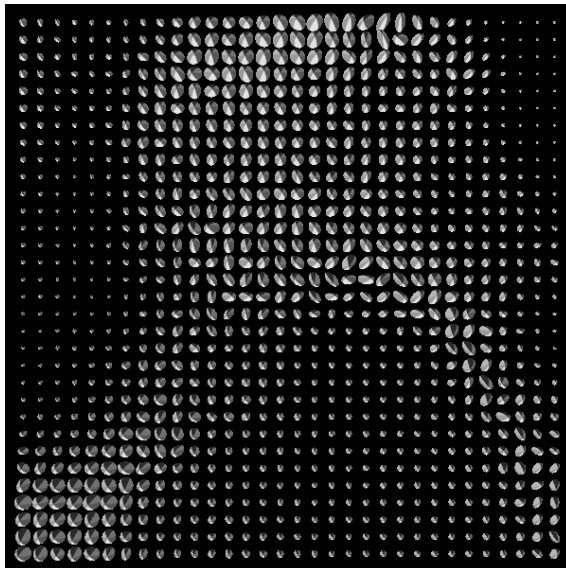
Ordering based approach (ball-shaped structuring element $BSE(\sqrt{2})$)



Closing



Opening

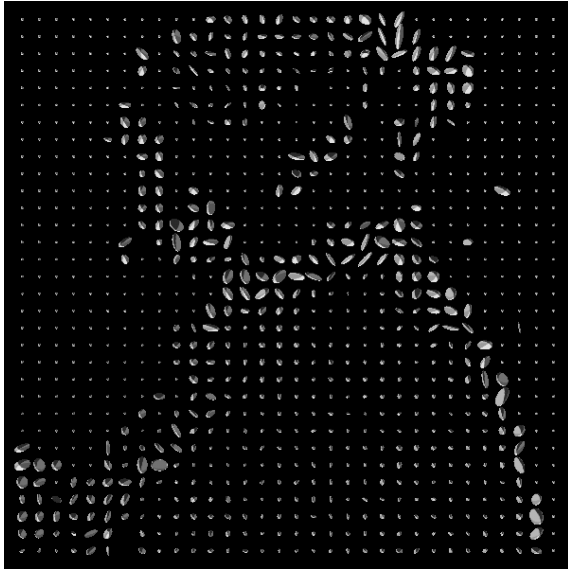


PDE based approach (stopping time 2)

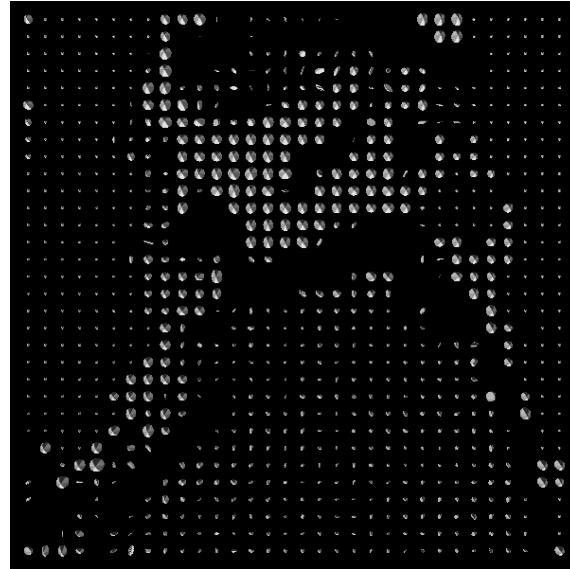
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Experiments: Ordering vs PDE IV

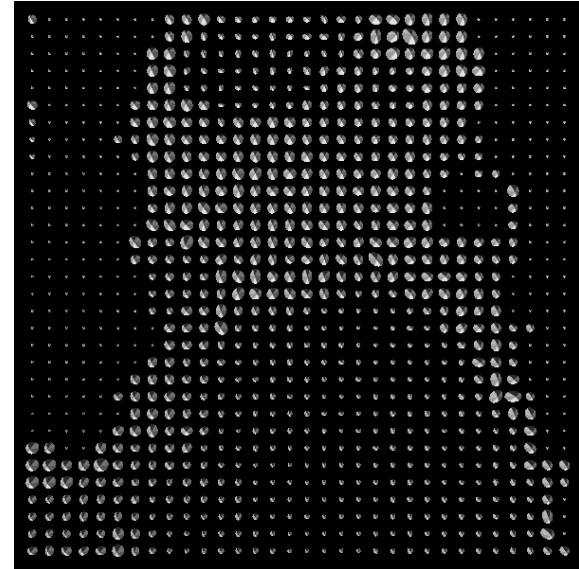
Ordering based approach (ball-shaped structuring element $BSE(\sqrt{2})$)



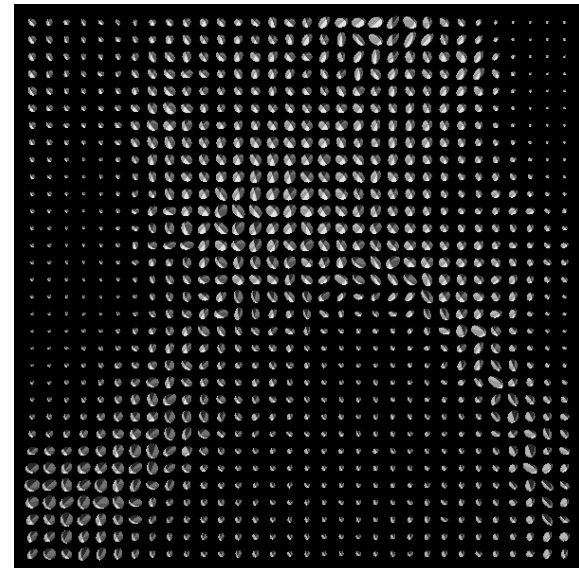
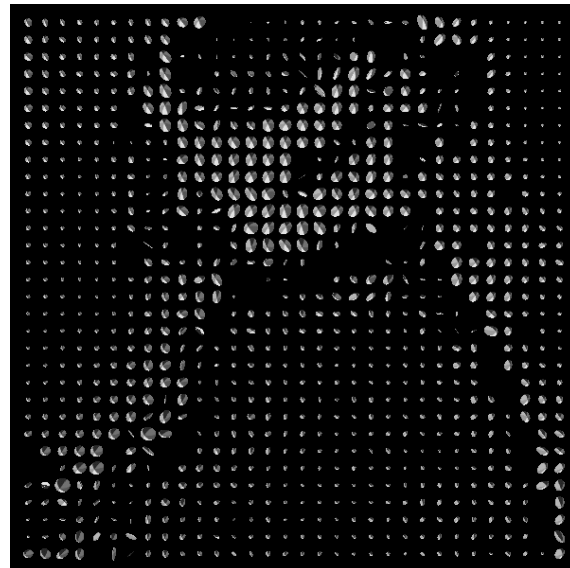
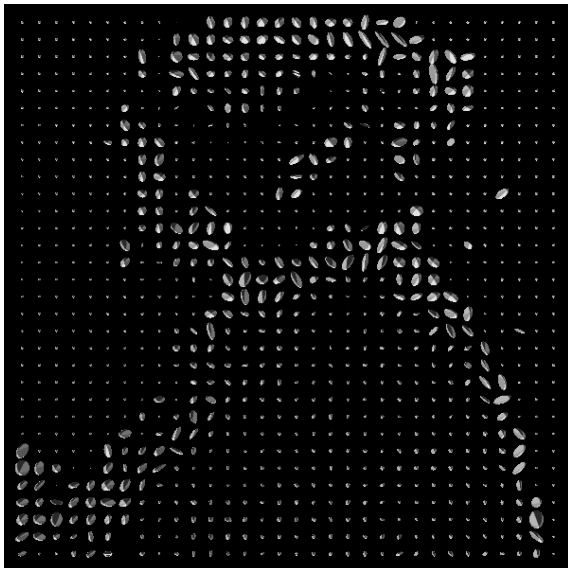
White Top Hat



Black Top Hat



Self-Dual Top Hat

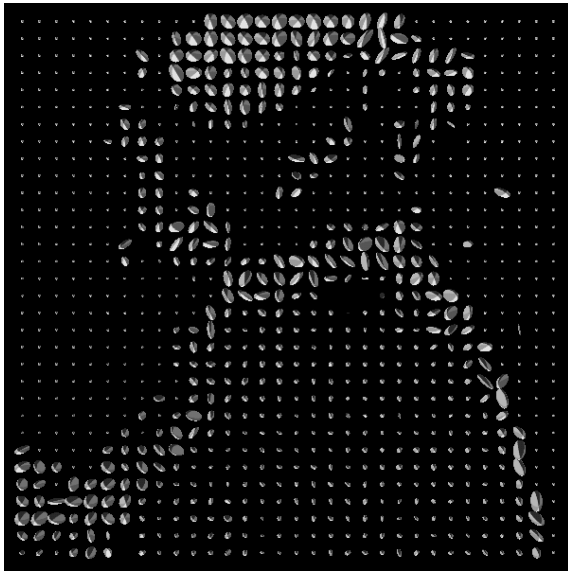


PDE based approach (stopping time 2)

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Experiments: Ordering vs PDE V

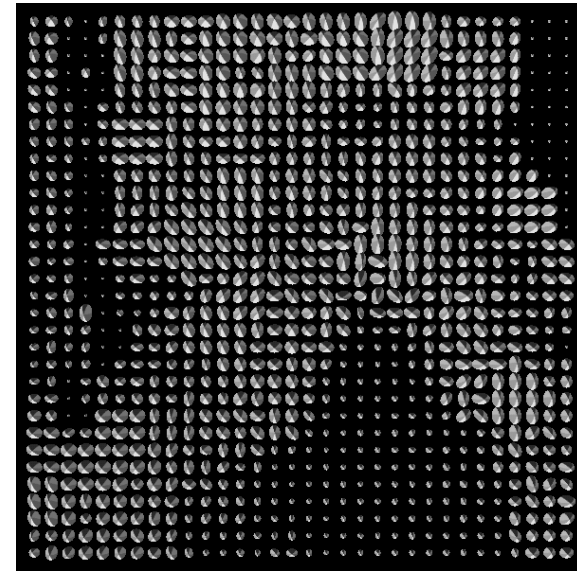
Ordering based approach (ball-shaped structuring element $BSE(\sqrt{2})$)



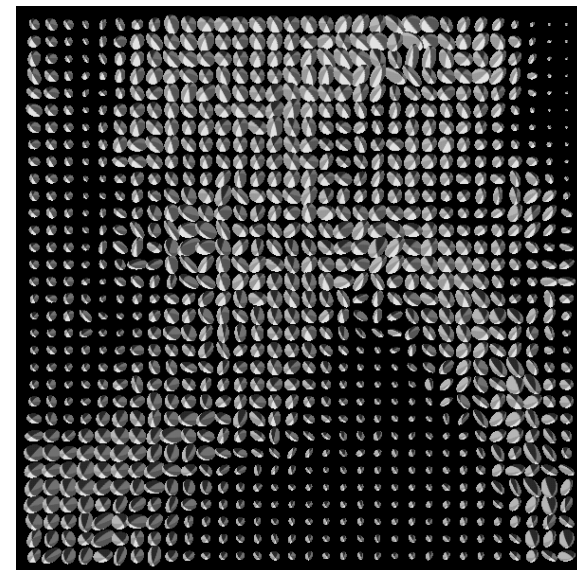
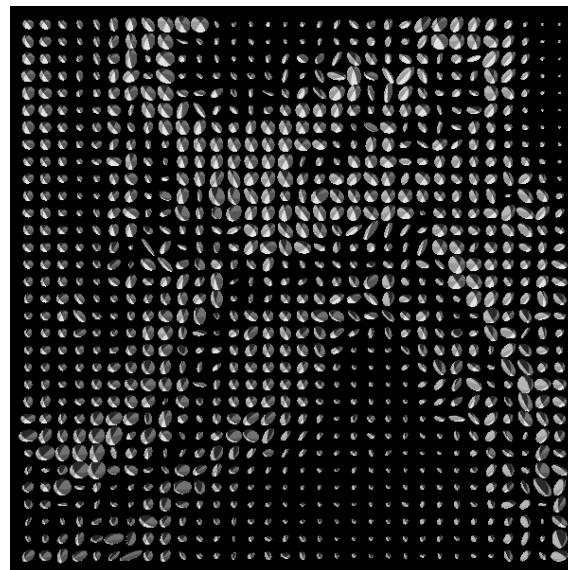
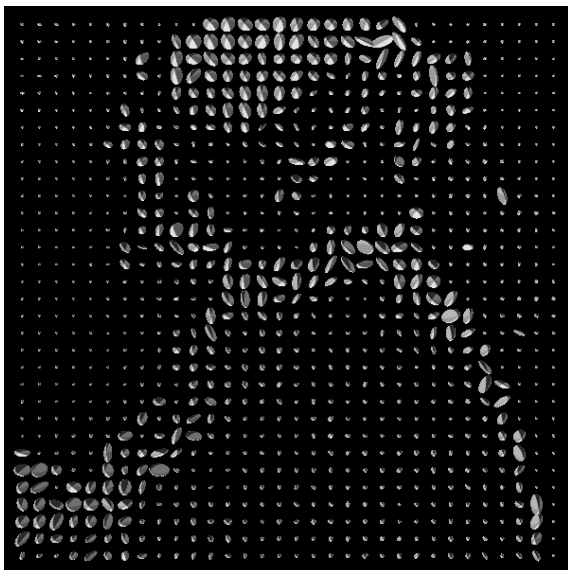
Internal Gradient



External Gradient



Beucher Gradient

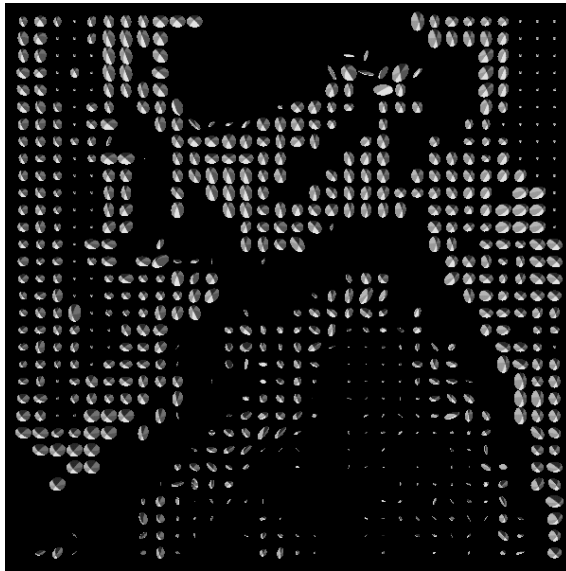


PDE based approach (stopping time 2)

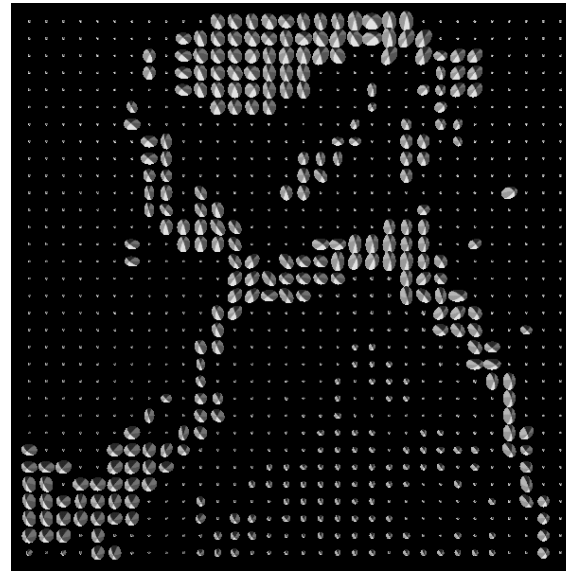
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Experiments: Ordering vs PDE VI

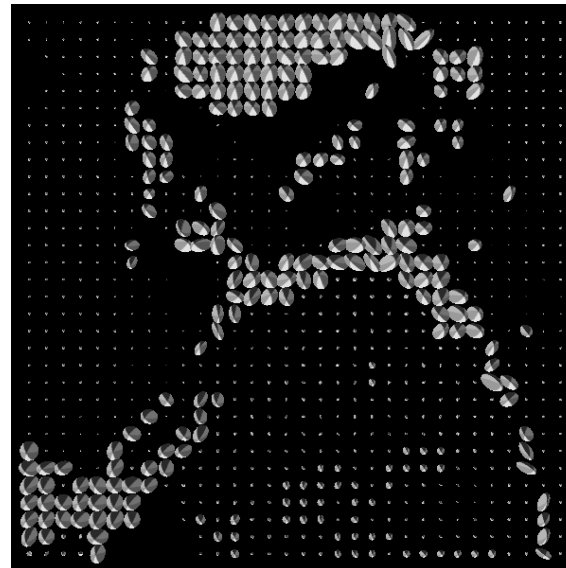
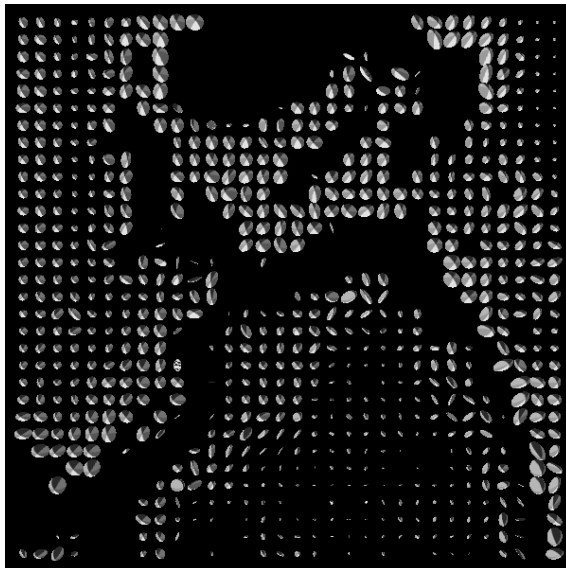
Ordering based approach (ball-shaped structuring element $BSE(\sqrt{2})$)



Morphological Laplacian



Shock Filter



PDE based approach (stopping time 2)

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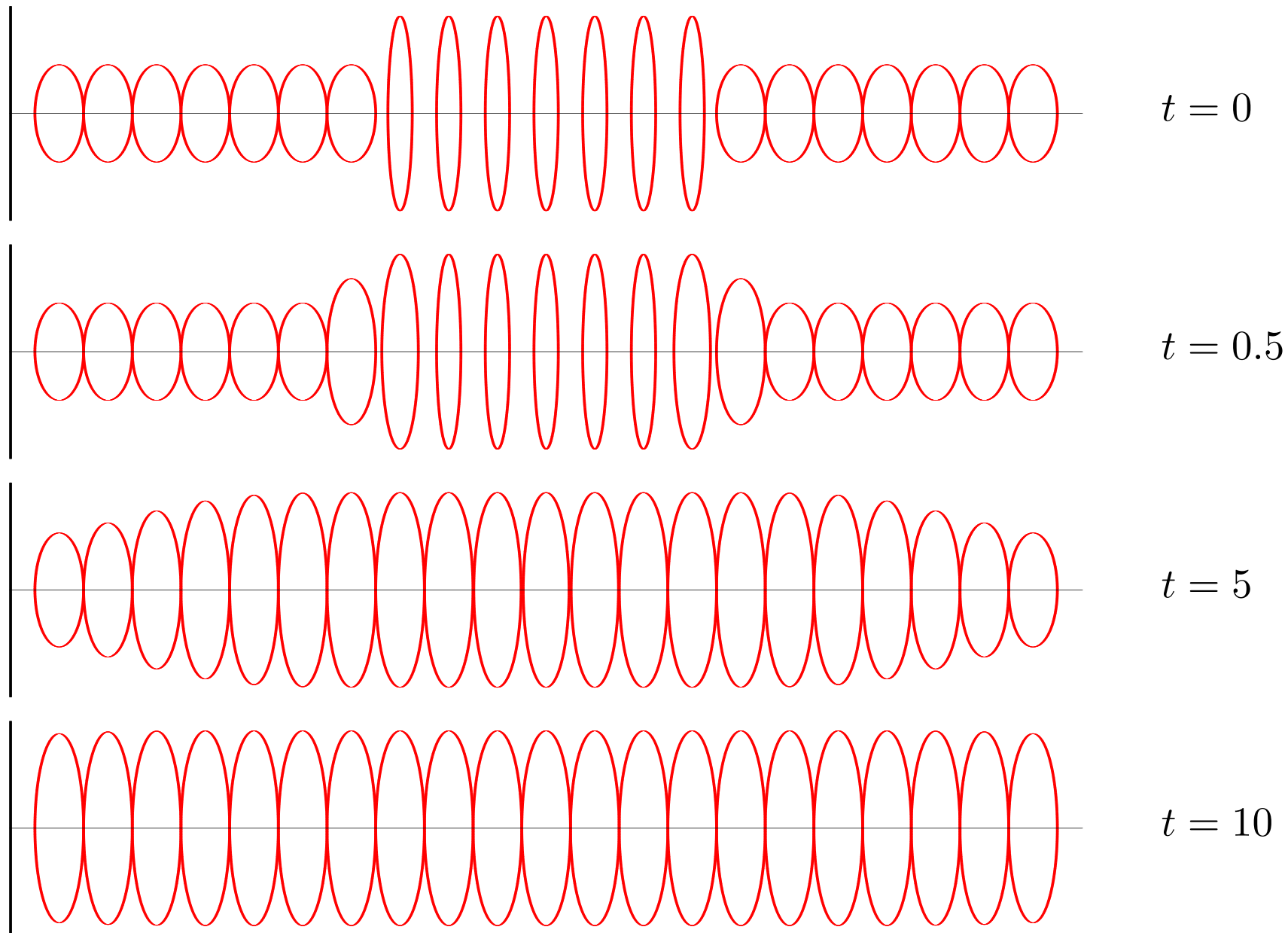
Concluding Remarks

- ◆ Two novel approaches to mathematical morphology for matrix fields:
 - A novel notion for the supremum and infimum of a set of matrices based on the **Loewner ordering**
 - A truly matrix-valued counterpart for nonlinear morphological **PDEs**
- ◆ **Numerical schemes** for scalar PDEs can be transferred to symmetric matrices.
- ◆ The properties of the proposed concepts allow for the application of
 - basic morphological operations as well as
 - morphological **derivatives**to matrix-valued data
- ◆ However, matrix data are “high dimensional” data and some scalar concepts might **not** be directly transferable (discontinuity, ordering, oscillation,...)
- ◆ **Ongoing research** concentrates on the development of more sophisticated operations for matrix fields based on the above notions.

Thank you very much for your attention!

Experiments I

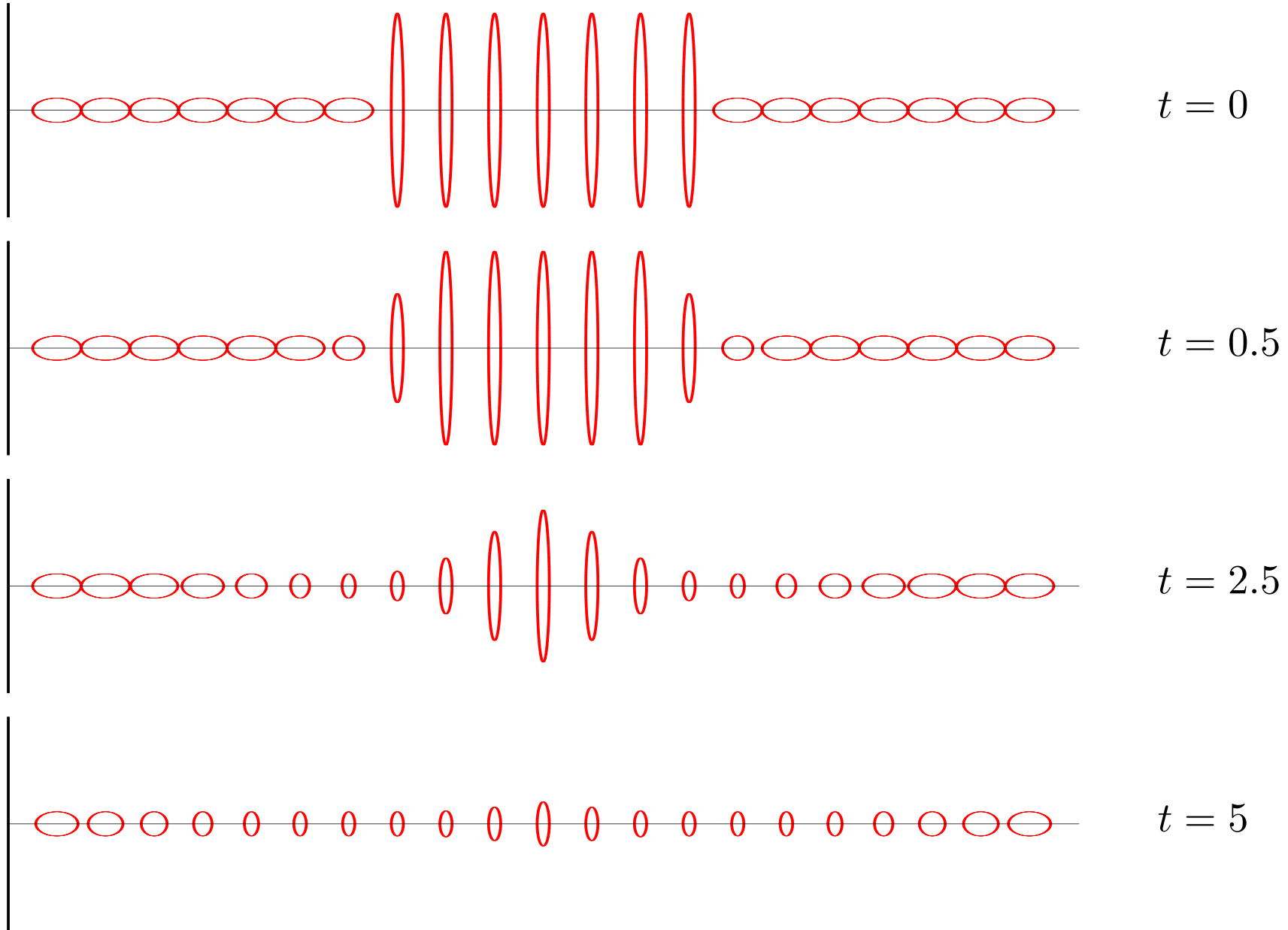
Experiments in 1D: PDE-driven dilation



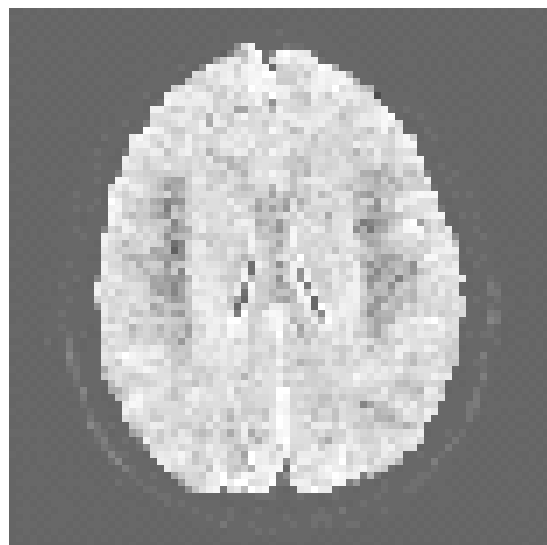
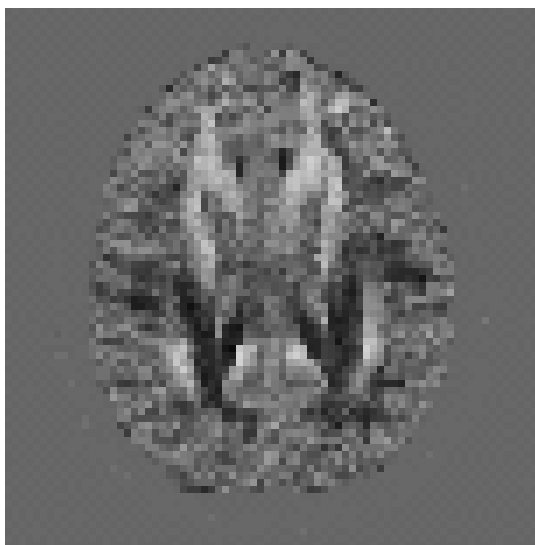
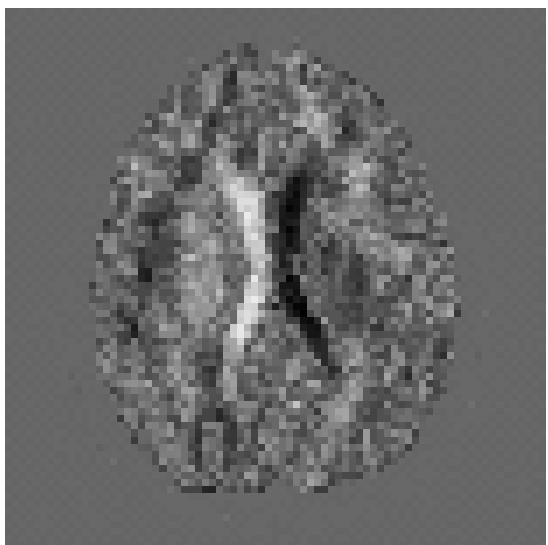
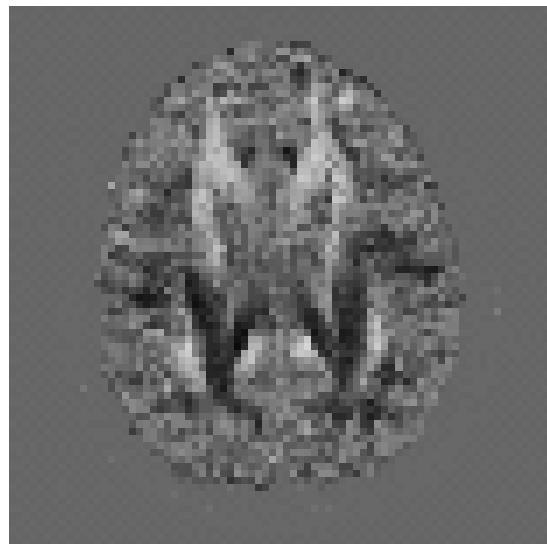
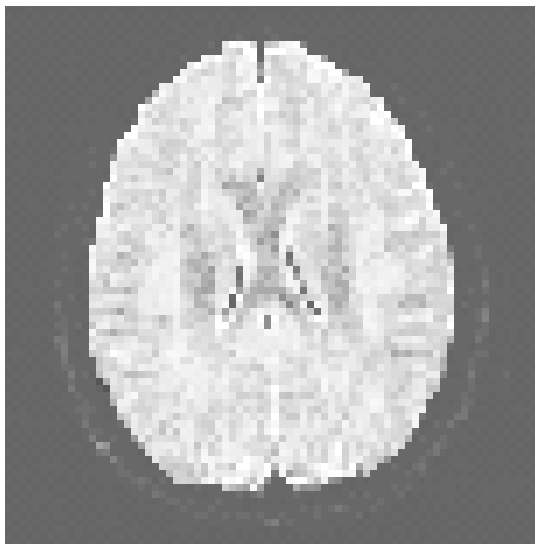
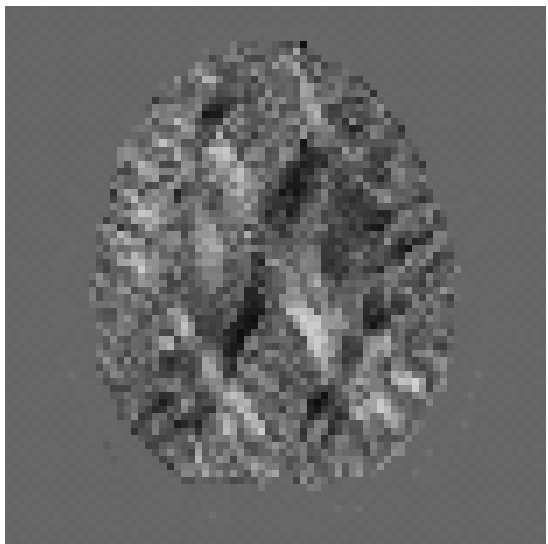
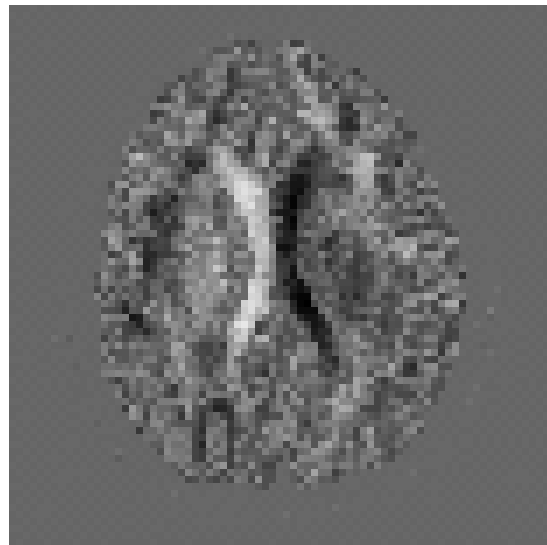
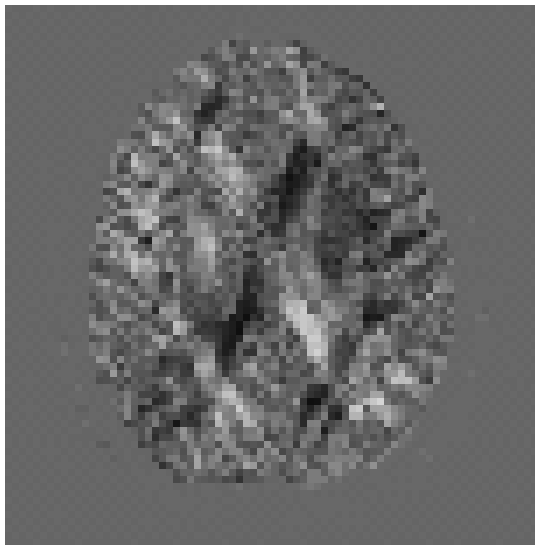
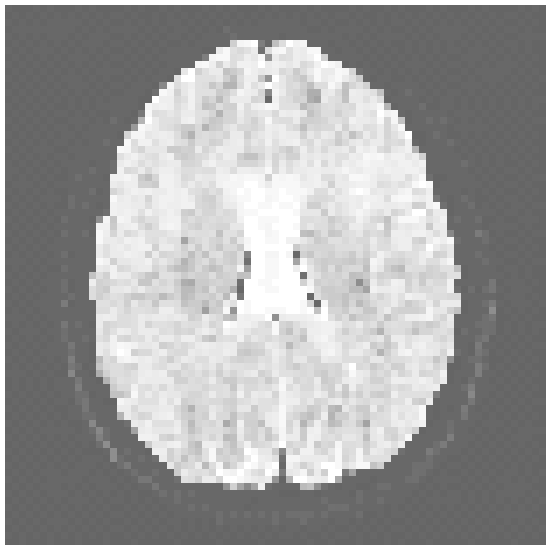
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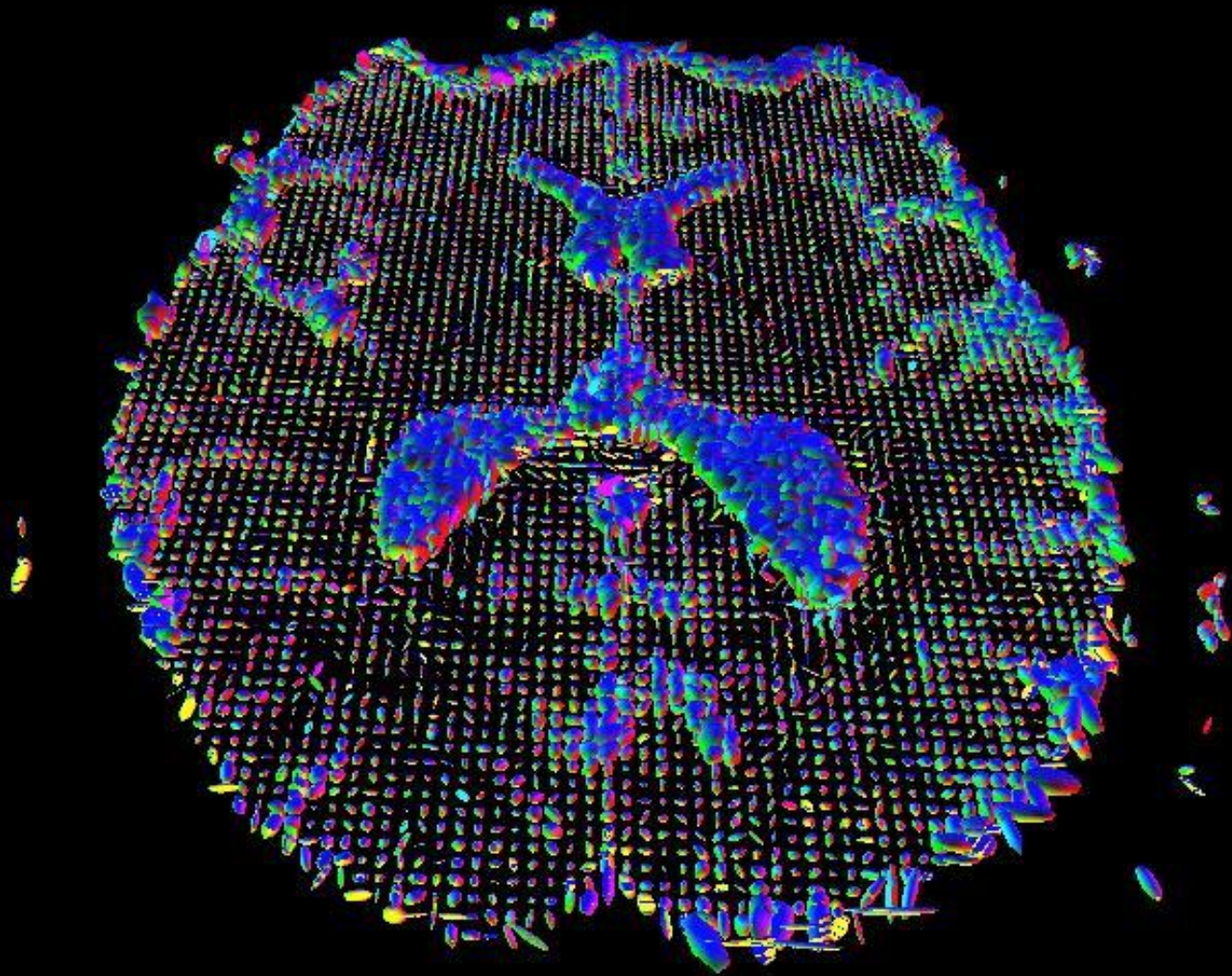
Experiments II

Experiments in 1D: PDE-driven erosion



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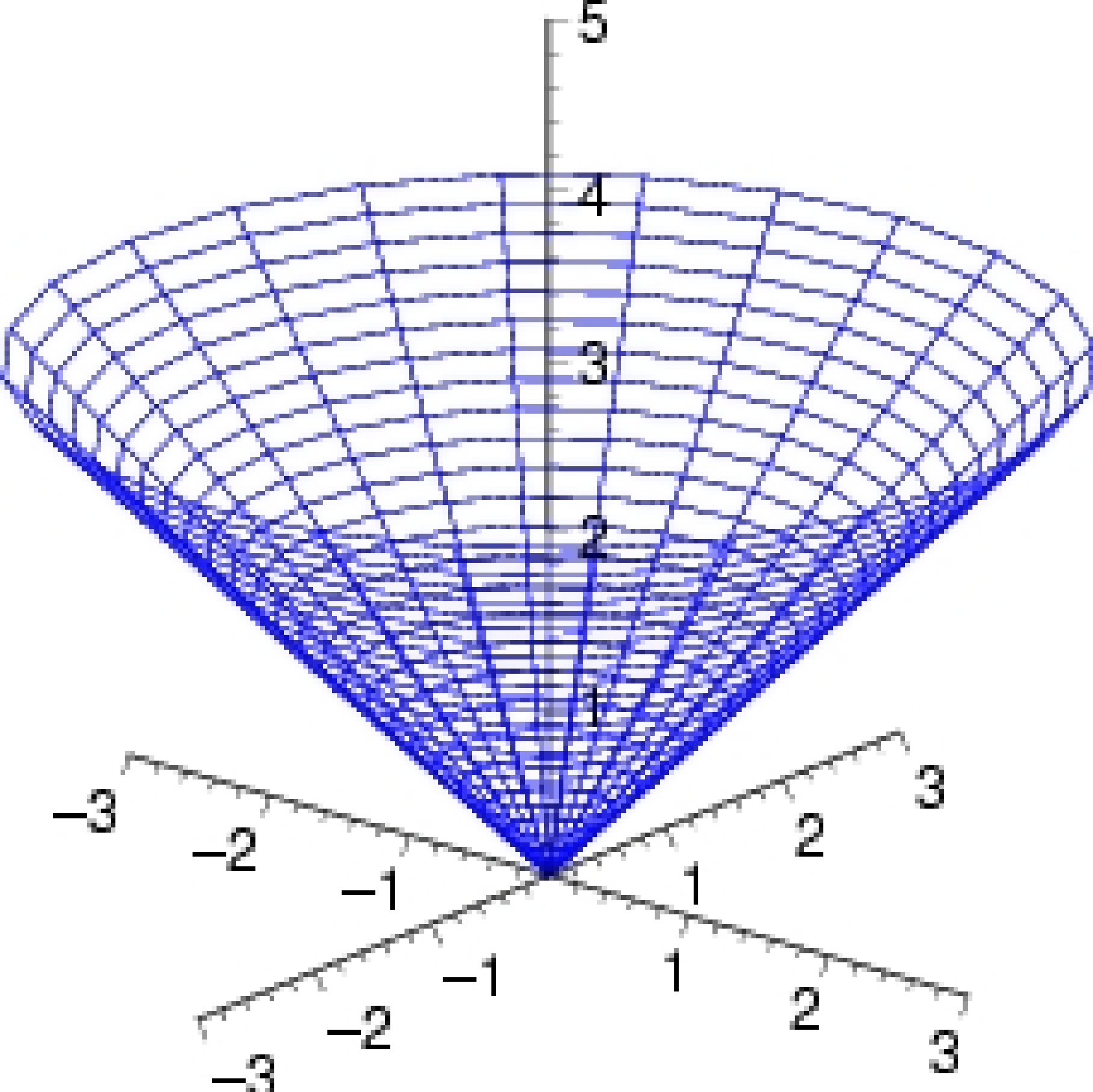


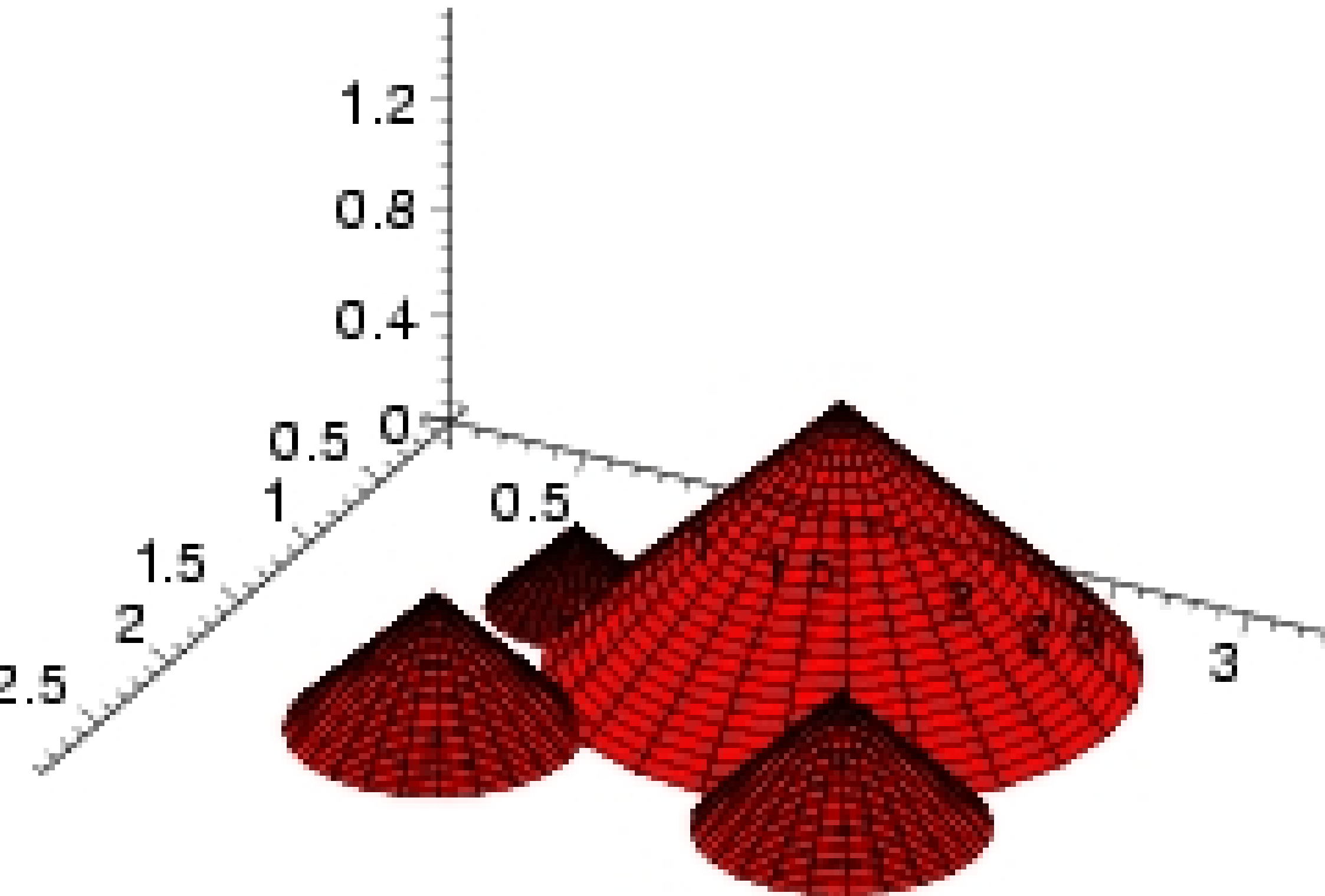


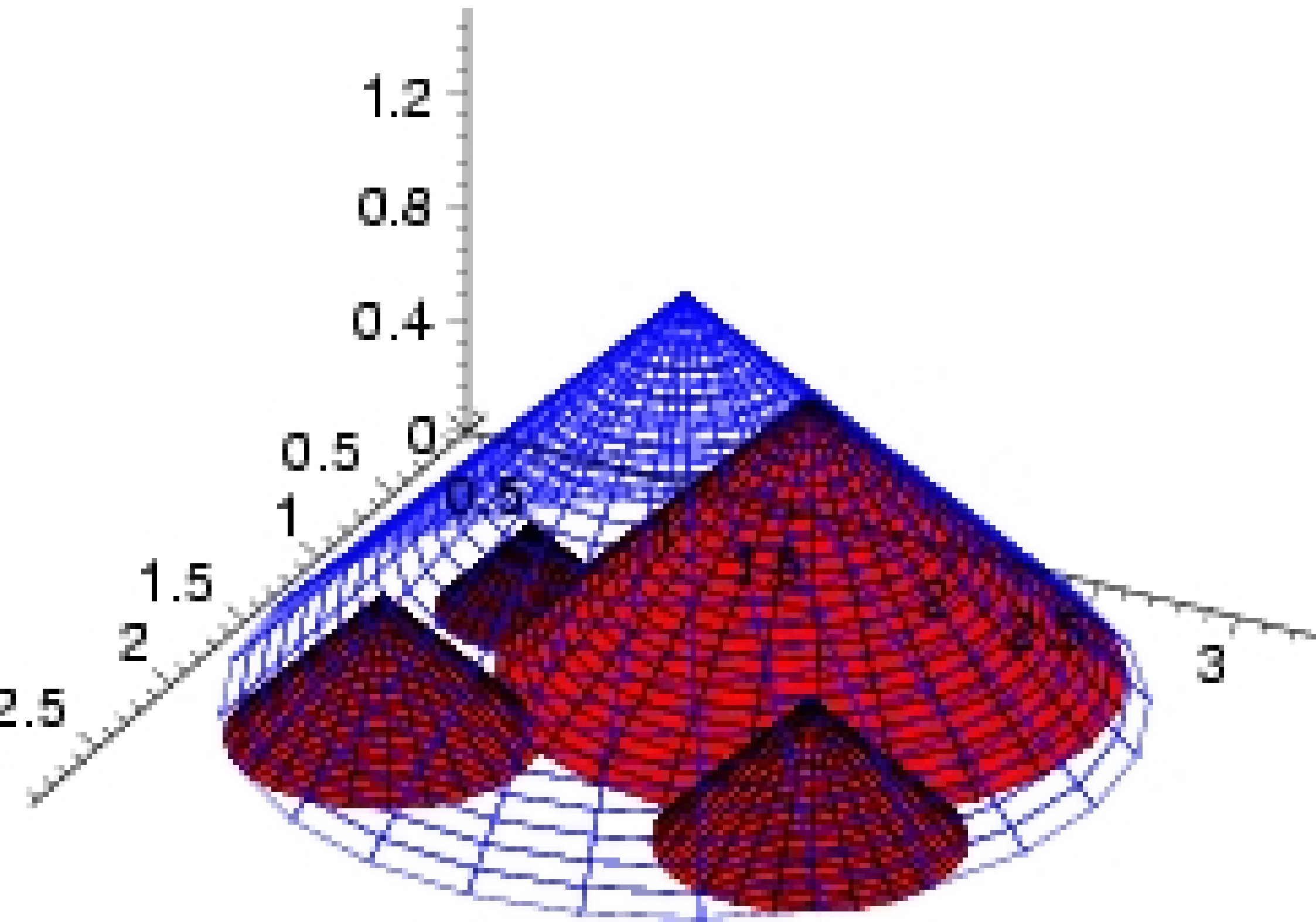


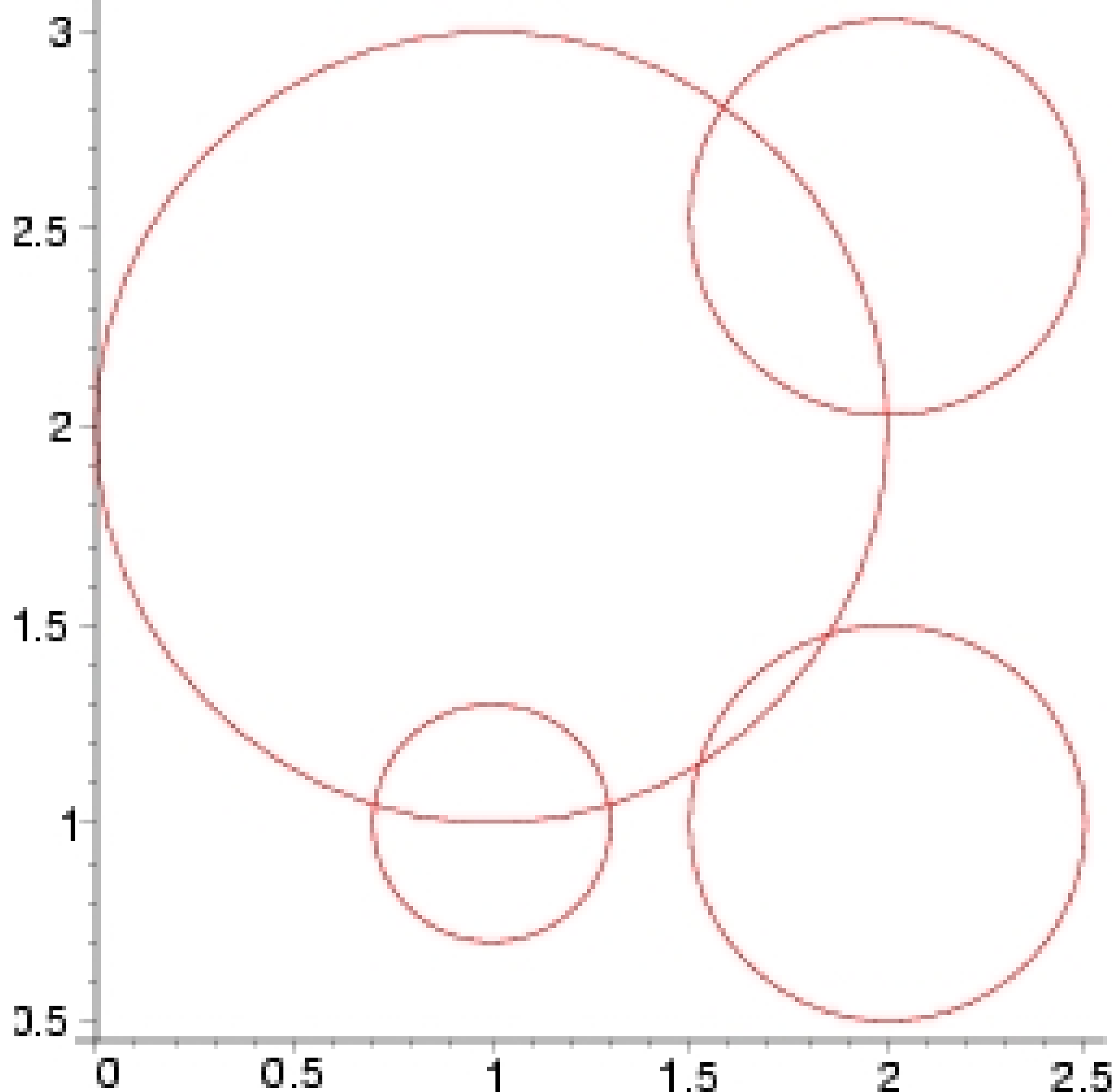


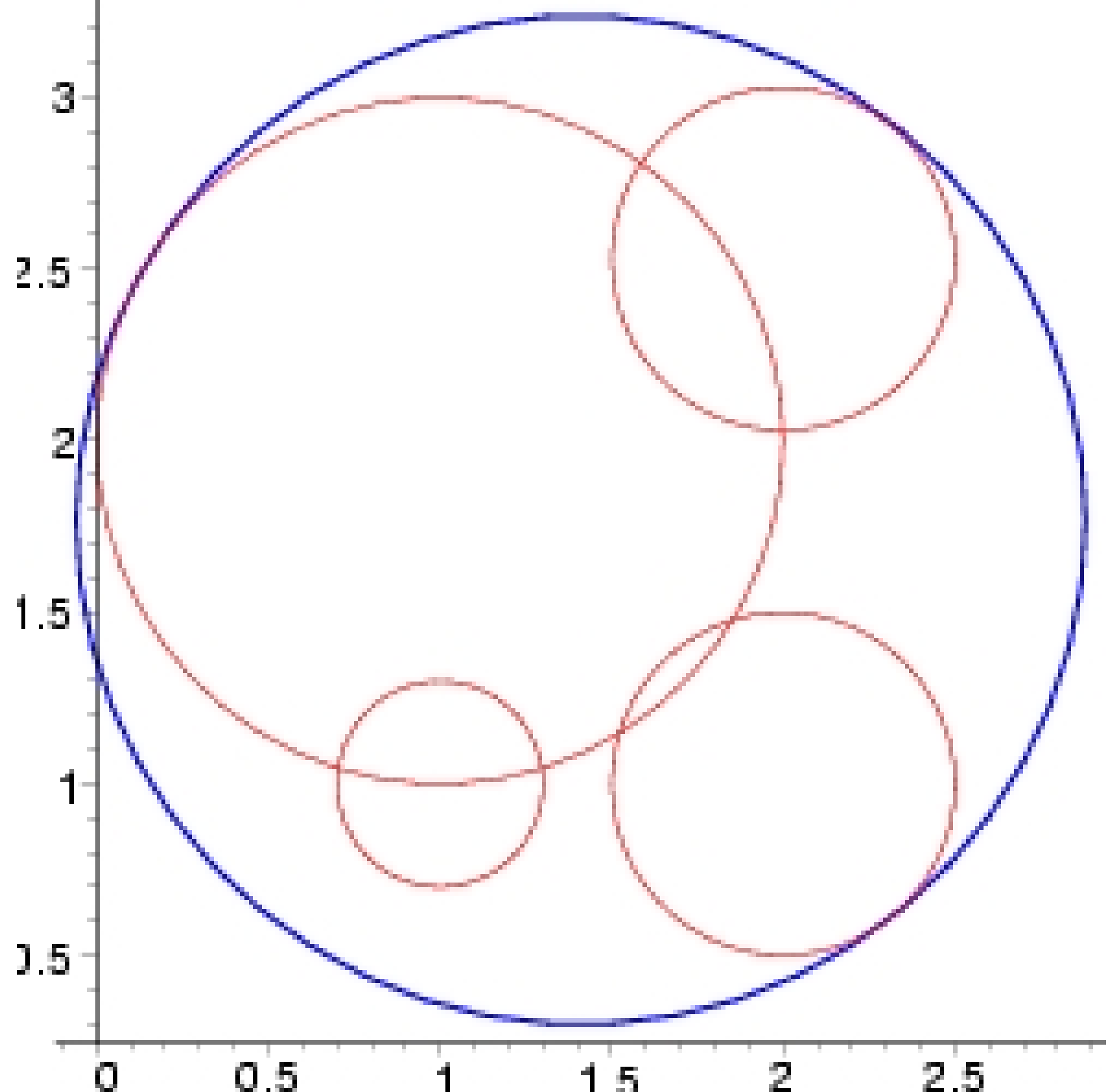


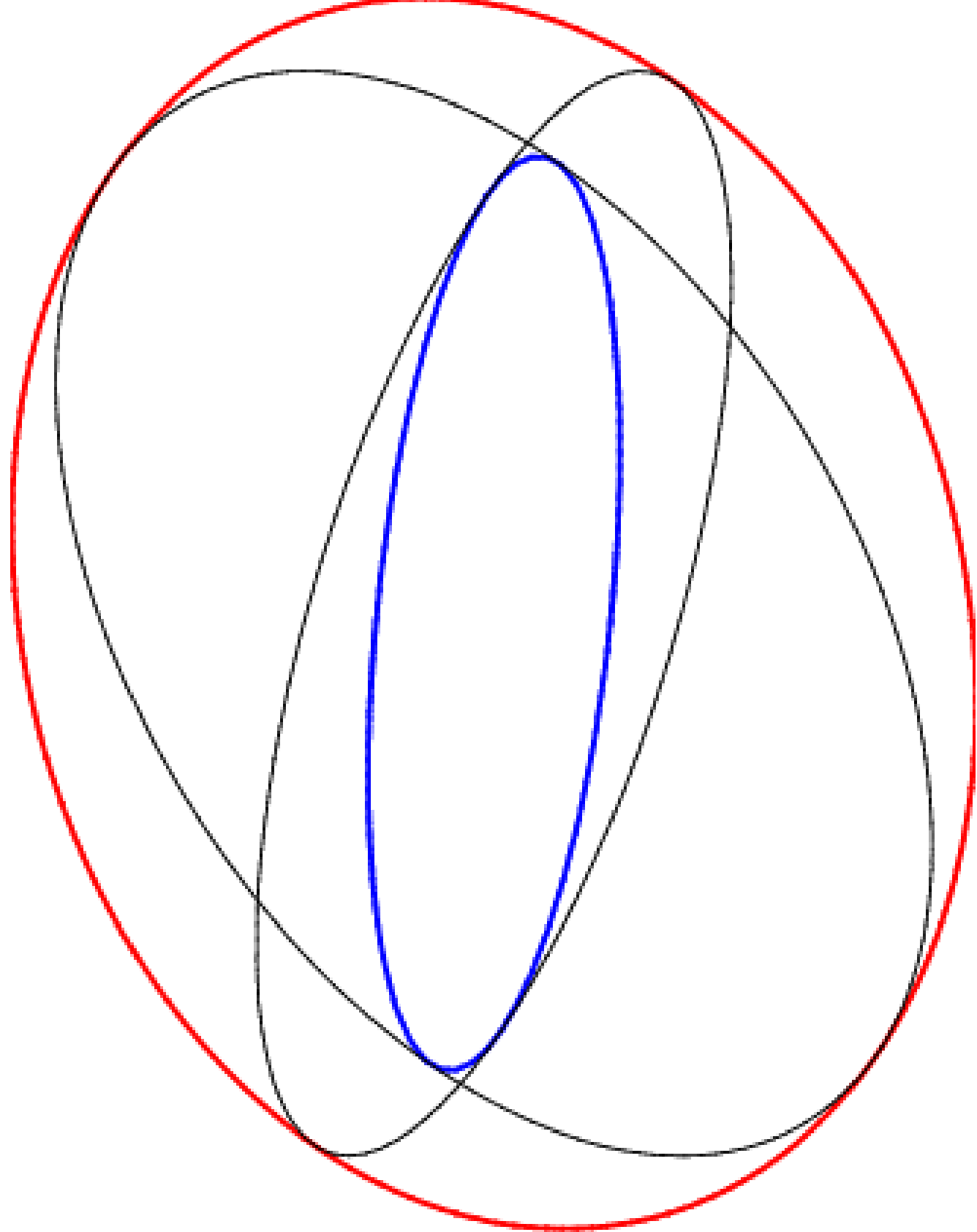








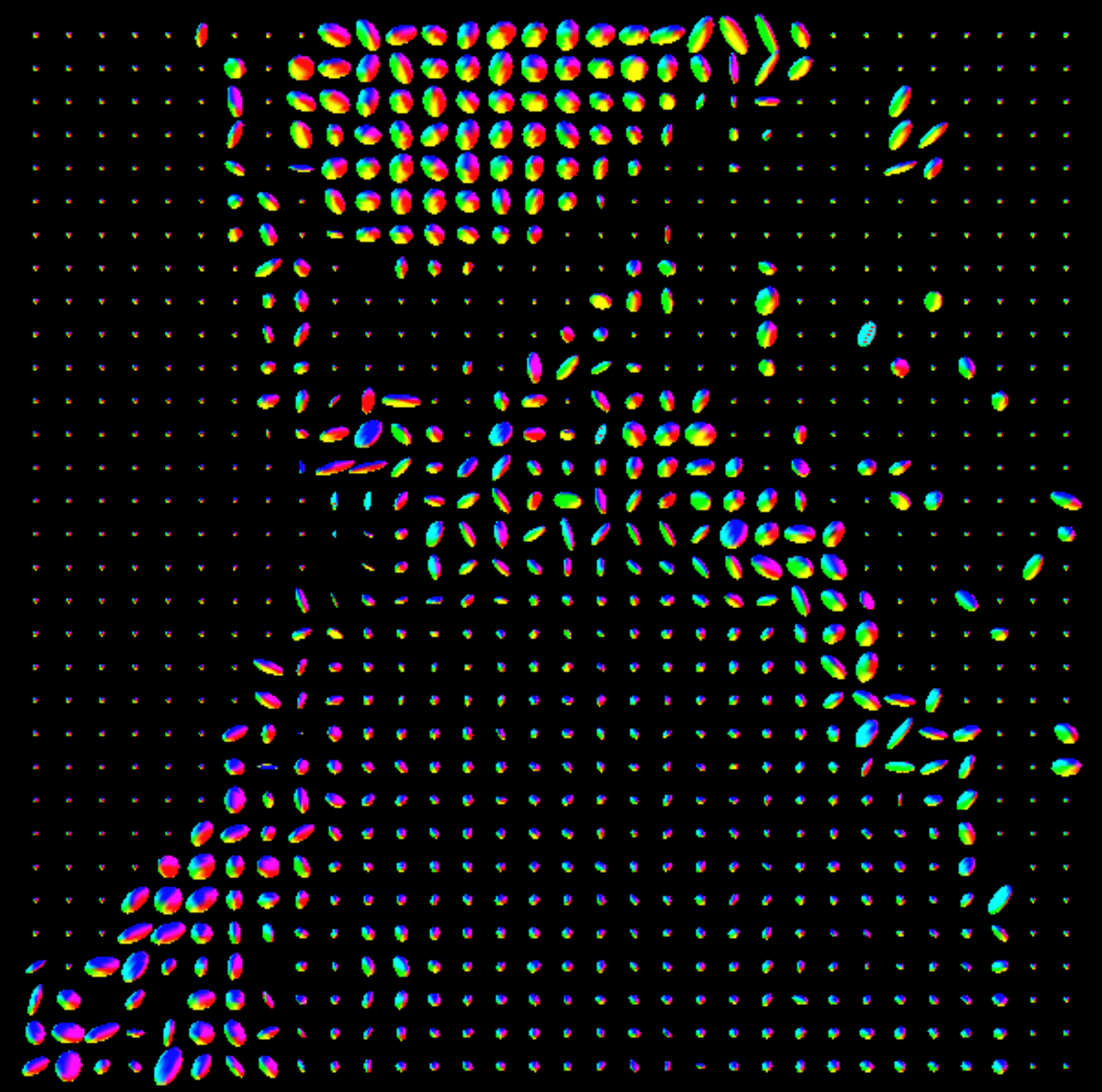




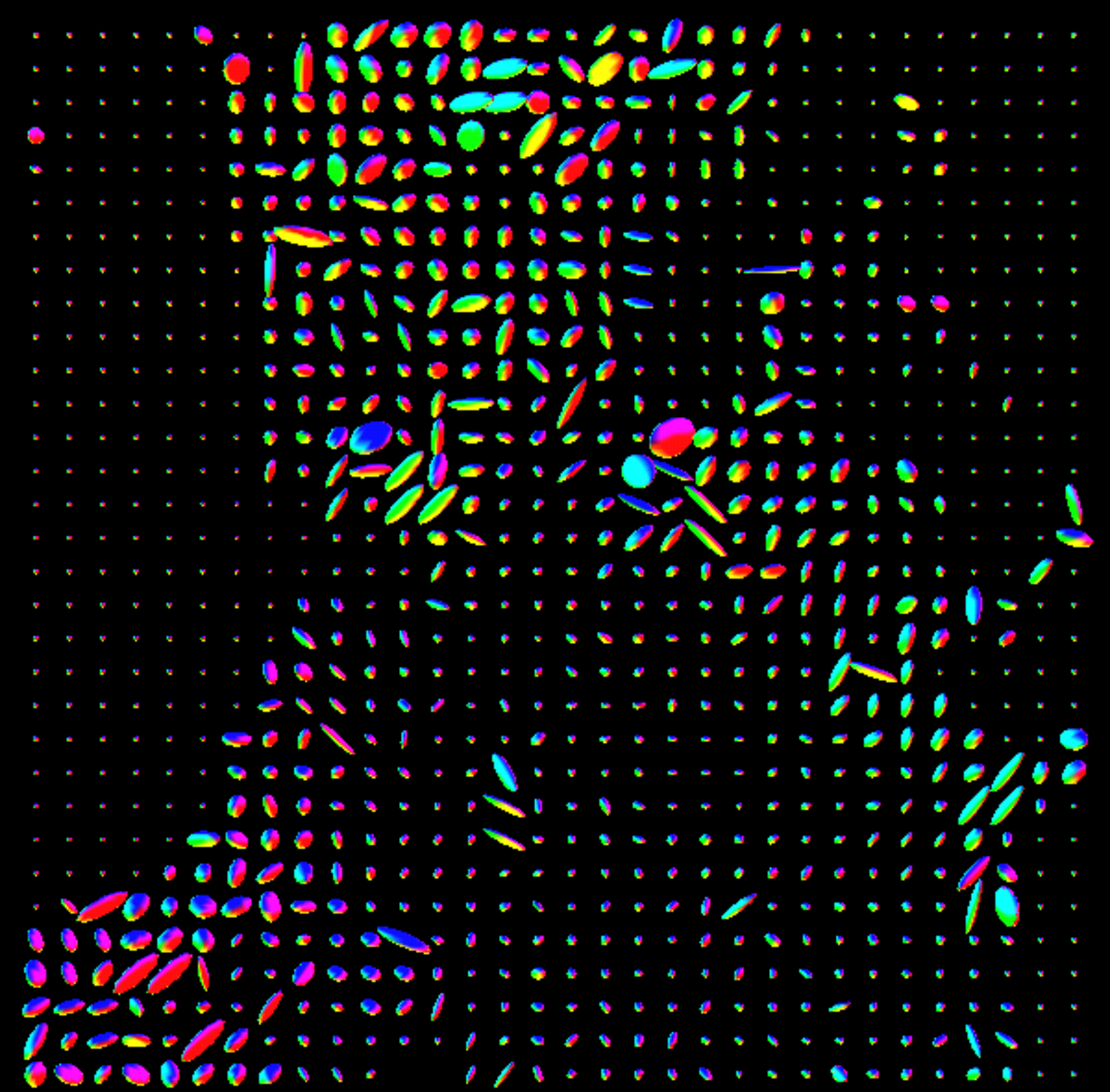


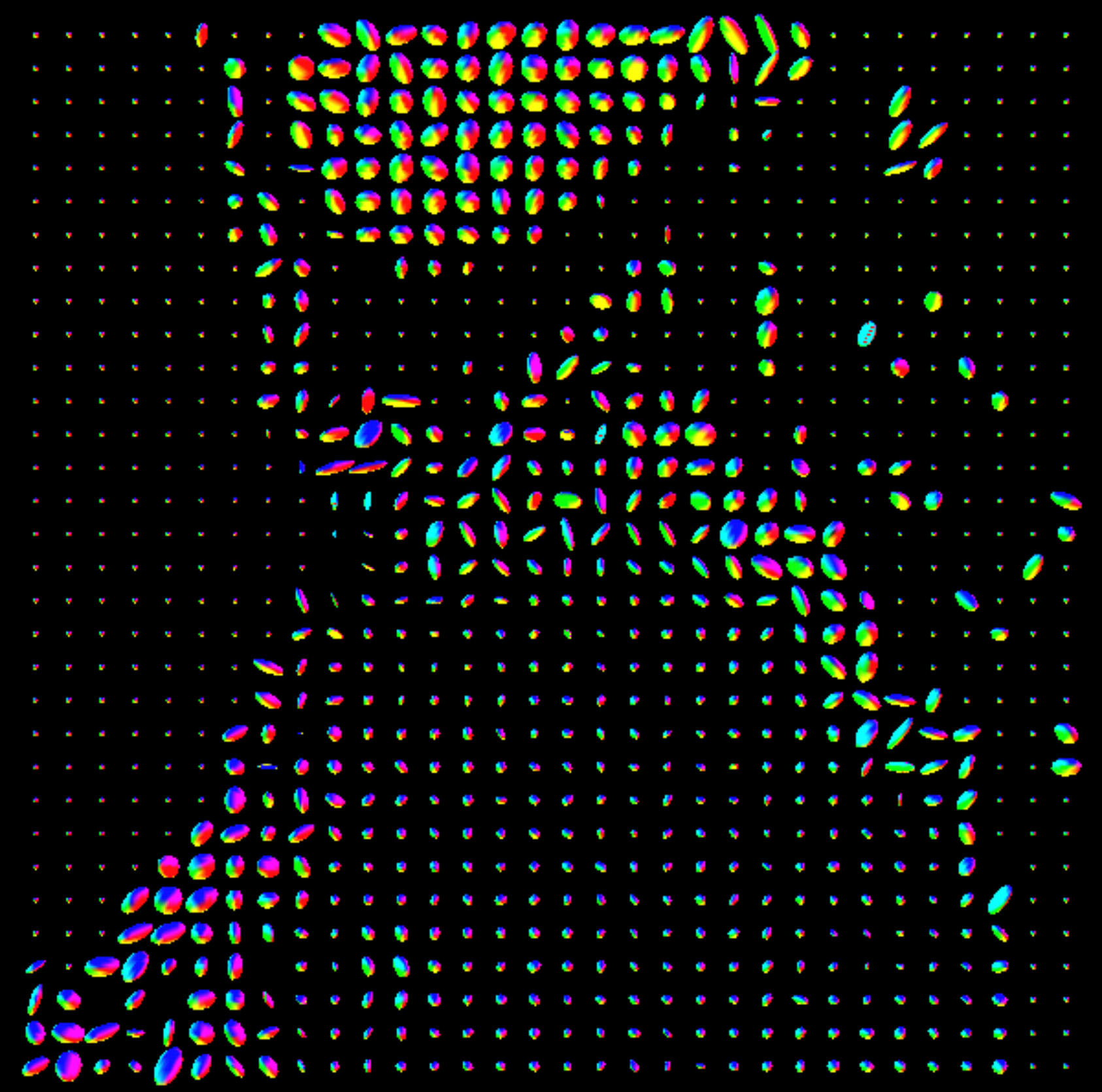


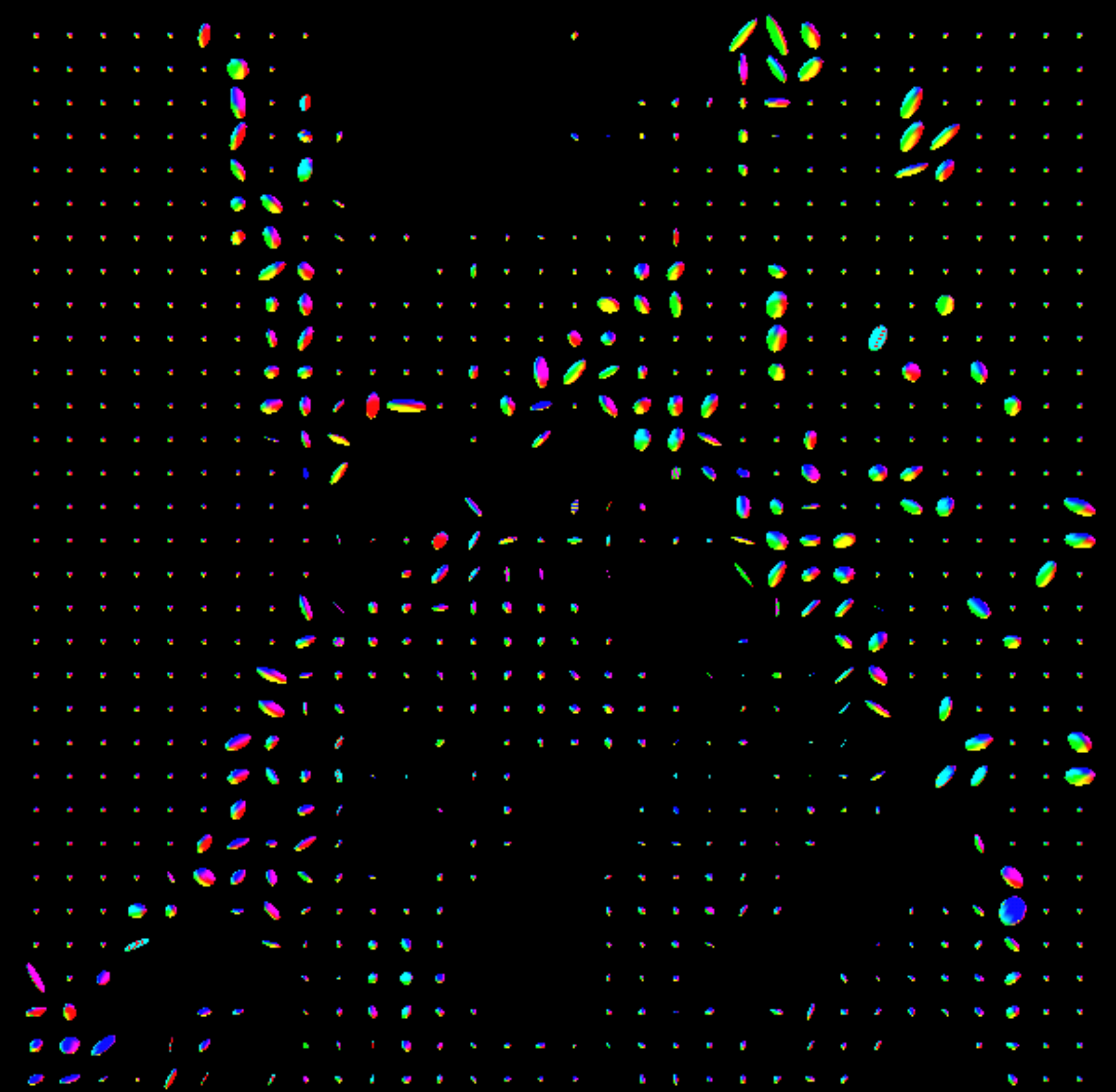




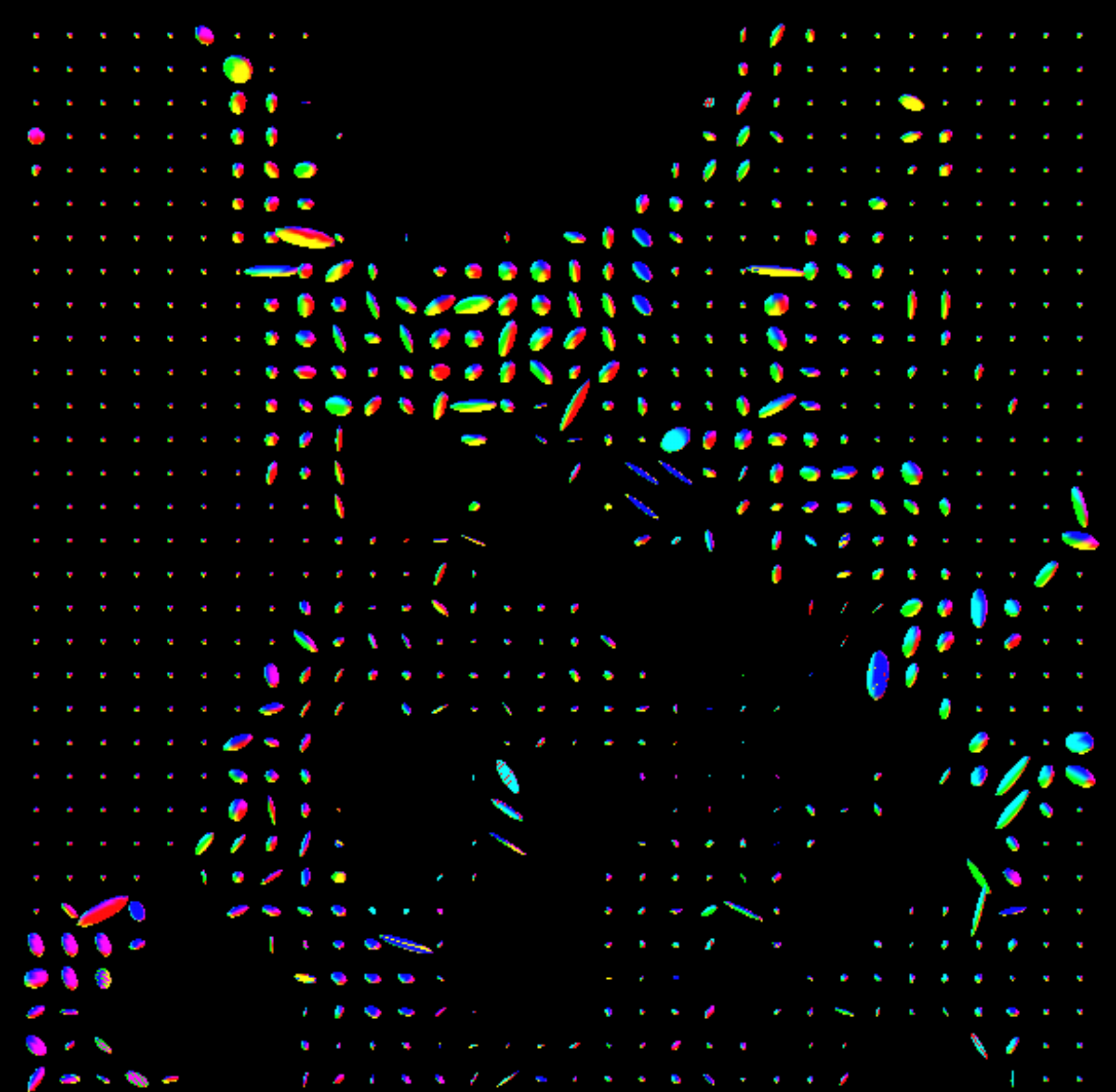


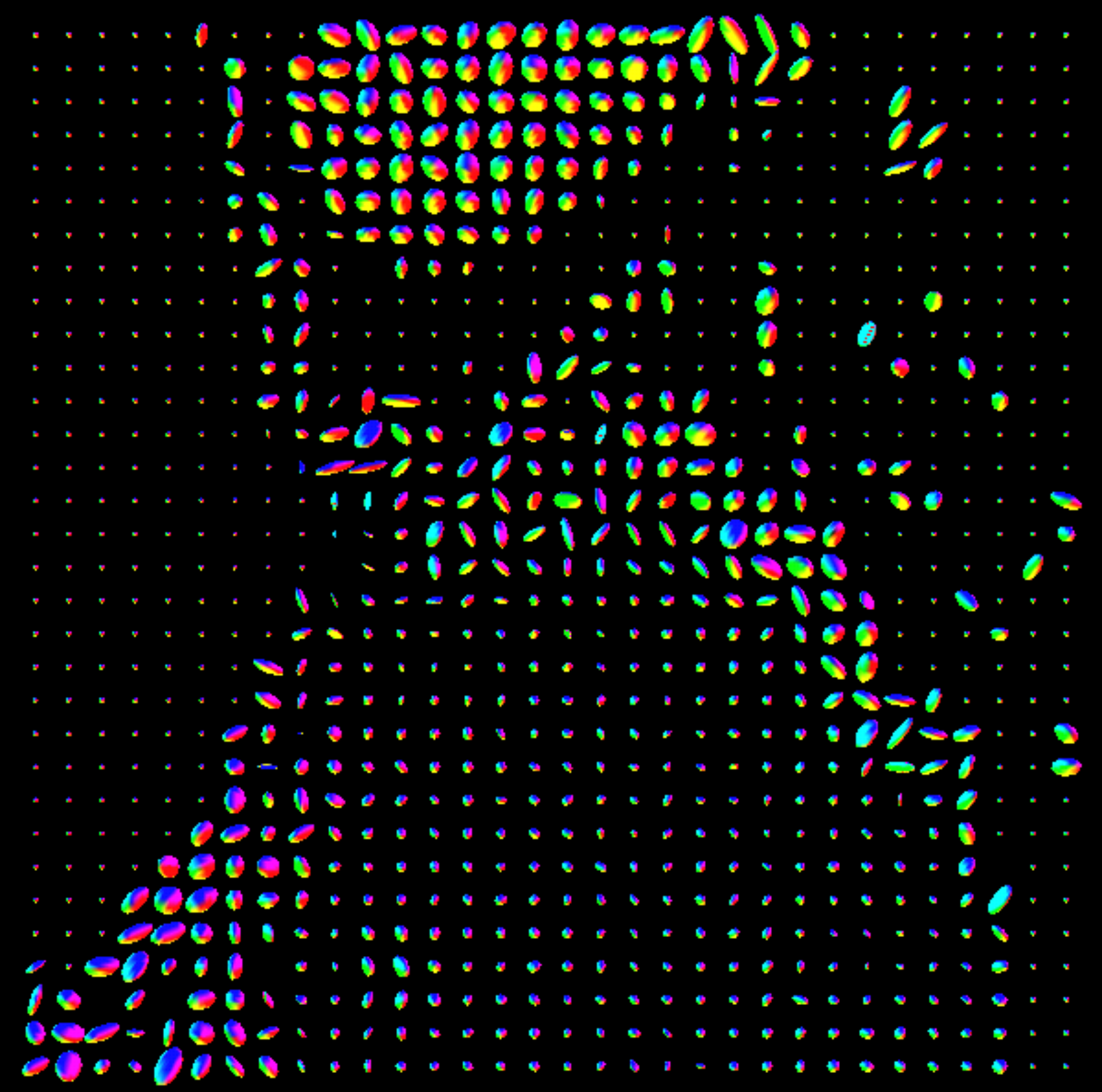


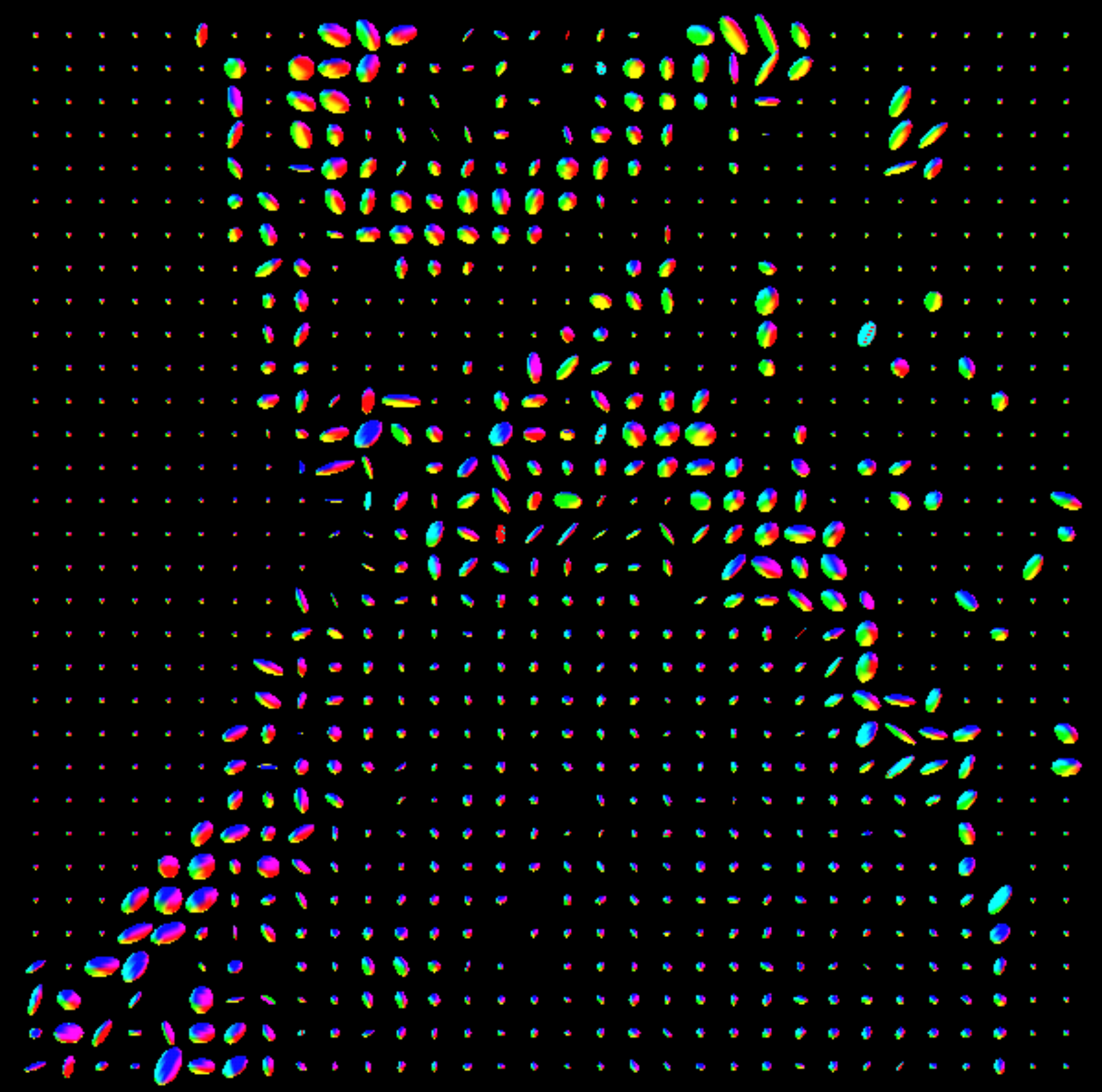


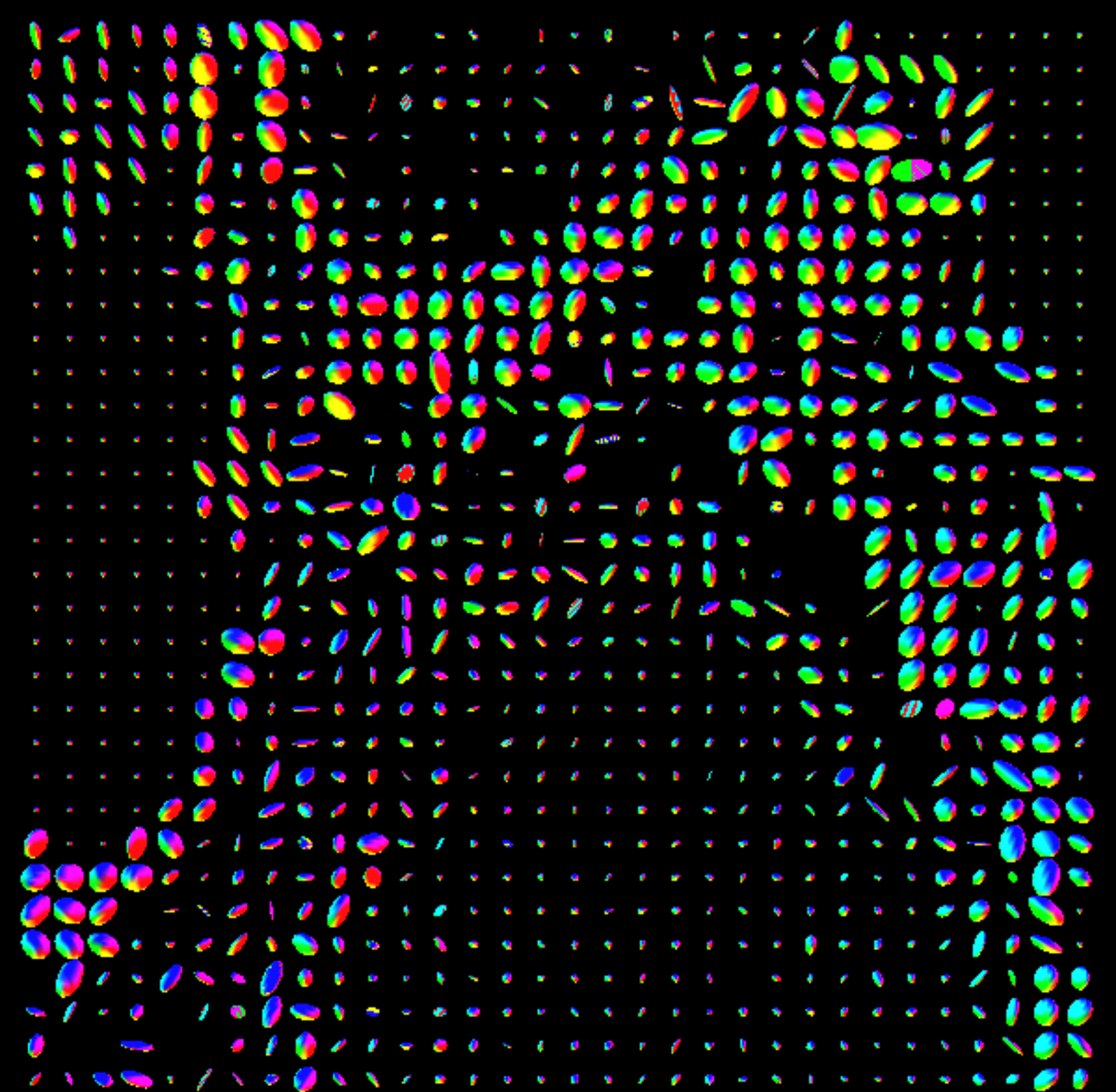


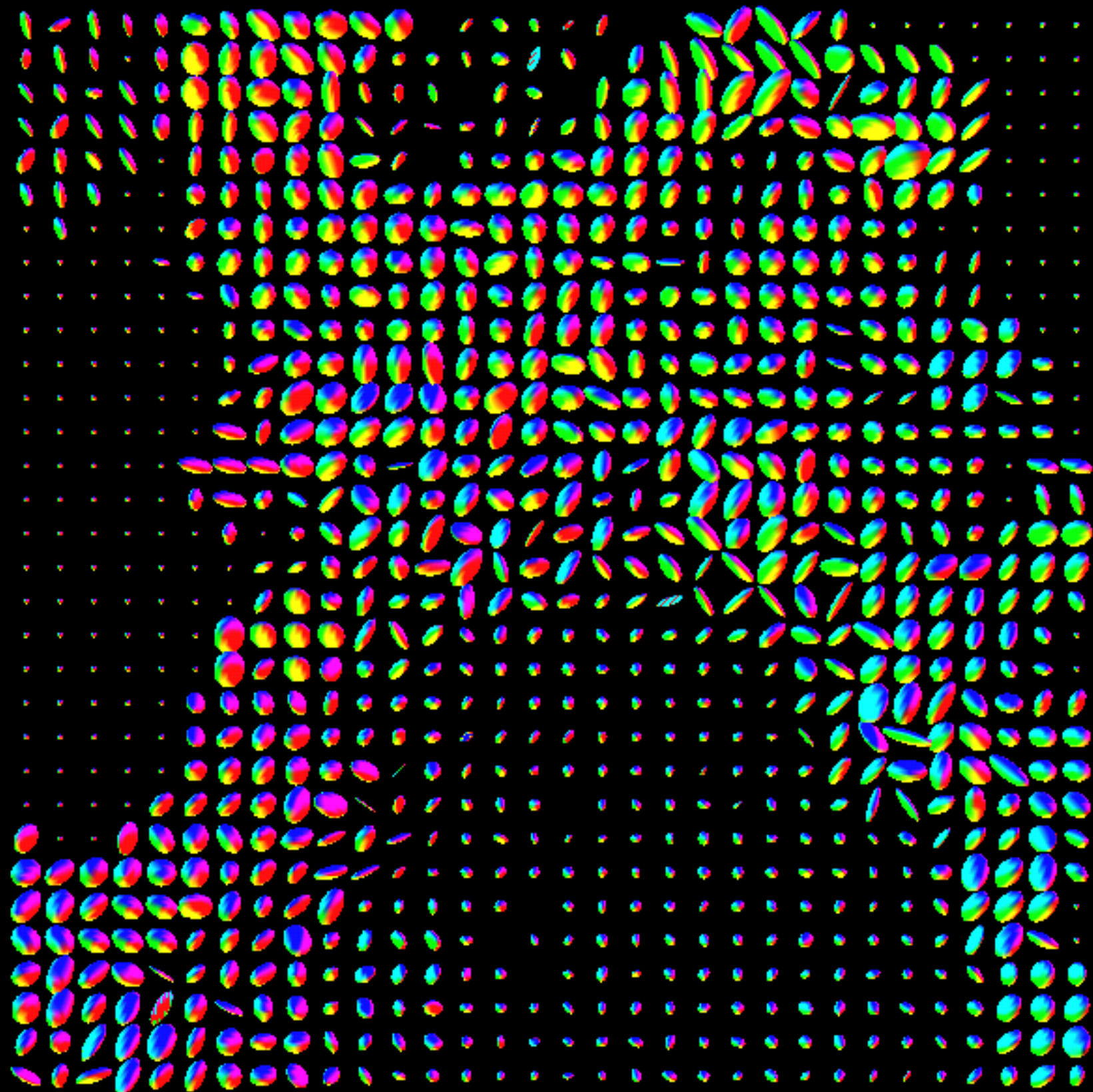


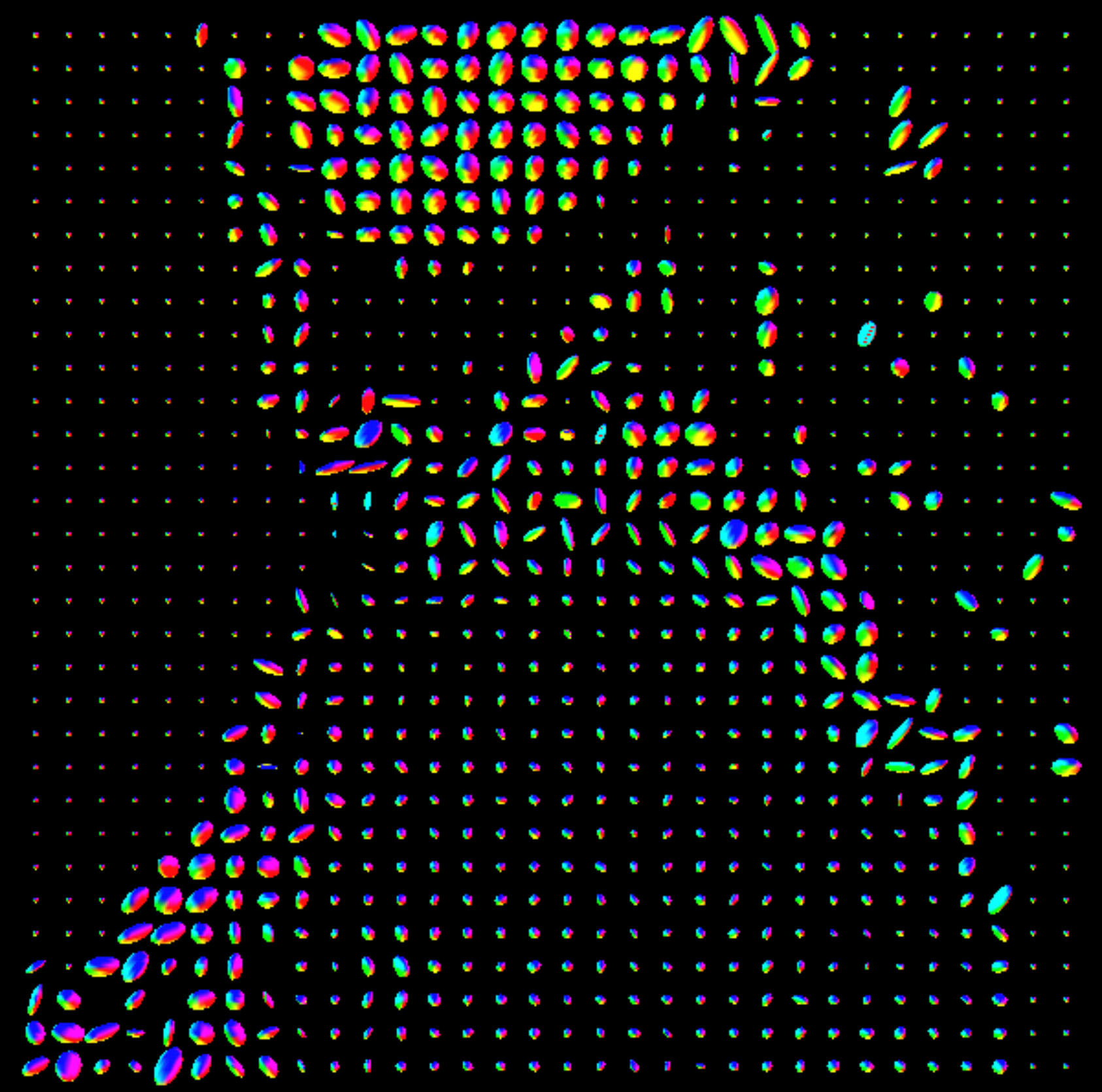


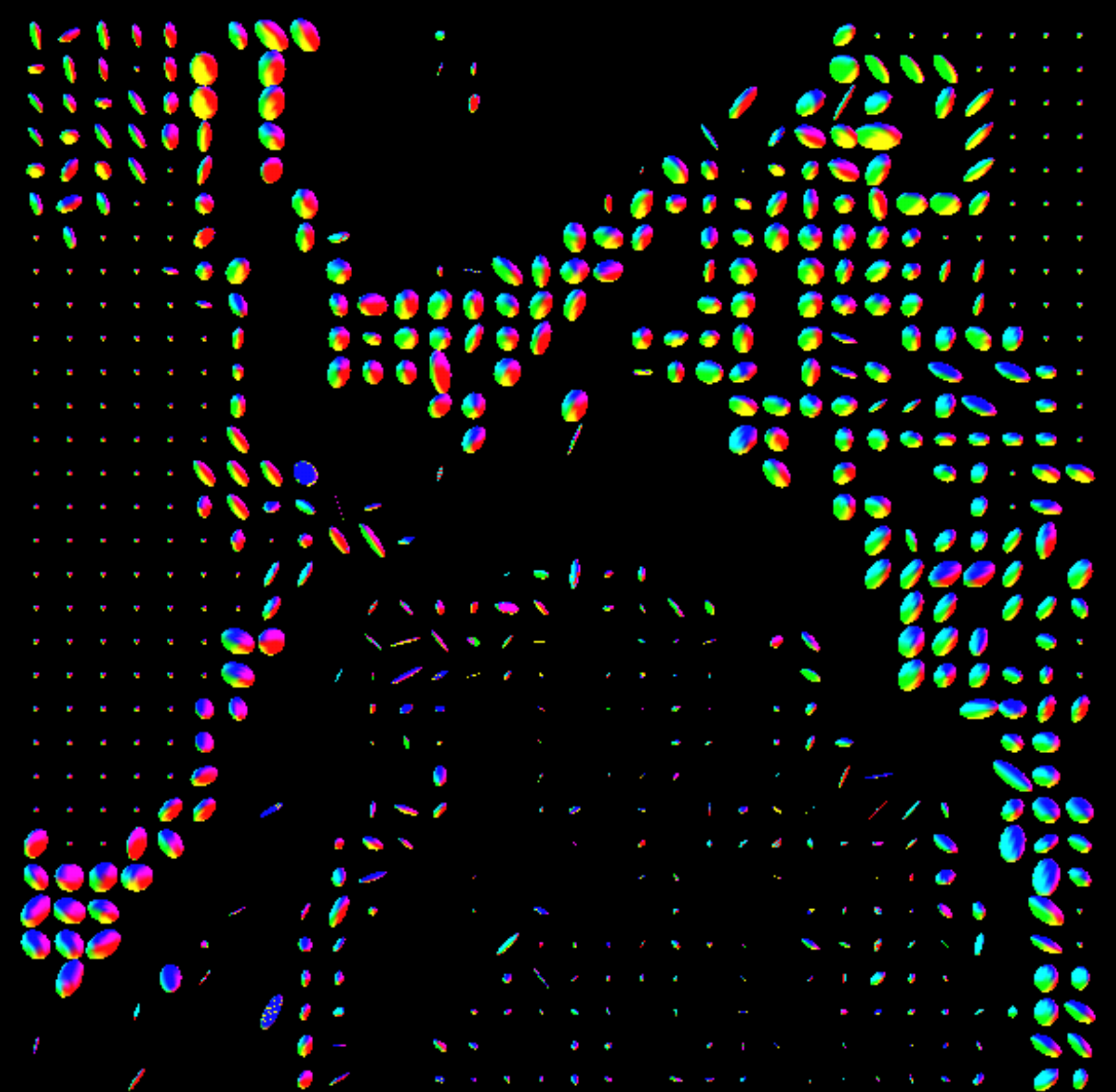


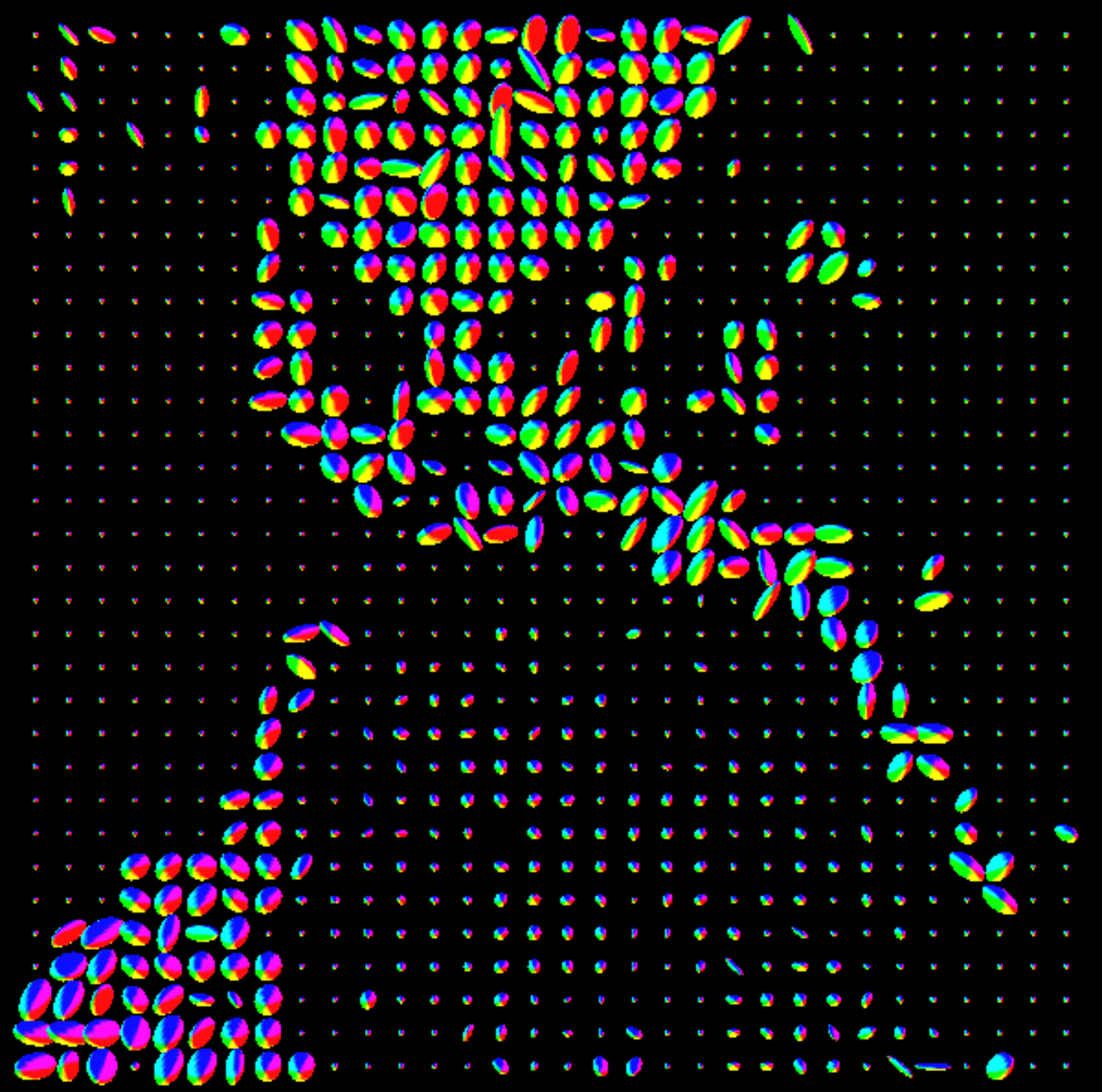


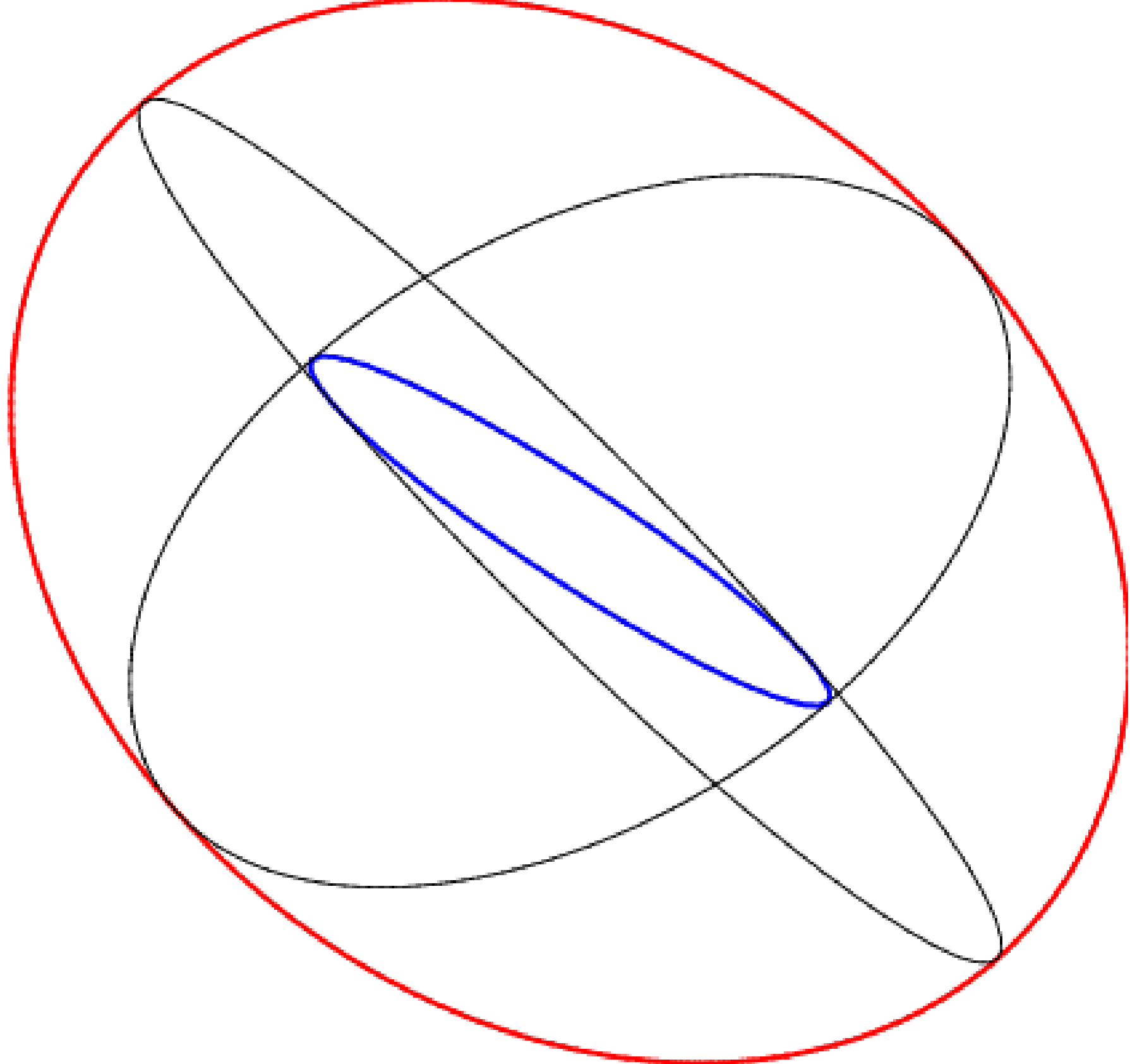


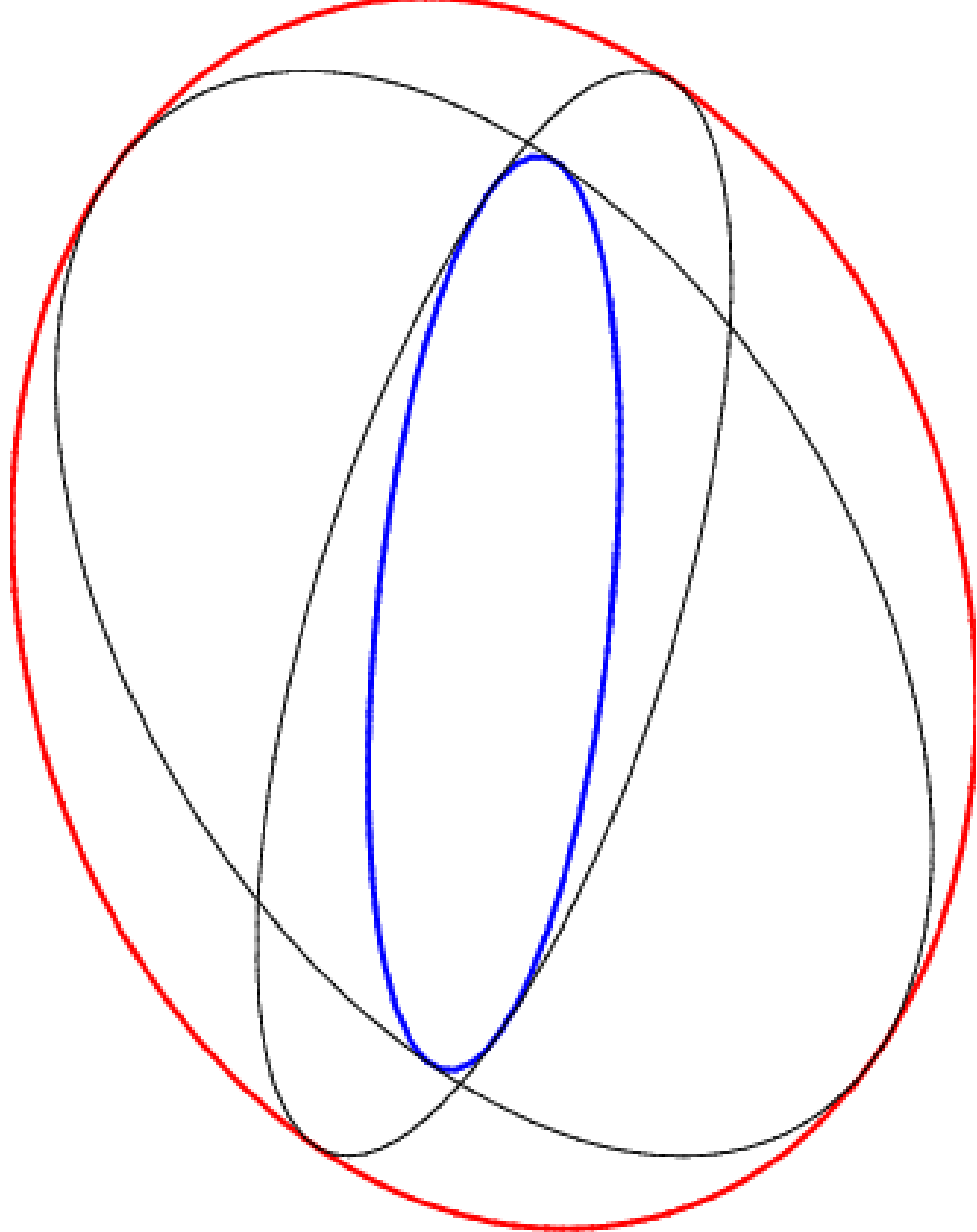


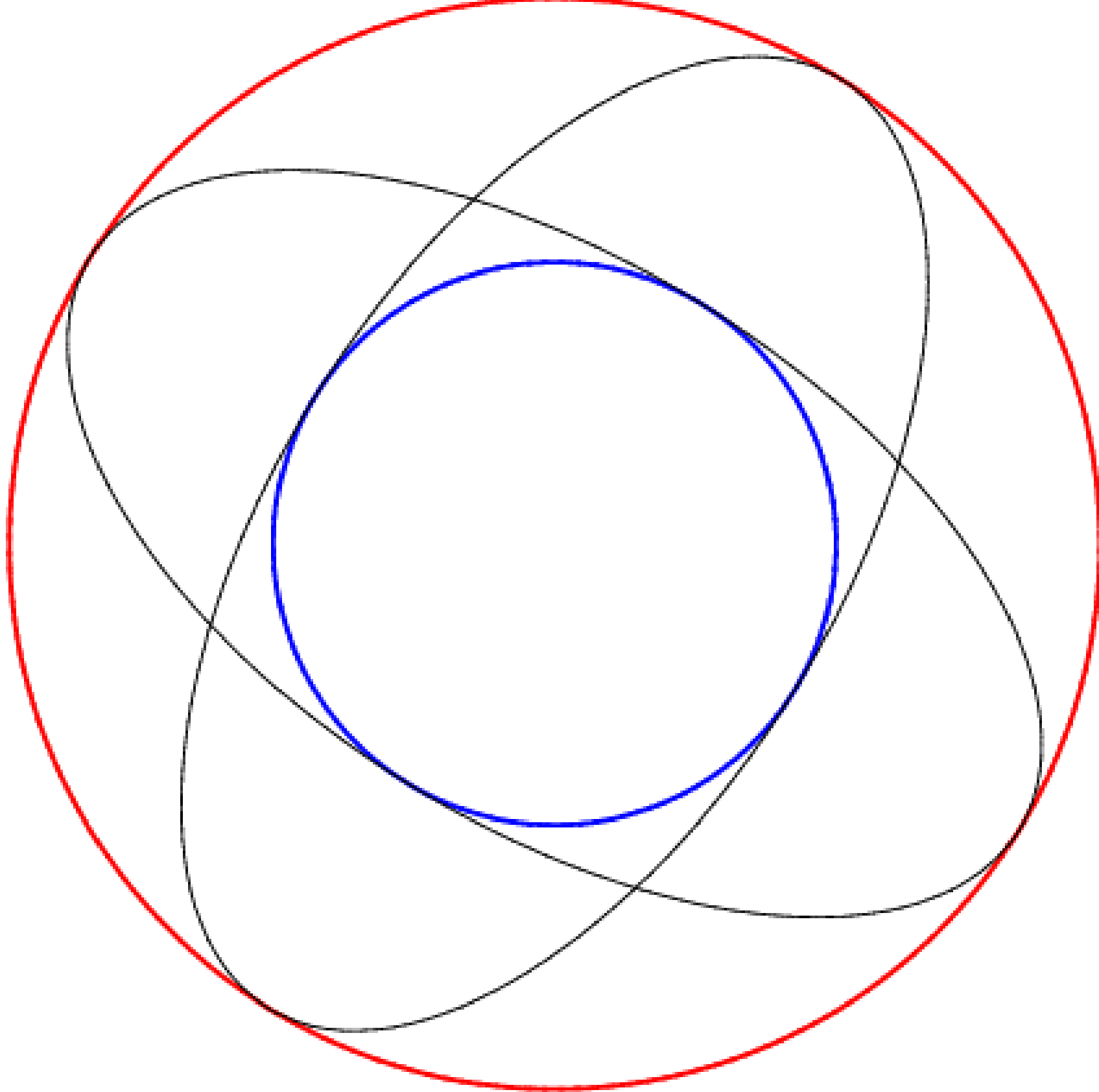


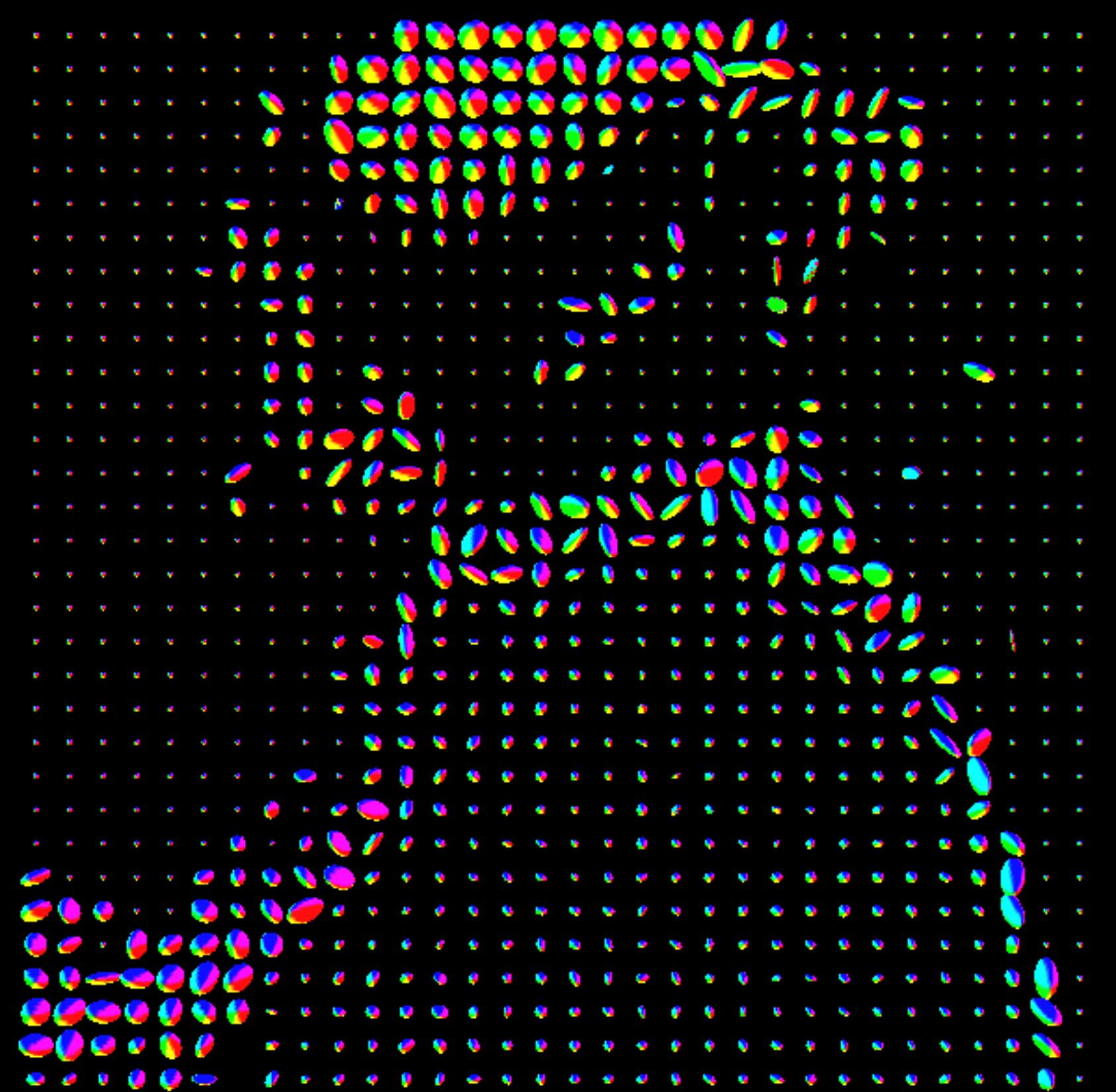






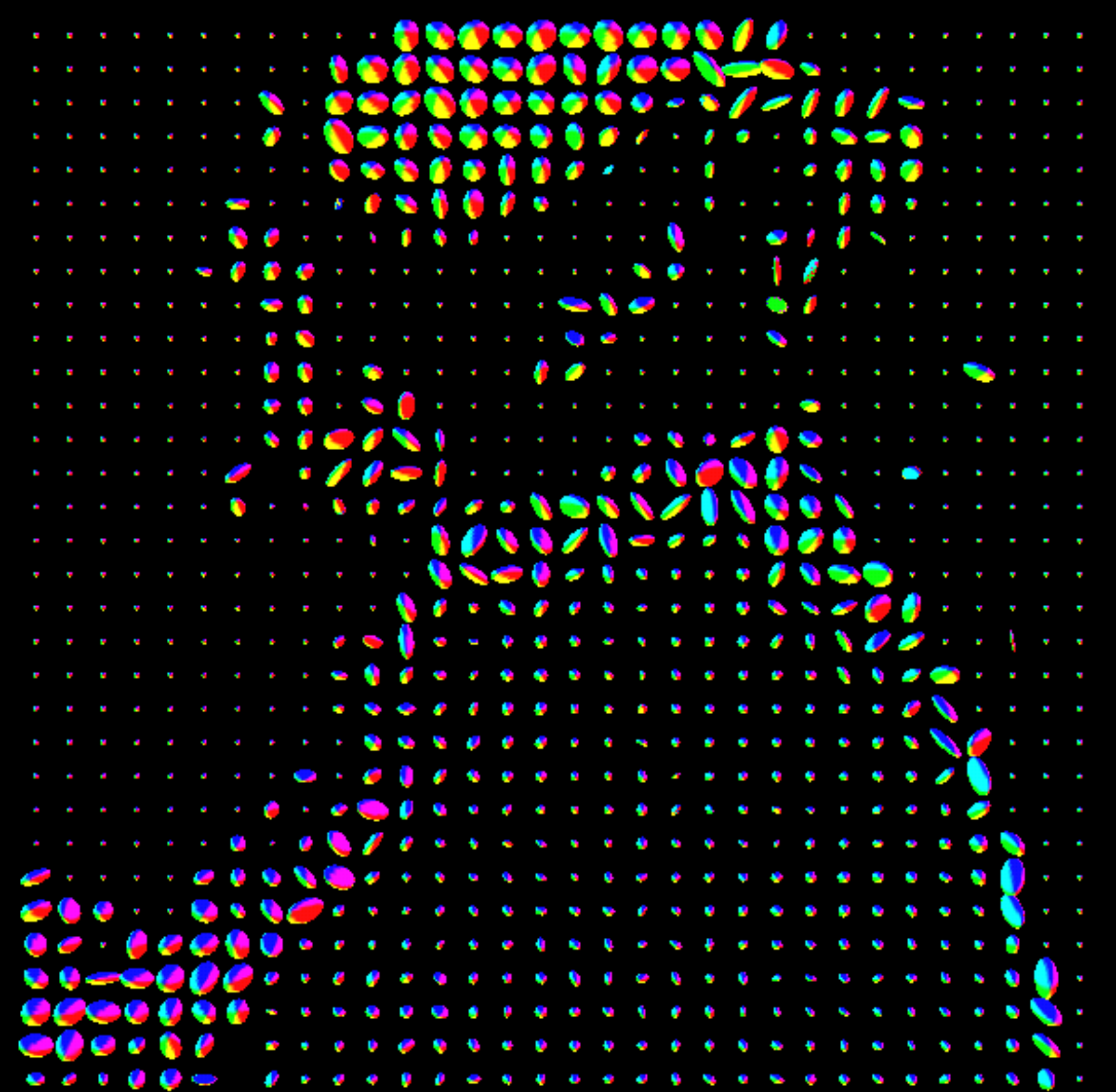


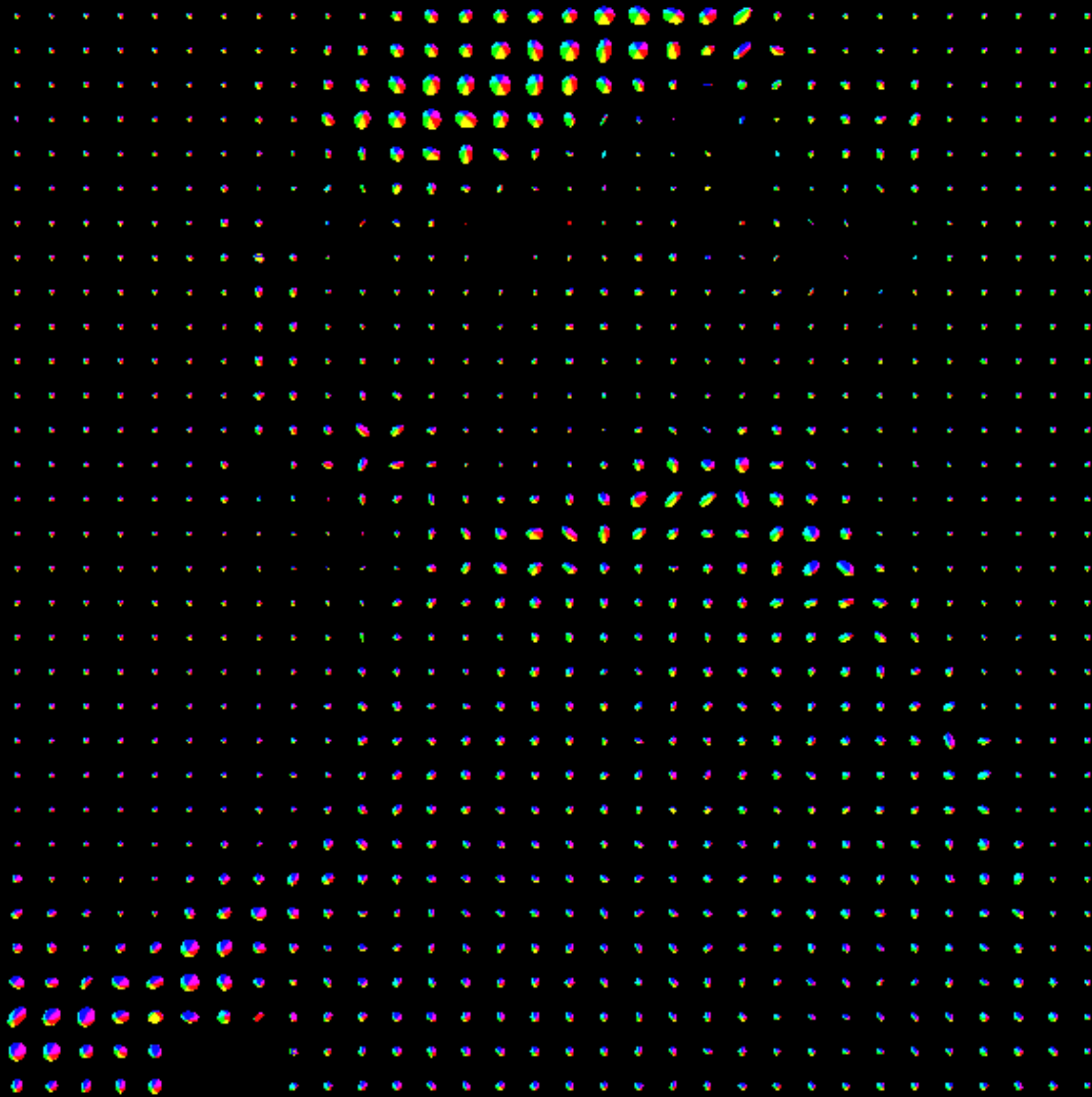




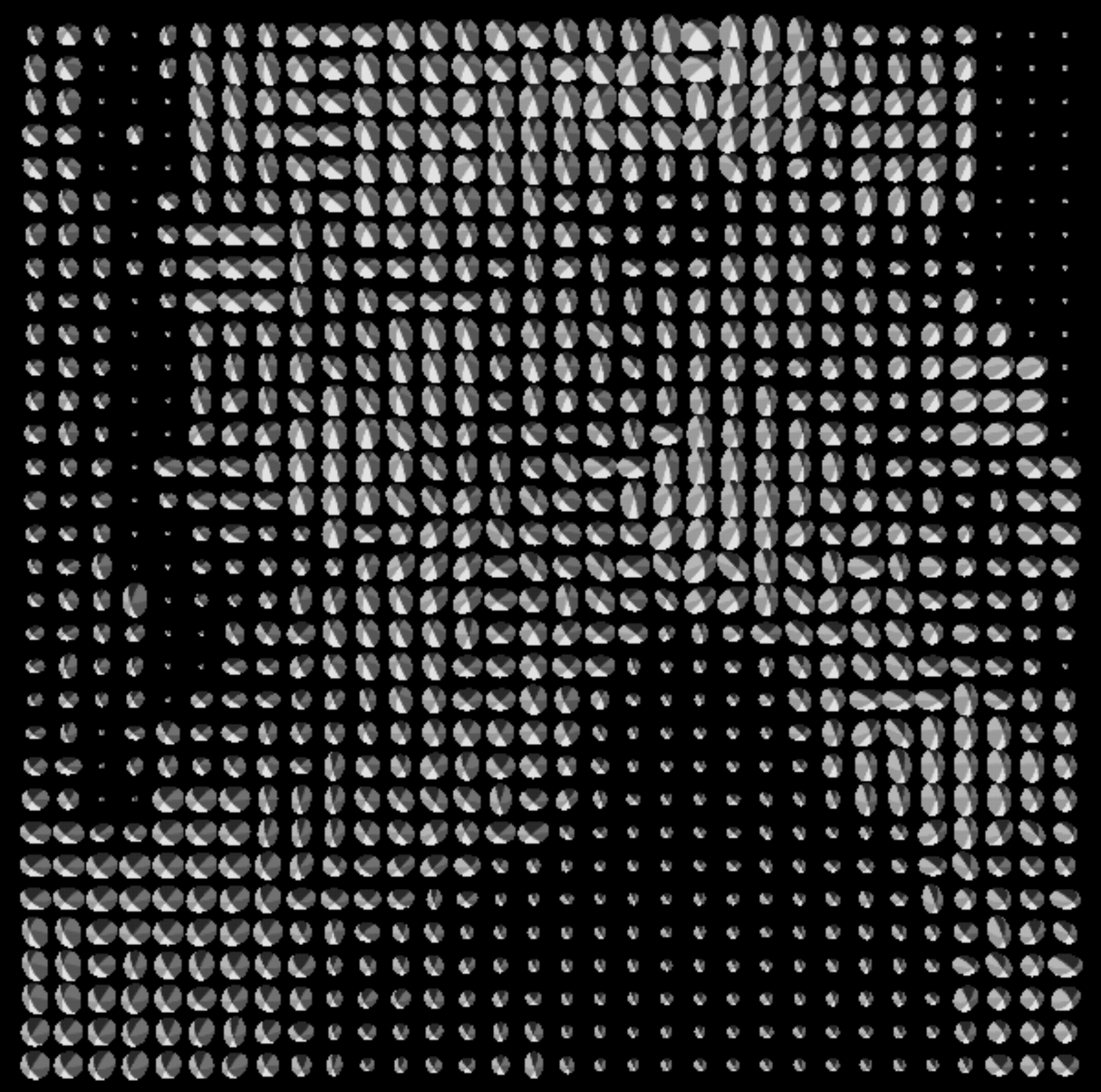




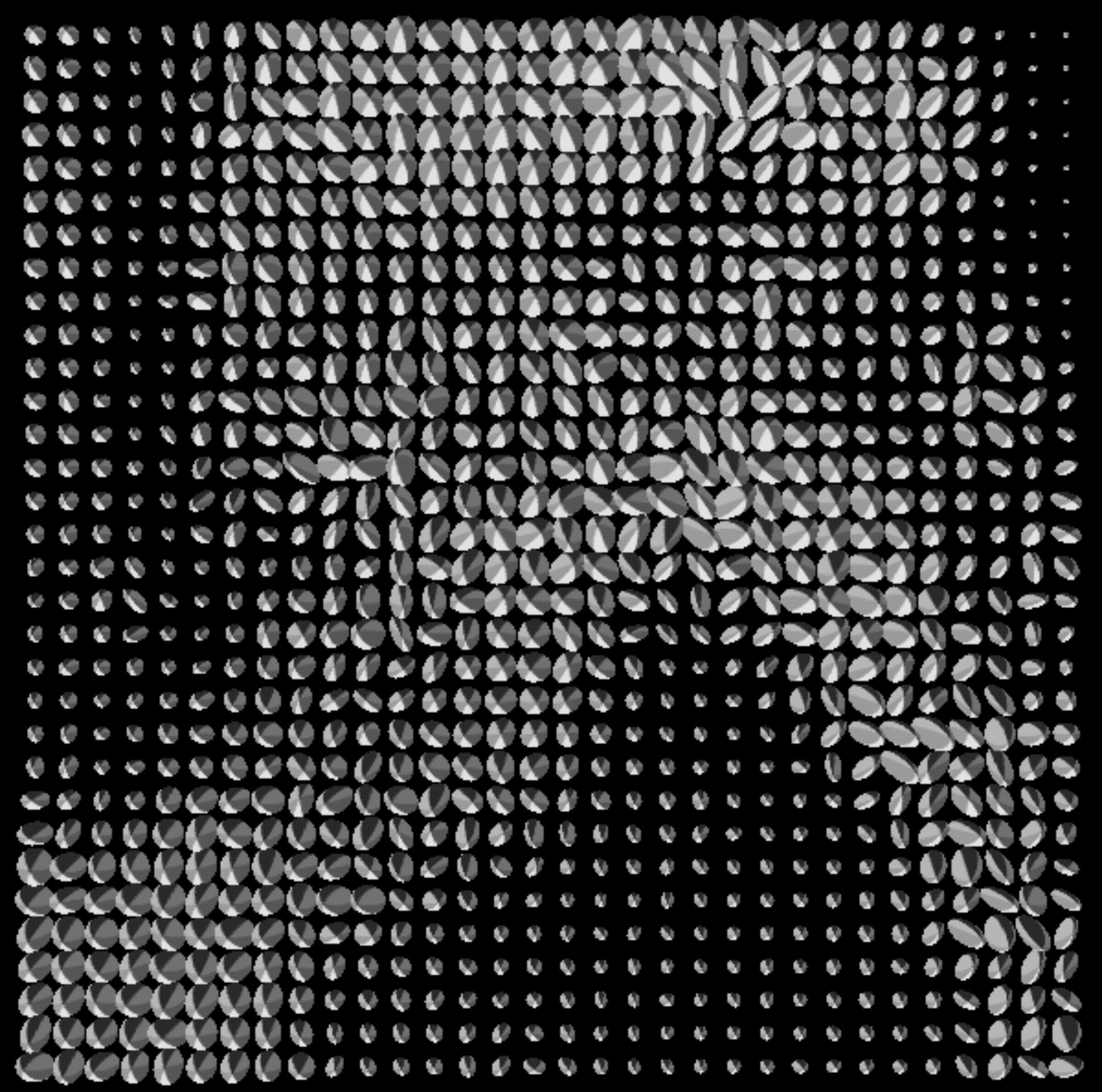












1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes that proper record-keeping is essential for transparency and accountability, particularly in financial matters. The text outlines various methods for organizing and storing these records, including digital databases and physical filing systems. It also highlights the need for regular audits and reviews to ensure the integrity and accuracy of the data.

2. The second part of the document focuses on the legal and regulatory requirements that govern record-keeping. It details the specific rules and standards that apply to different industries and sectors, such as healthcare, finance, and government. The text explains how these regulations are designed to protect the privacy and security of sensitive information, as well as to ensure that organizations are held to a high standard of ethical conduct. It also discusses the consequences of non-compliance, including potential fines and legal action.

3. The third part of the document explores the role of technology in modern record-keeping. It discusses the benefits of using digital tools and software to manage records, such as increased efficiency, reduced risk of loss, and improved accessibility. The text also addresses the challenges associated with digital records, such as data security, interoperability, and the need for robust backup and recovery plans. It provides practical advice on how to choose the right technology solutions for an organization's needs.

4. The fourth part of the document discusses the importance of training and education in ensuring that all staff members are properly equipped to handle records. It emphasizes that record-keeping is not just a technical task, but also a responsibility that requires a high level of attention to detail and a strong understanding of the organization's policies and procedures. The text provides guidance on how to develop and implement effective training programs, and how to foster a culture of accountability and responsibility among all employees.

5. The fifth and final part of the document provides a summary of the key points discussed throughout the document. It reiterates the importance of maintaining accurate records, the need to comply with legal and regulatory requirements, the role of technology in modern record-keeping, and the importance of training and education. The text concludes by encouraging organizations to take a proactive approach to record-keeping, and to view it as a critical component of their overall operational and strategic success.

