Morphology for Matrix-Fields: Ordering vs PDE

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in collaboration with

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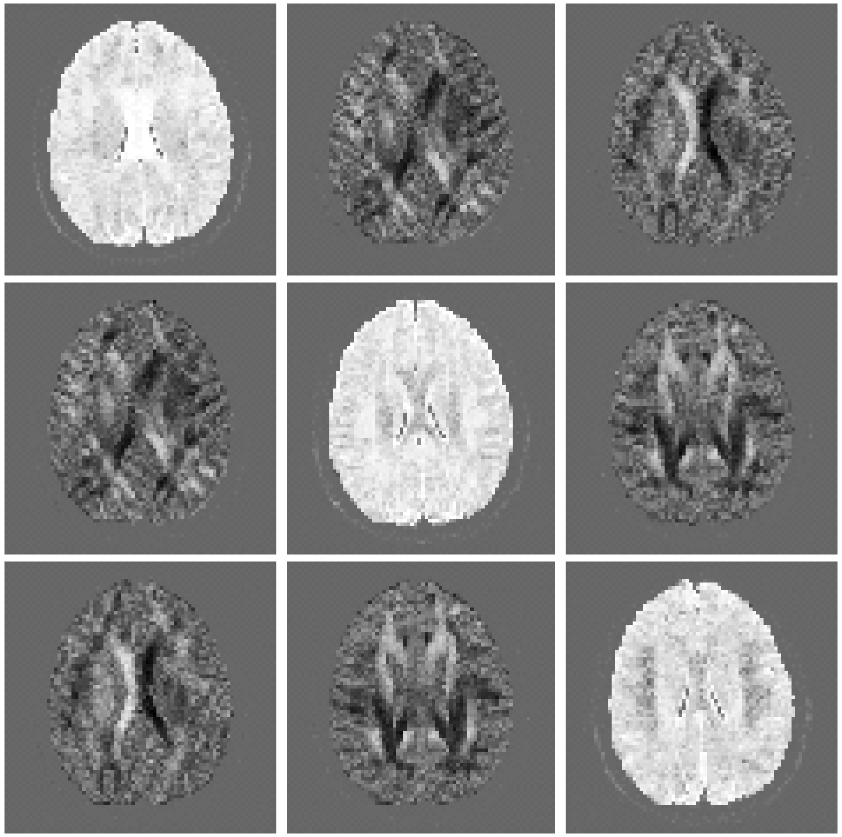
- Description: **Matrix-valued image (or matrix field)**:
  function with values in $\mathrm{Sym}_n(\mathbb{R})$, the set of real, symmetric $n \times n$-matrices

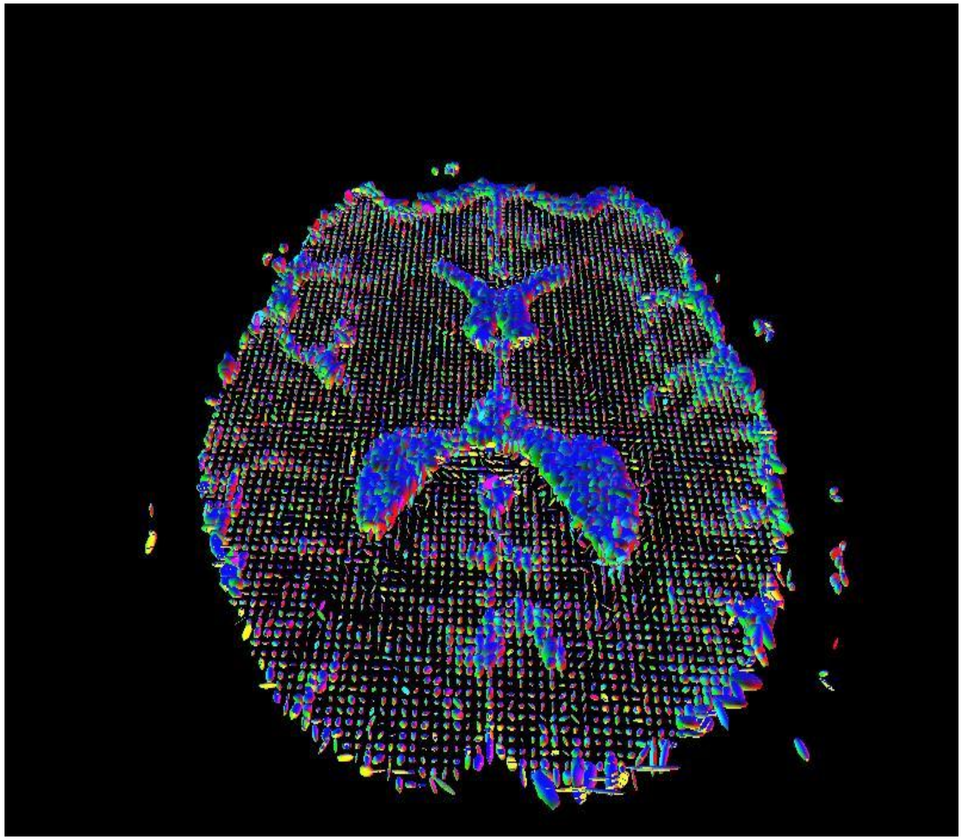
$$F : \Omega \subset \mathbb{R}^3 \longrightarrow \mathrm{Sym}_n(\mathbb{R})$$

- Sources:
  - in civil engineering and solid mechanics: **diffusion** and **permittivity tensors** and stress-strain relationships describe anisotropic behaviour
  - in image analysis: **structure tensor** (also called Förstner interest operator)
  - **diffusion tensor magnetic resonance imaging (DT-MRI)**
- Properties:
  - $A \in \mathrm{Sym}_n^+(\mathbb{R})$ are **positive (semi-)definite**:

$$q_A(x) := x^\top A x \geq 0 \qquad \text{for all } x \in \mathbb{R}^n.$$

  - quadratic form $q_A(x)$ describes isoprobability surface, $q_A(x) = 1$
  - reflects the diffusive property of water molecules in tissue





Slice of 3D DT-MRI data of a human head.

Left: Channelwise, tiled view. **Right:** Visualisation by ellipsoids via quadratic form

DT-MRI data: Courtesy of Anna Villanova, TU Eindhoven

Content

- Matrix-valued data
- Morphology for matrix-fields via Loewner ordering
  - Basic idea in the 2×2-matrix case
  - Extensions to 3×3- and larger matrices
  - Experiments
- Mathematical Morphology via PDEs
  - Matrix-valued morphological PDEs
  - Matrix-valued solution schemes
  - Experiments
- Concluding Remarks

Basic morphological operations in the scalar case:

- Greyscale dilation $\oplus$ replaces the greyvalue of the image $f(x, y)$ by its supremum within a mask defined by B:

$$(f \oplus B)(x, y) \;:=\; \sup\,\{f(x - x',\, y - y') \mid (x', y') \in B\}$$

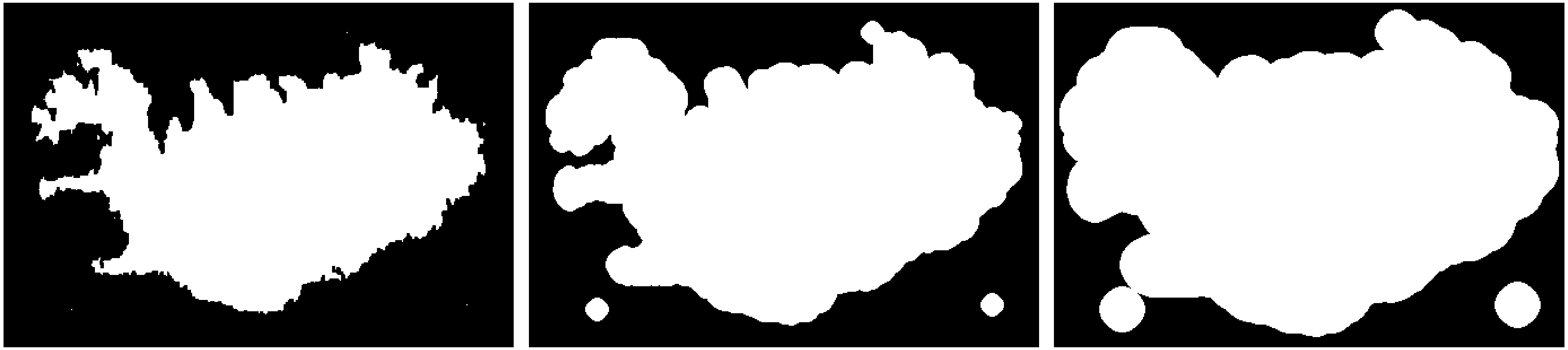
- while erosion $\ominus$ is determined by

$$(f \ominus B)(x, y) \;:=\; \inf\,\{f(x + x',\, y + y') \mid (x', y') \in B\}$$

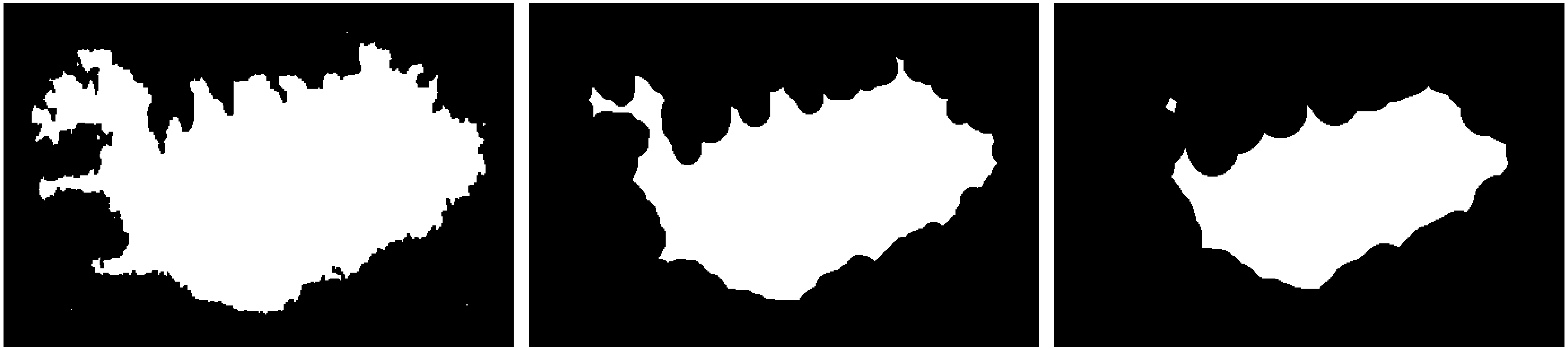
Morphological Operations II

Erosion and dilation of an image with disc-shaped structuring elements

Top: Dilation



Original radius = 10 radius = 20



Bottom: Erosion

- Combinations of erosion and dilation operations lead to
  - opening, closing
  - top hats
  - derivatives
- Morphological "Laplacian"

$$\Delta_B F := (f \oplus B) - 2 \cdot F + (f \ominus B)$$

- **Interpretation:** It approximates the second directional derivative $\partial_{\eta\eta} f$ where η denotes the direction of the steepest slope

Morphological Laplacians are useful for designing so-called **shock filters**

- **Idea**: Apply dilations around maxima and erosions around minima:

$$S_B f := \begin{cases} f \oplus B & \text{if } \Delta_B f < 0) \\ f & \text{if } \Delta_B f = 0) \\ f \ominus B & \text{if } \Delta_B f > 0) \end{cases}$$

- experimentally their iterates converge towards a *steady state* given by a *piecewise constant segmented image*

- discontinuities (“shocks”) between the segments

The basic morphological operations of **dilation** and **erosion** rely on the definition of **infimum** and **supremum**

Problem: What is the right notion of infimum and supremum for matrices, the right matrix-infimum (MI), the right matrix-supremum (MS)?

- In the **scalar** case: Infimum and supremum are based on an **ordering**
- In the **vectorial** case: generally **no suitable ordering** on vector spaces!
- In the **matrix-valued** case:
  - **Plus:** - There is a **partial ordering** on Sym(n),
    the so-called Loewner ordering
  - **Minus:** - It is **not** a lattice ordering.
    - MI / MS must be **rotationally invariant**
    - MI / MS must **preserve positive definiteness**
    - MI / MS must **depend continuously on input** matrices

"Loewner approach" for 2×2-matrices: the basic idea

Definition: (Loewner ordering)
Let $A, B \in$ Sym(n). Then $A \leq B$ if and only if $B - A$ is positive semidefinite.

How does the corresponding **ordering cone** Sym$^+$(2) in Sym(2) look like ?

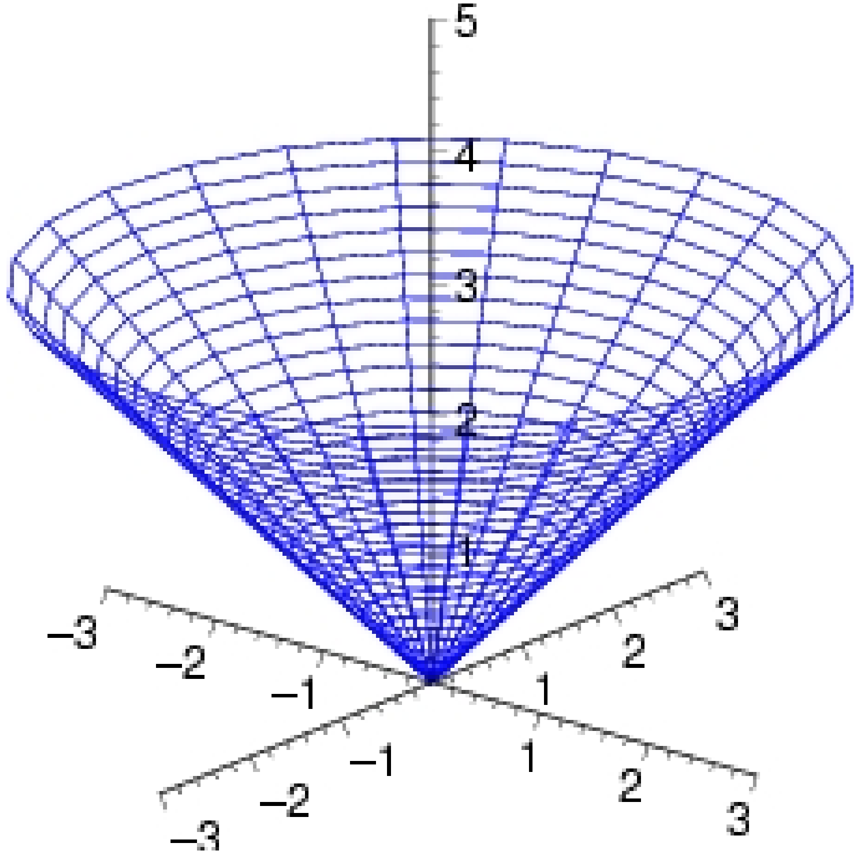
The mapping

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(2\beta, \gamma - \alpha, \gamma + \alpha)^\top$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} z - y & x \\ x & z + y \end{pmatrix} \longleftrightarrow (x, y, z)^\top$$

creates an **isomorphic image** of the **cone Sym$^+$(2)** in the Euclidean $\mathbb{R}^3$

The convex cone $\mathrm{Sym}^+(2)$ in $\mathbb{R}^3$ corresponding to the Loewner ordering in Sym(2):



Loewner ordering cone with 90° angle at its vertex

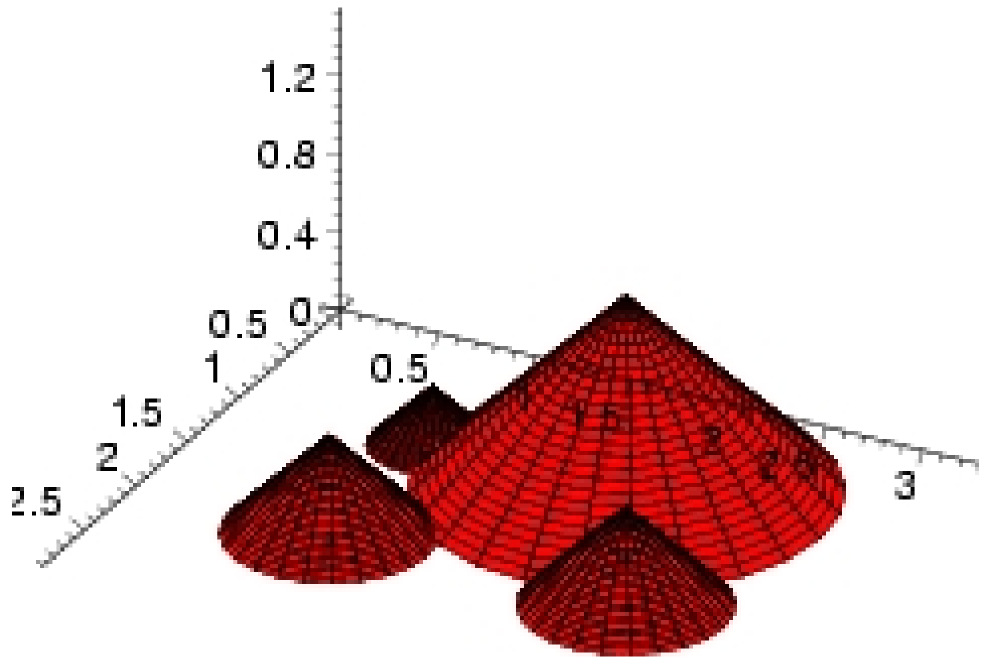
How can this cone be used to find a matrix-supremum?

Matrix-Supremum via Loewner Ordering I

In order to find matrix-supremum $M = MS(A_1, \dots, A_n)$ of a set of matrices $A_1, \dots, A_n \in \text{Sym}(2)$ consider

- the penumbra of each matrix A_i:

$$A_i - \text{Sym}^+(2)$$



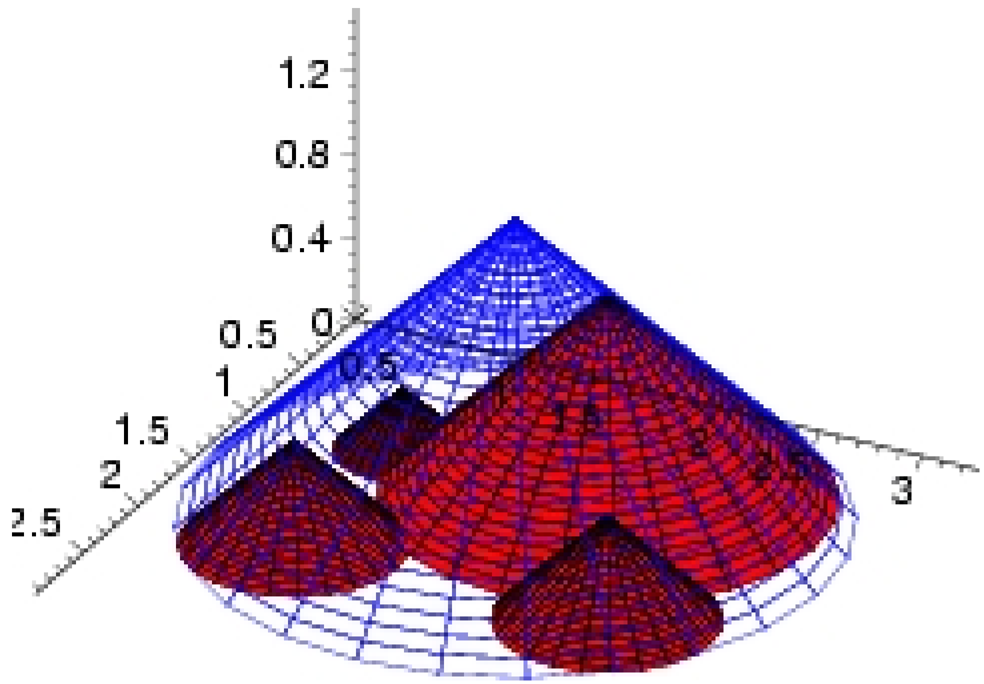
Penumbras of the matrices A_i

- **Note:** The **vertex** of each penubral cone specifies a matrix **uniquely**

Matrix-Supremum via Loewner Ordering II

In order to find the matrix-supremum $M = \max(A_1, \ldots, A_n)$ of a set of matrices $A_1, \ldots, A_n \in$ Sym(2) consider

- the penumbra of each matrix A_i
- and find the "covering" cone.

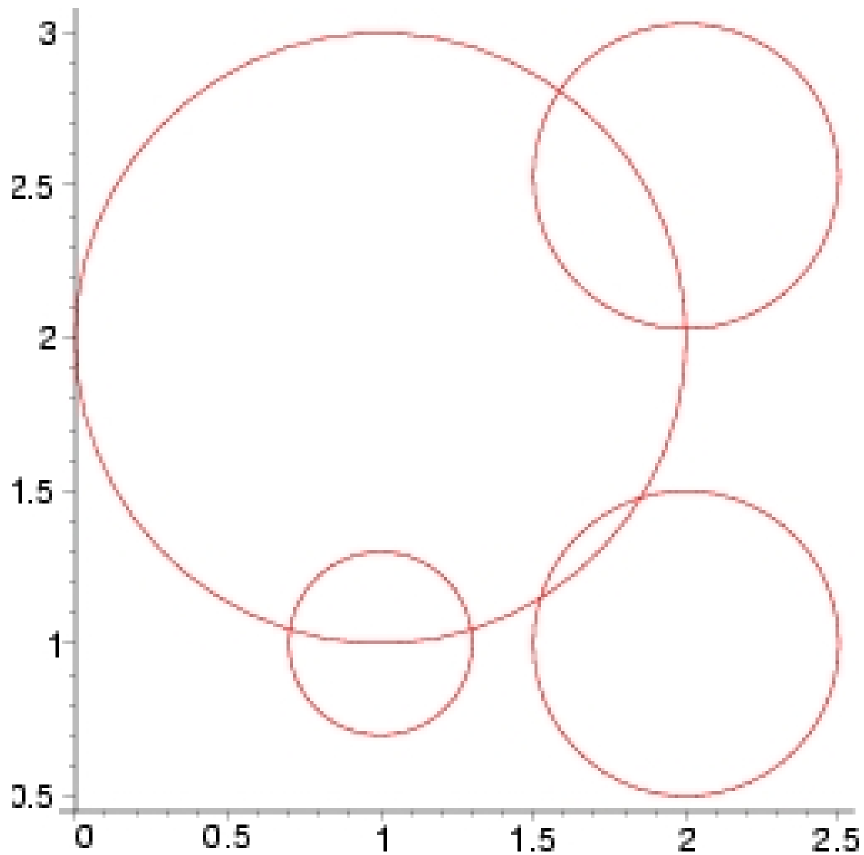


Covering cone encasing the penumbral cones of the A_i's

How to find this covering cone **computationally** ?

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41

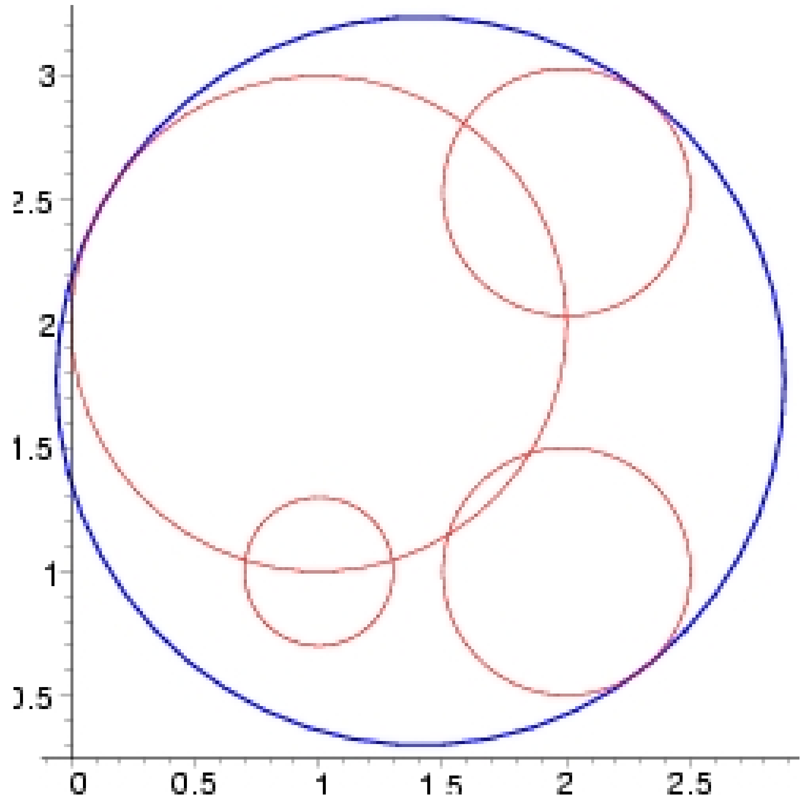
- The **bases of the penumbral cones** are circles C_i in the x-y-plane



Circles as bases of cones

- **Note:** A circle (center and radius) determines the penumbra **uniquely**

- The bases of the penumbral cones are circles C_i in the x-y-plane
- **Goal:** find the smallest circle C enclosing the circles C_i



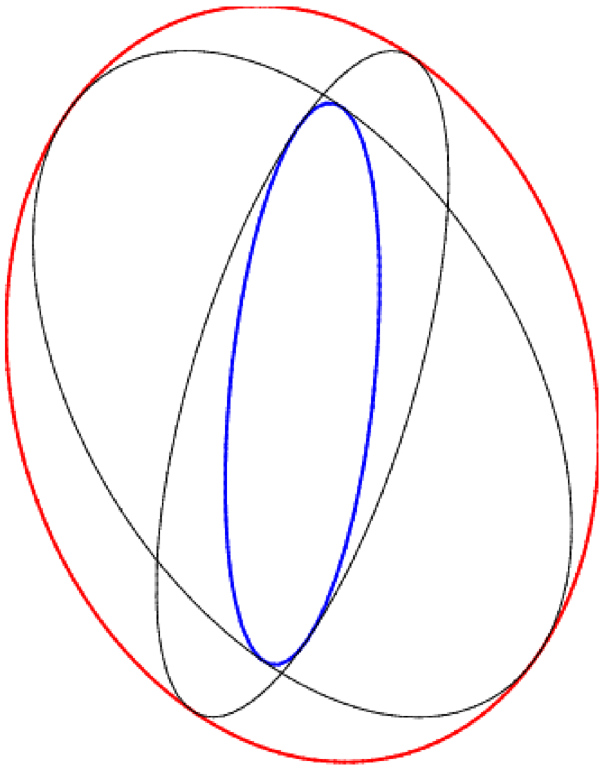
Minimal enclosing circle C

- An algorithm of B. Gärtner (ETH Zürich, 1999) finds this circle C
- This enclosing circle C determines the matrix-supremum

- The matrix-infimum m is obtained via the matrix-supremum of $A_1^{-1}, \ldots, A_n^{-1}$:

$$m := \mathrm{MI}(A_1, \ldots, A_n)) := \left(\mathrm{MS}(A_1^{-1}, \ldots, A_n^{-1})\right)^{-1}$$

Example: Loewner approach,
maximal and minimal ellipses



How does all this generalise to 3×3- or larger matrices ?

Answer:

- The basic idea carries over in spirit to the higher order case.
- No 'visualising' mapping is known
- The base of the cone is much more complicated, it is **not** a **strictly convex set**
- Sample the **extreme points** of the base and find the smallest enclosing (higher dimensional) ball
- The center and the radius of this ball determine the penumbral cone, that is, the matrix-supremum MS
- MI via MS

- We obtain simple formulas with I as $n \times n$-identity matrix:
  - the **center** c_M of the circumfering ball associated with M is given by

$$c_M := M - \frac{\text{trace}(M)}{n} \, I$$

  - its **radius** r satisfies $(v \in \mathbb{R}^3, \|v\| = 1)$

$$r := \|M - \text{trace}(M) v \, v^\top - c_M\| = \text{trace}(M) \sqrt{1 - \frac{1}{n}}$$

  - the **vertex** M of the associated penumbra is obtained by

$$M = c_M + \frac{r}{n} \frac{1}{\sqrt{1 - \frac{1}{n}}} I$$

Properties of the approach based on the Loewner ordering:

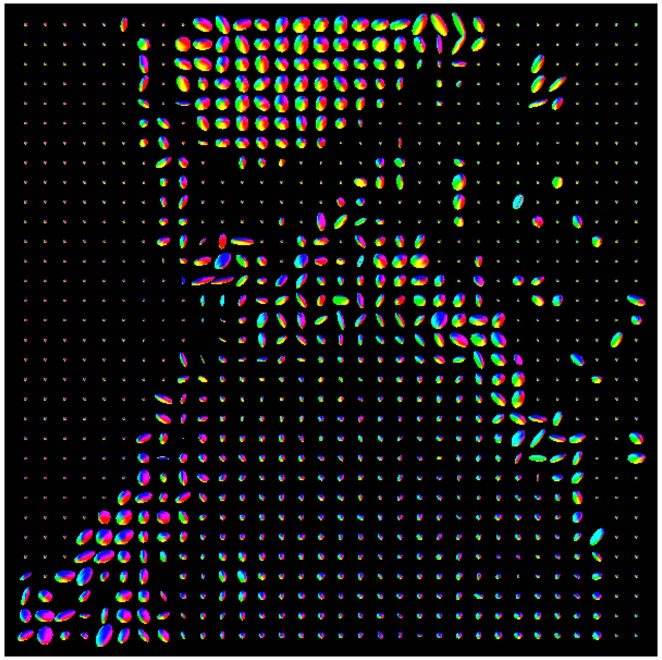
- rotationally invariant,
- preserves positive definiteness,
- continuous dependence on the input matrices A_i,
- extendable to indefinite matrices,
- extendable to higher order matrices.

For more details on Loewner based matrix morphology:

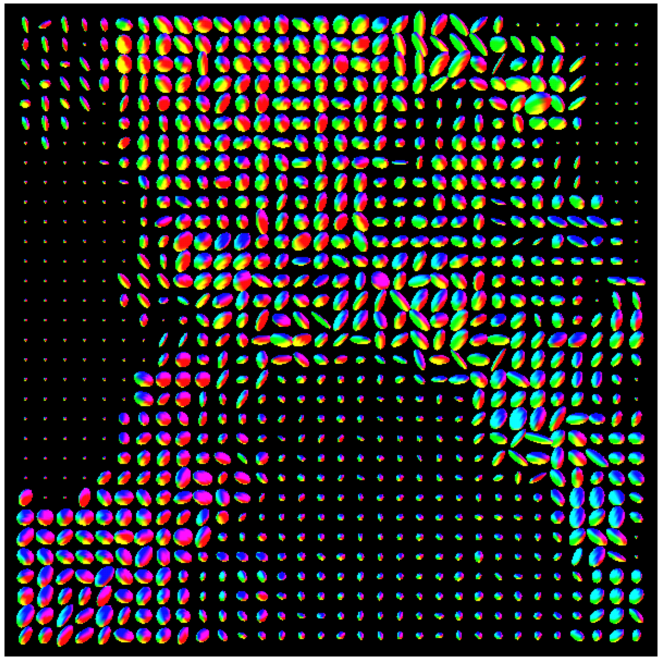
B.B. et al., Mathematical Morphology for Tensor Data Induced by the Loewner Ordering in Higher Dimensions. Preprint 2005 (to be published in IEEE, Sig. Proc.)

Experiments

Dilation and erosion
of a 3D matrix field F with a ball-shaped structuring element B of radius 2.

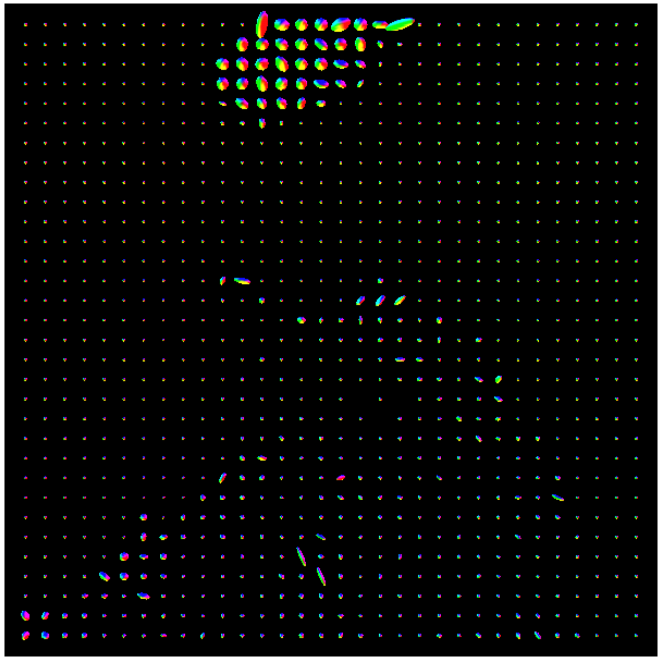


Original



Dilation

$F \oplus B$



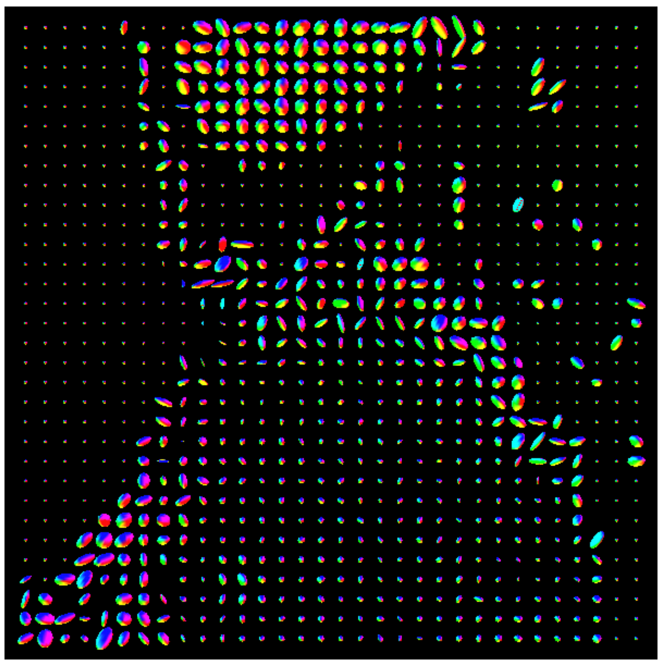
Erosion

$F \ominus B$

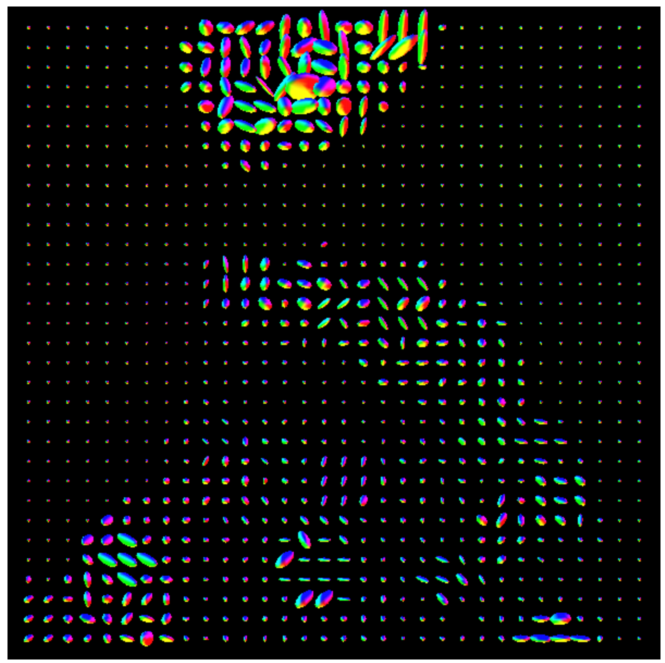
Experiments

MIA
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41

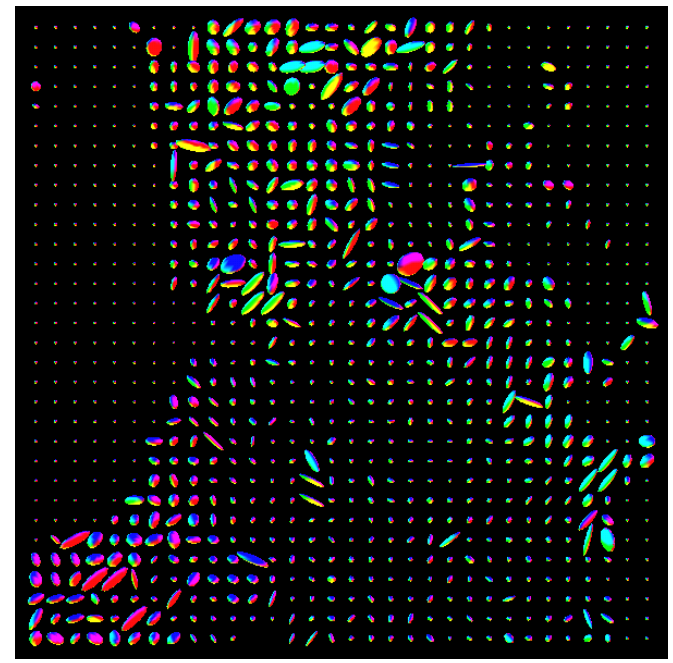
Opening and closing
of a 3D matrix field F with a ball-shaped structuring element B of radius 2.



Original



Opening
$$F \circ B = (F \ominus B) \oplus B$$

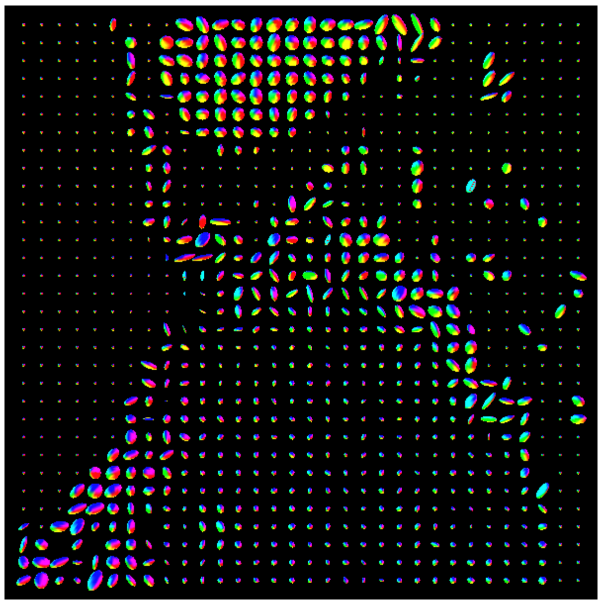


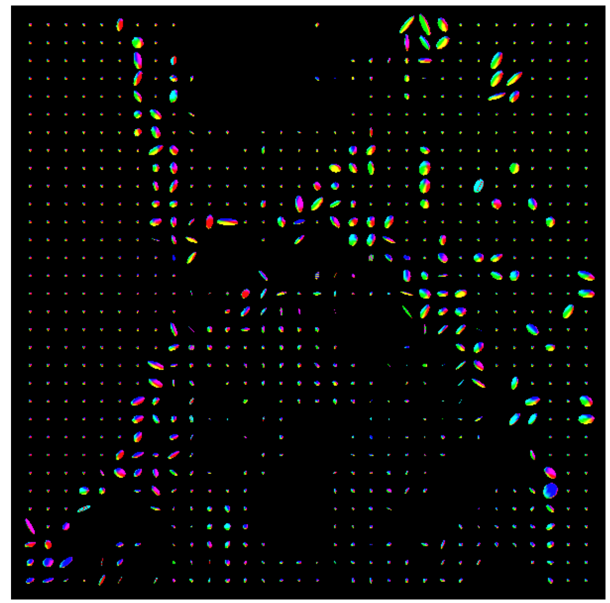
Closing
$$F \bullet B = (F \oplus B) \ominus B$$

Top Hats

of a 3D matrix-field F with a ball-shaped structuring element B of radius 2.

Original



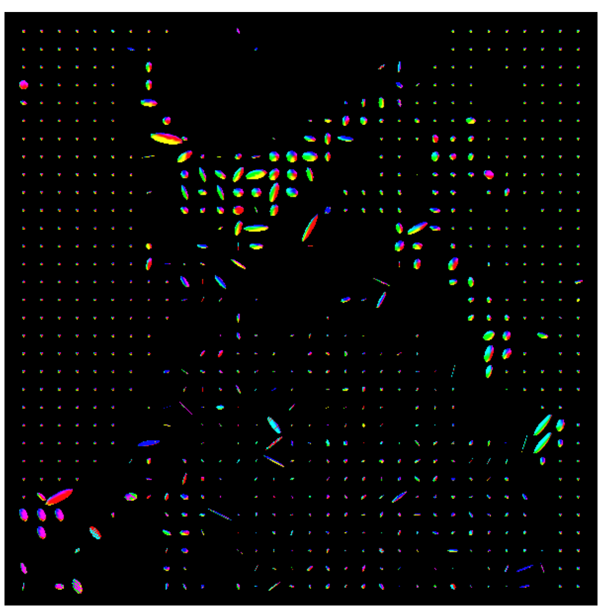


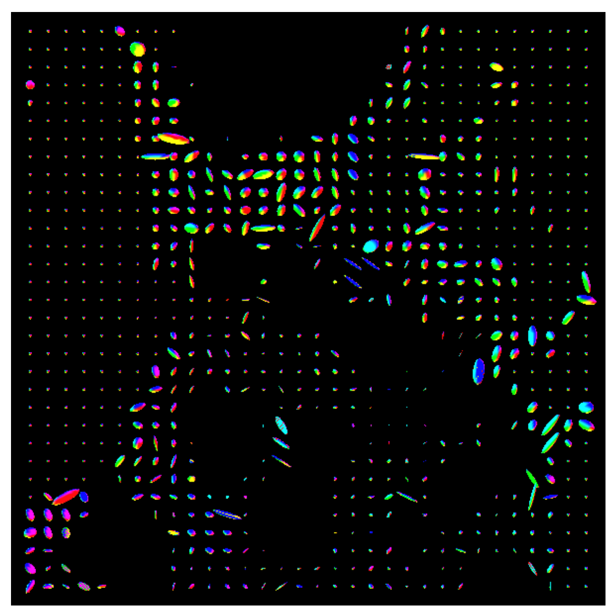
White top hat

$F - (F \circ B)$

Black top hat

$(F \bullet B) - F$





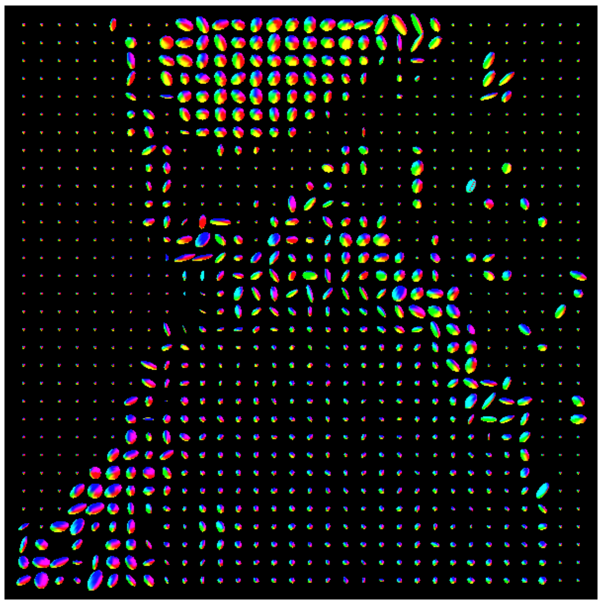
Self-dual top hat

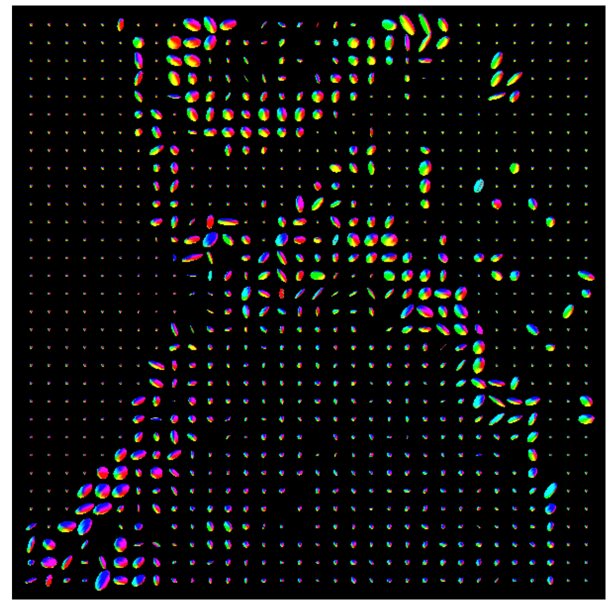
$(F \bullet B) - (F \circ B)$

Morphological derivatives

of a 3D matrix-field F with a ball-shaped structuring element B of radius 2.

Original



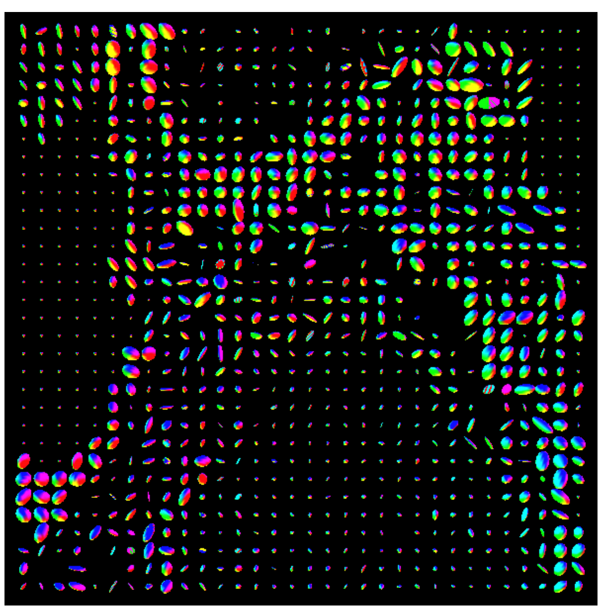


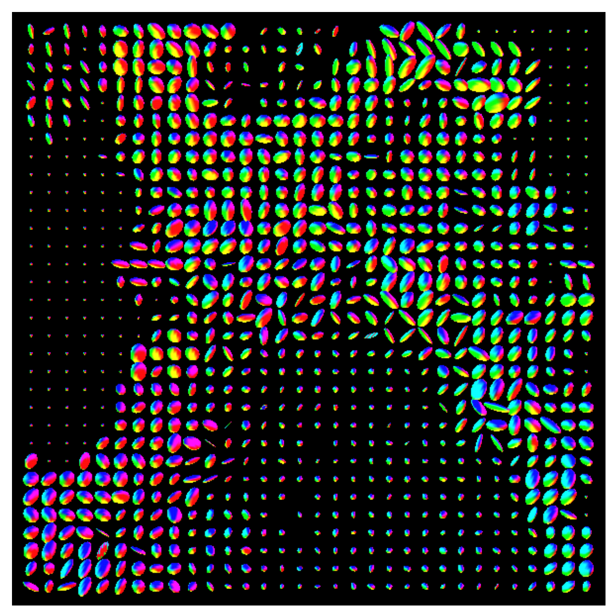
External Gradient

$(F \oplus B) - F$

Internal Gradient

$F - (F \ominus B)$





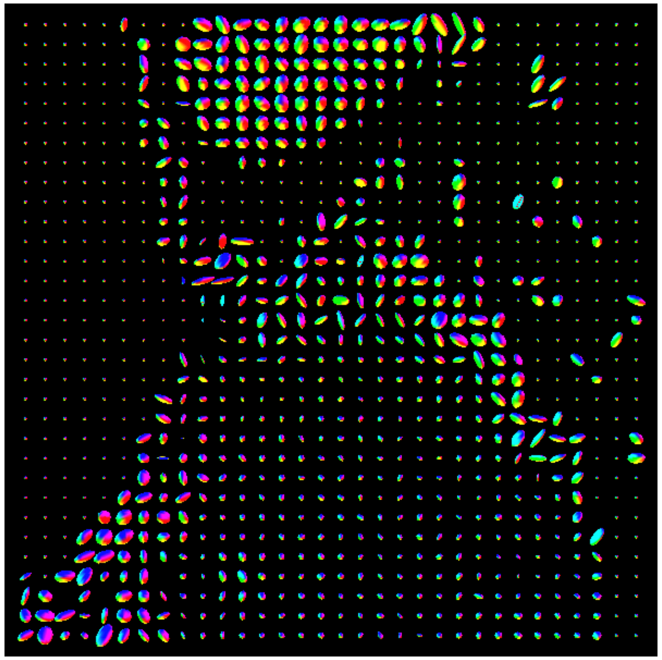
Beucher Gradient

$(F \oplus B) - (F \ominus B)$

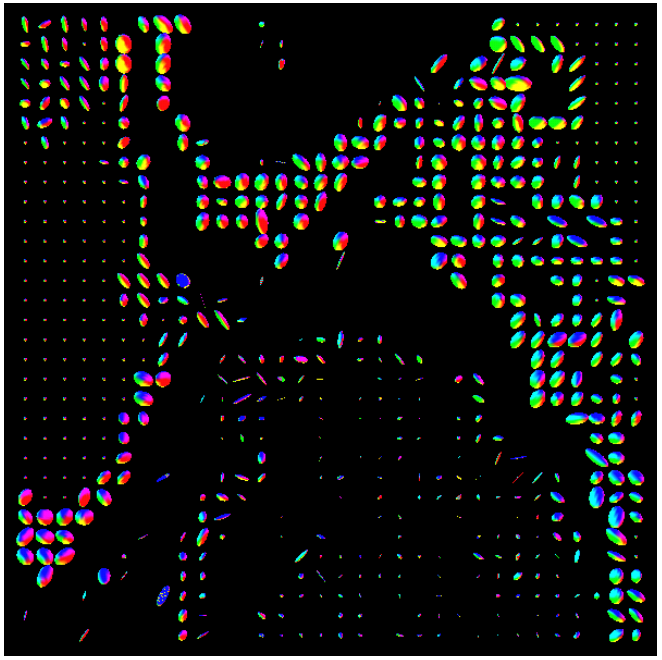
Experiments

Morphological Laplacian and shock filtering

of a 3D matrix field F with a ball-shaped structuring element of radius 2.

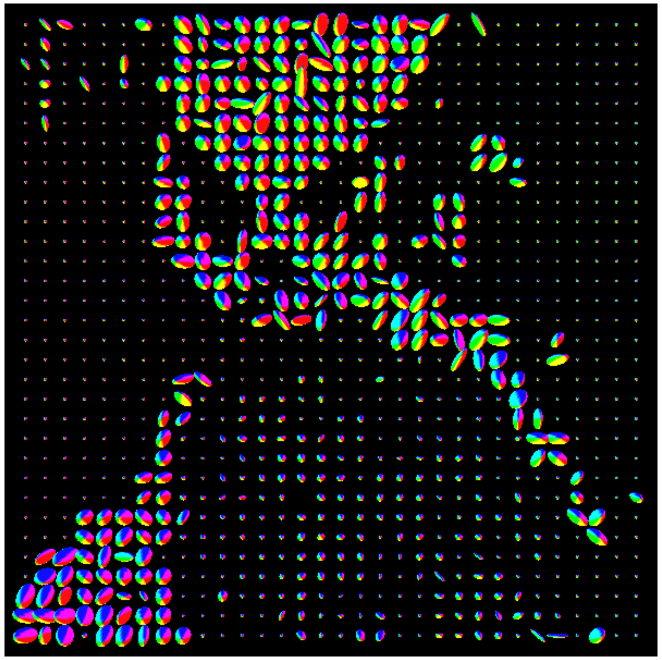


Original



Morphological Laplacian

$$(F \oplus B) - 2 \cdot F + (F \ominus B)$$



Shock filtering

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Continuous Morphology

Basic Approach (Boomgaard/Dorst '92): Nonlinear partial differential equations that mimic the process of dilation and erosion.

- **Situation:** Original image $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, transformed version u
- **Dilation** with a ball-shaped structuring element:

$$\boxed{\partial_t u = \|\nabla u\|}$$

- **Erosion** with a ball-shaped structuring element:

$$\boxed{\partial_t u = -\,\|\nabla u\|}$$

  with initial condition $u(x, y, 0) = f(x, y)$.

Advantages of PDE framework:

- Sophisticated machinery of numerical solution methods for PDEs is available
- Continuous approach allows for sub-pixel accuracy

Scalar PDE: $\partial_t u = \|\nabla u\| = \sqrt{(\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2}$

How to find a PDE for matrix-valued data $U = (u_{ij})_{ij} \in \mathrm{Sym}_n(\mathbb{R})$?

Scalar PDE: $\partial_t u = \|\nabla u\| = \sqrt{(\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2}$

How to find a PDE for matrix-valued data $U = (u_{ij})_{ij} \in \mathrm{Sym}_n(\mathbb{R})$?

- Define functions on $\mathrm{Sym}_n(\mathbb{R})$: If $U = V^\top \mathrm{diag}(\lambda_1, \ldots, \lambda_n) V$ and $h : I \subset \mathbb{R} \to \mathbb{R}$,

$$h(U) := V^\top \mathrm{diag}(h(\lambda_1), \ldots, h(\lambda_n)) V$$

- Generalise partial derivatives ∂_ω, with $\omega \in \{t, x_1, \ldots, x_d\}$:

$$\overline{\partial}_\omega U := (\partial_\omega u_{ij})_{ij}$$

- Generalise gradient ∇:

$$\overline{\nabla} U := (\overline{\partial}_{x_1} U, \ldots, \overline{\partial}_{x_d} U)^\top \in \left(\mathrm{Sym}_n(\mathbb{R})\right)^d$$

Matrix PDE:

$$\overline{\partial}_t U = |\overline{\nabla} u|_2 = \sqrt{(\overline{\partial}_x U)^2 + (\overline{\partial}_y U)^2 + (\overline{\partial}_z U)^2}$$

How to solve the morphological matrix PDE ?

Numerical solution through the matrix-valued counterparts of the scalar schemes for the scalar PDEs

Example: OS-scheme by Osher & Sethian (1997)

- Osher-Sethian scheme, **scalar-valued** numerical approximation in 1D:

$$\frac{u(i)^{(n+1)} - u(i)^{(n)}}{\tau} =$$

$$= \left[\left(\min \left(\frac{u(i)^{(n)} - u(i-1)^{(n)}}{h}, 0 \right) \right)^2 + \left(\max \left(\frac{u(i+1)^{(n)} - u(i)^{(n)}}{h}, 0 \right) \right)^2 \right]^{1/2}$$

- **matrix-valued** counterpart, numerical approximation in 1D:

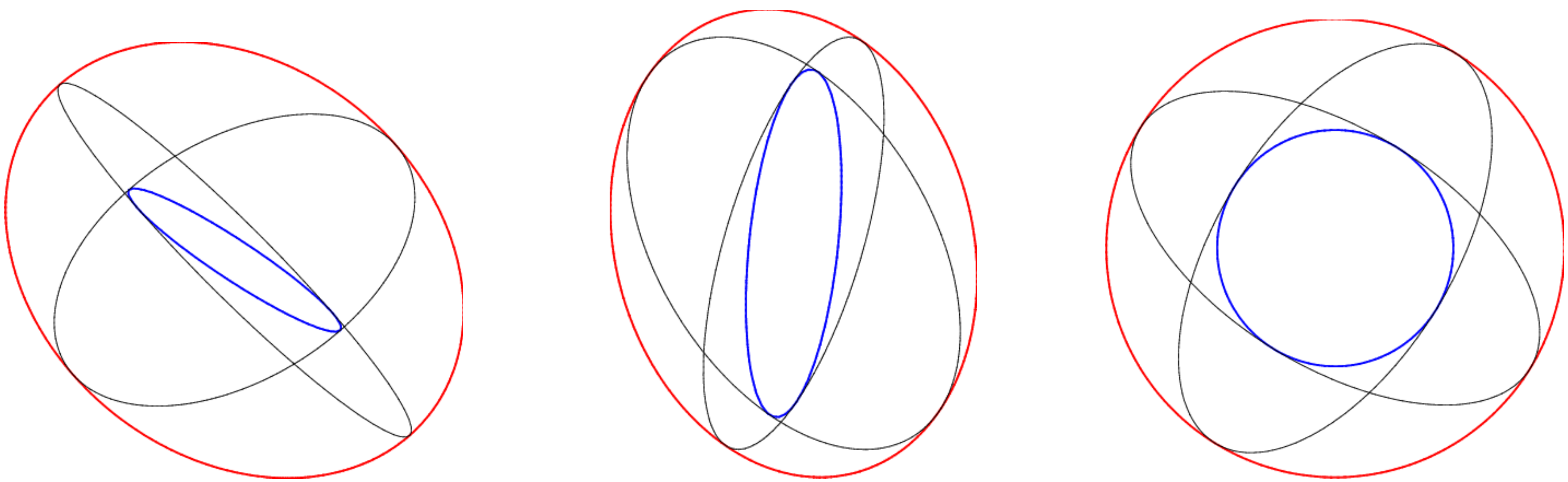
$$\frac{U(i)^{(n+1)} - U(i)^{(n)}}{\tau} =$$

$$= \left[\left(\min \left(\frac{U(i)^{(n)} - U(i-1)^{(n)}}{h}, 0 \right) \right)^2 + \left(\max \left(\frac{U(i+1)^{(n)} - U(i)^{(n)}}{h}, 0 \right) \right)^2 \right]^{1/2}$$

Definition: Let $A, B \in \text{Sym}_n(\mathbb{R})$ then

$$\max(A,B) := \frac{1}{2}\big(A+B+|A-B|\big)$$

$$\min(A,B) := \frac{1}{2}\big(A+B-|A-B|\big)$$



Maximal and minimal matrices $\in \text{Sym}_2(\mathbb{R})$

Remark: The maximal and minimal matrices are the one induced by the **Loewner ordering:** $A \geq B :\Longleftrightarrow A - B$ positive semidefinite

For comparison of ordering- and PDE-based matrix-morphology:

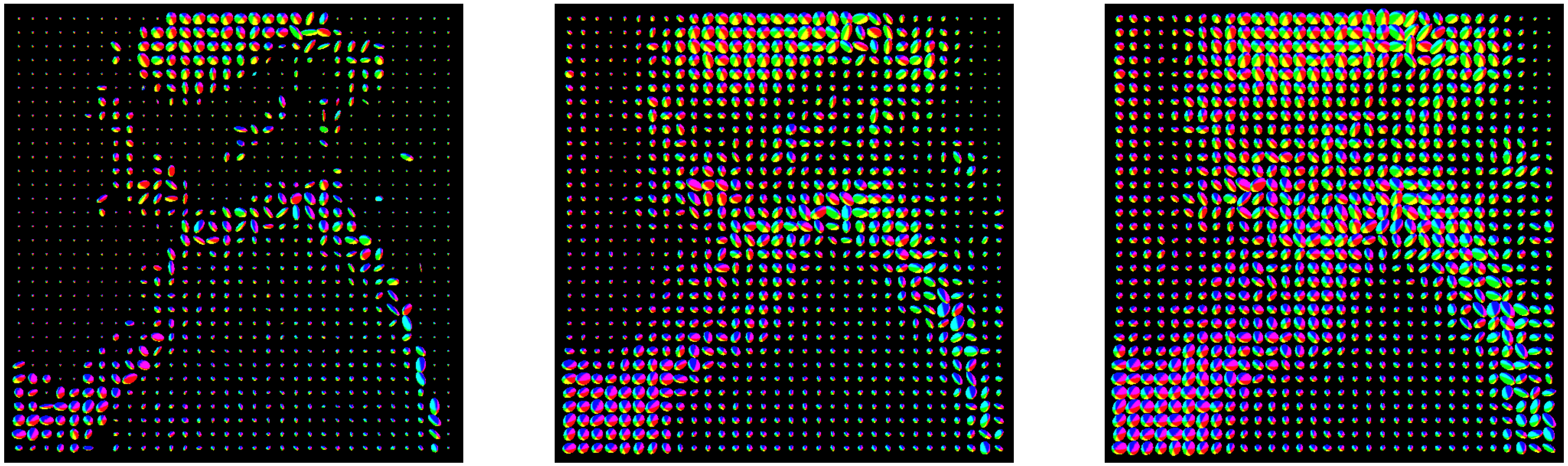
B.B. et al., Morphology for Tensor Data: Ordering versus PDE-Based Approach.

To be published in JMIV 2006.

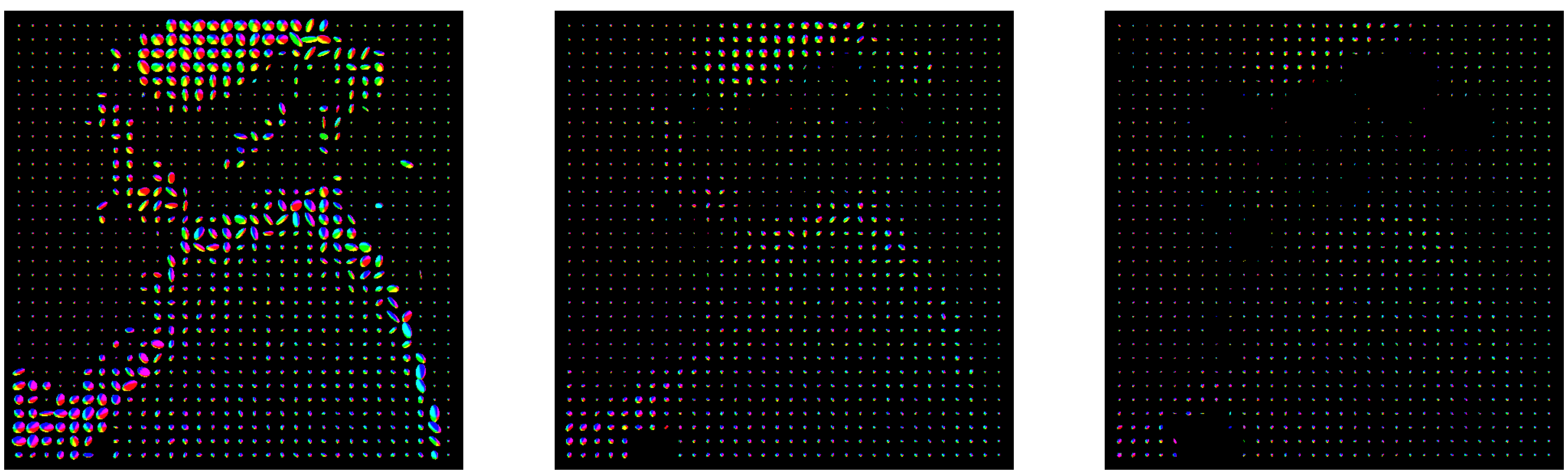
Experiments: PDE-based Dilation and Erosion

Erosion and dilation of matrix-valued images by matrix-valued OS-scheme

Top: Dilation



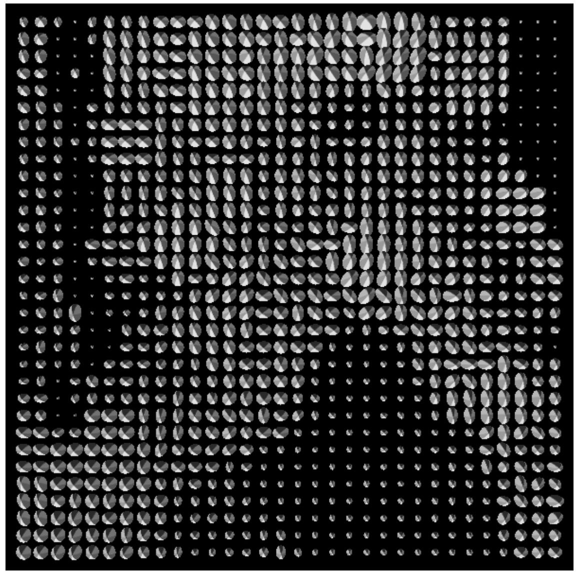
Original $\quad$ $t = 4$ $\quad$ $t = 10$

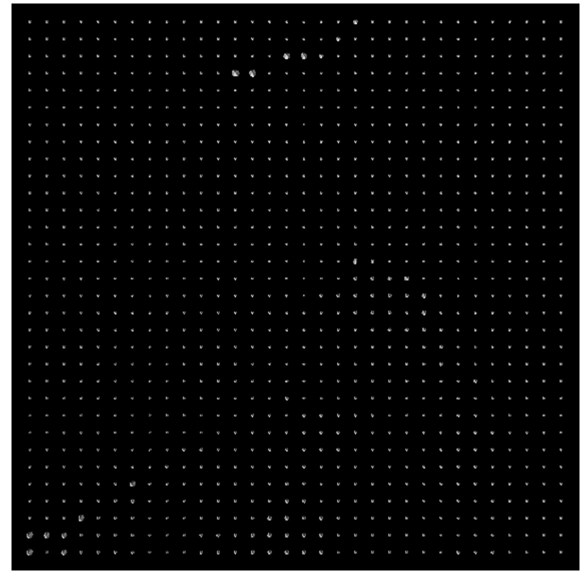


Bottom: Erosion

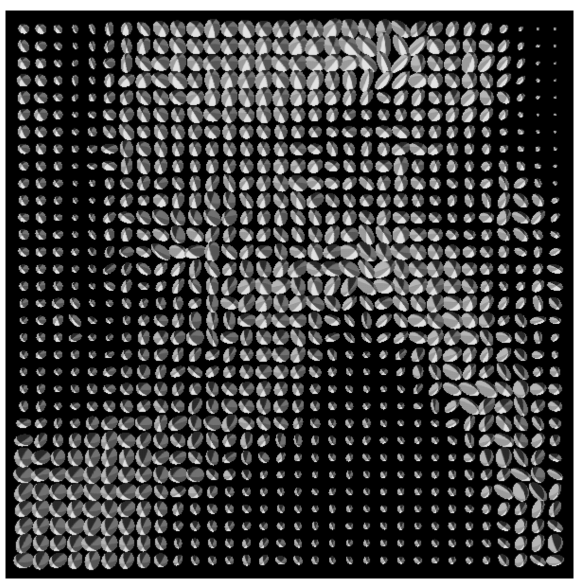
Experiments: Ordering vs PDE II

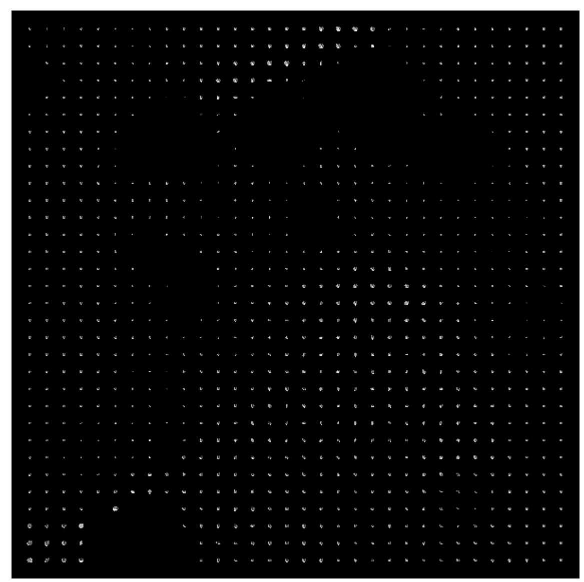
Ordering based approach (ball-shaped structuring element BSE($\sqrt{2}$))





Dilatation | Erosion

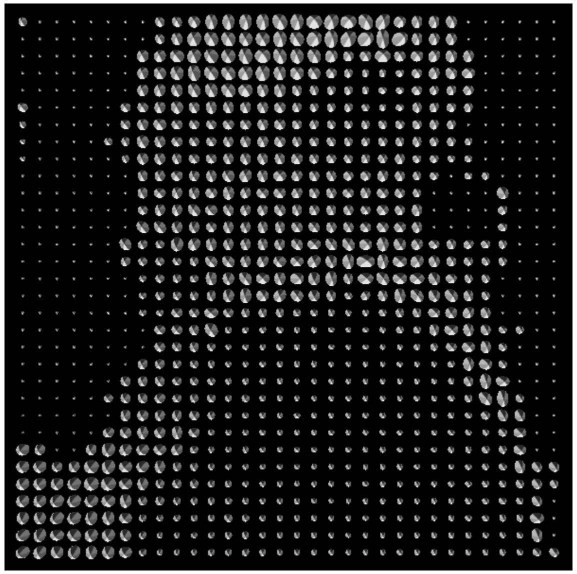
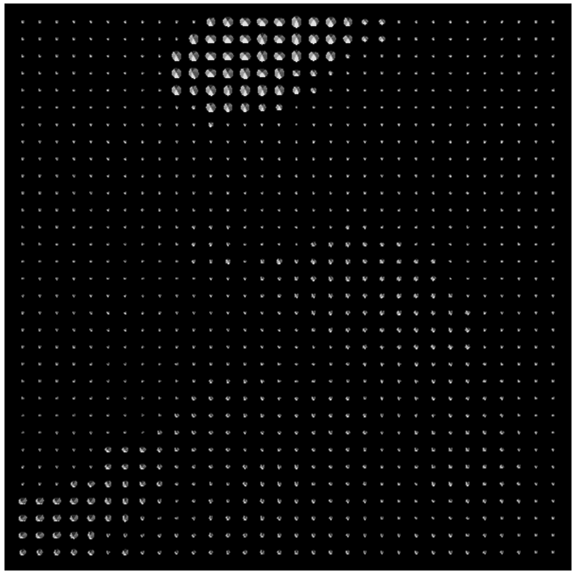




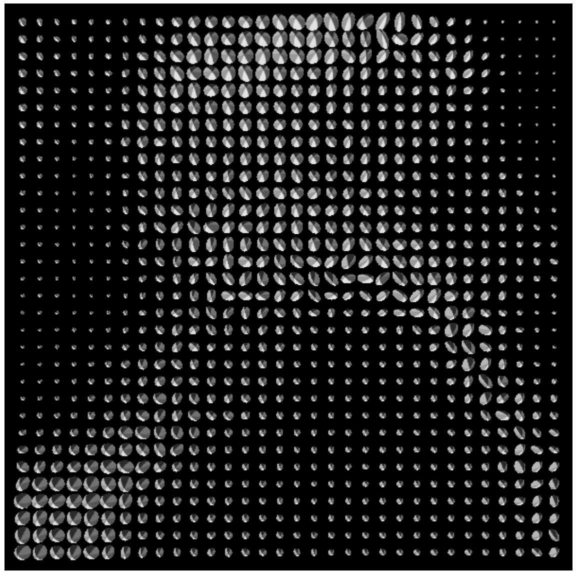
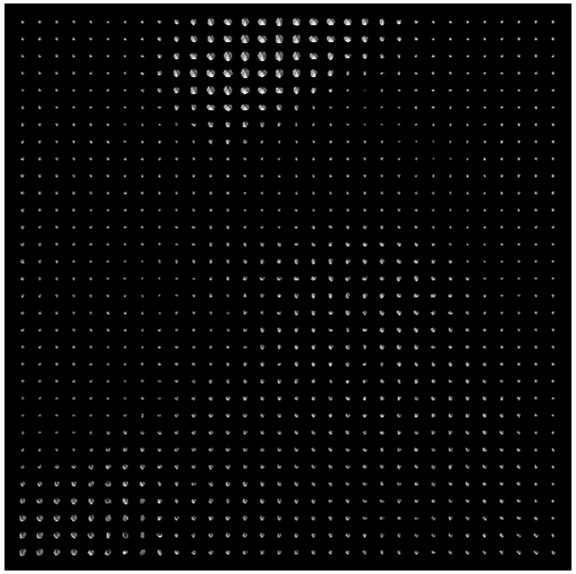
PDE based approach (stopping time 2)

Experiments: Ordering vs PDE III

Ordering based approach (ball-shaped structuring element BSE($\sqrt{2}$))

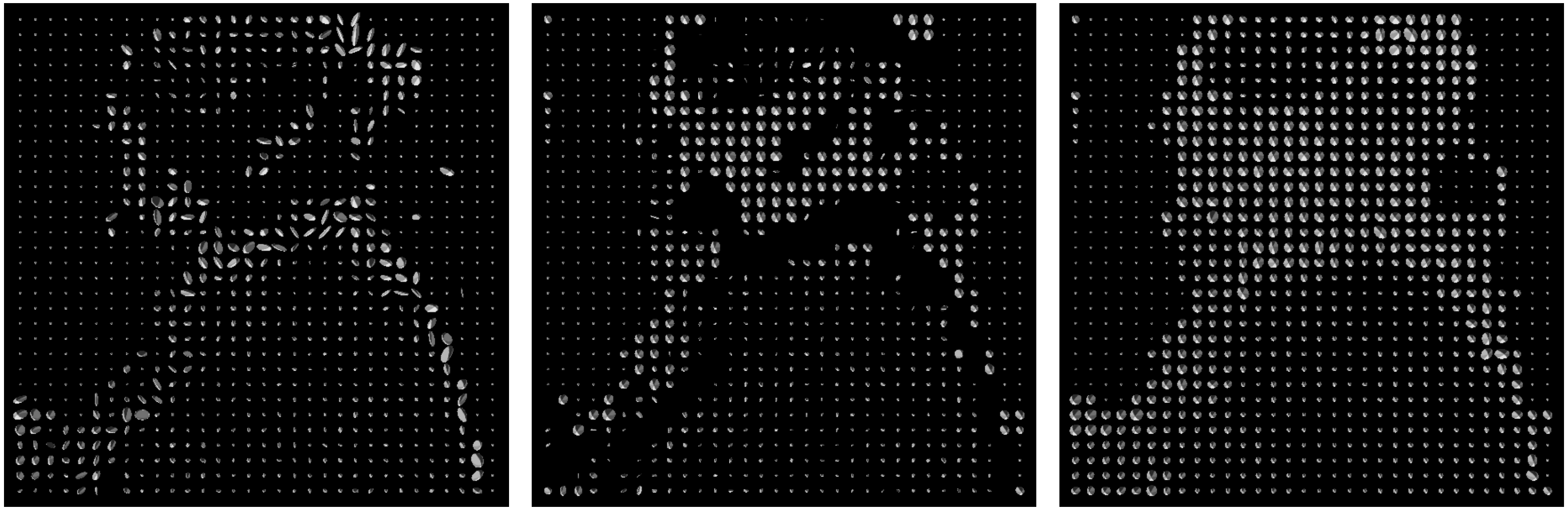
Closing | Opening

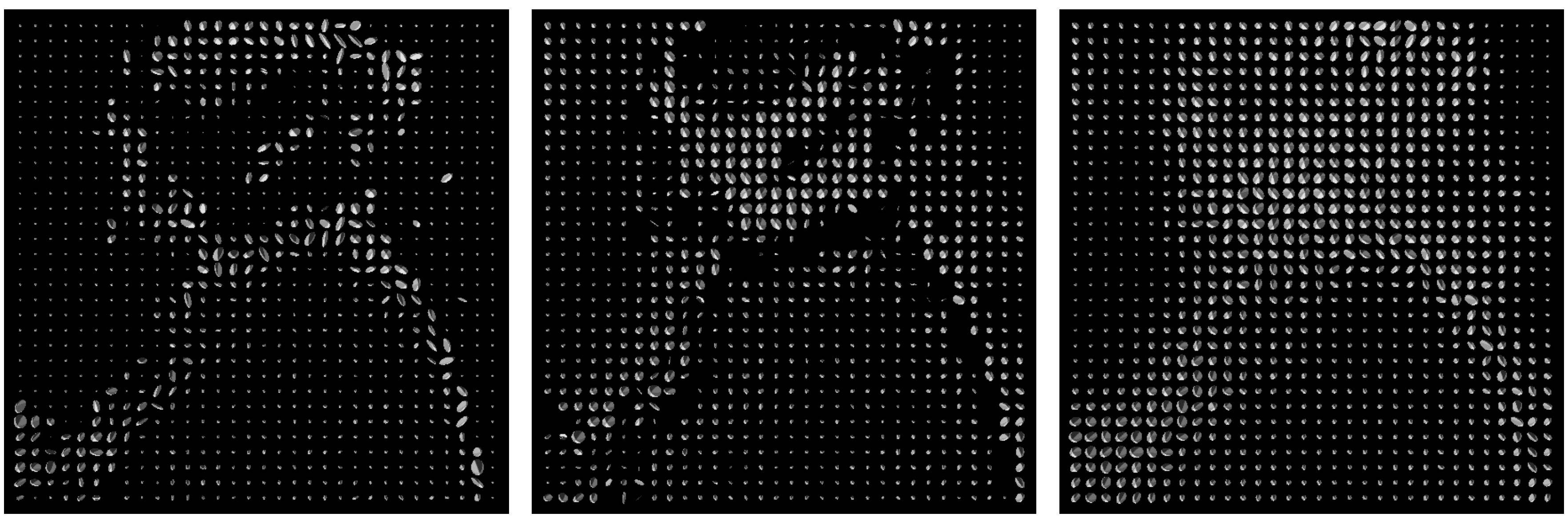
PDE based approach (stopping time 2)

Experiments: Ordering vs PDE IV

Ordering based approach (ball-shaped structuring element BSE($\sqrt{2}$))



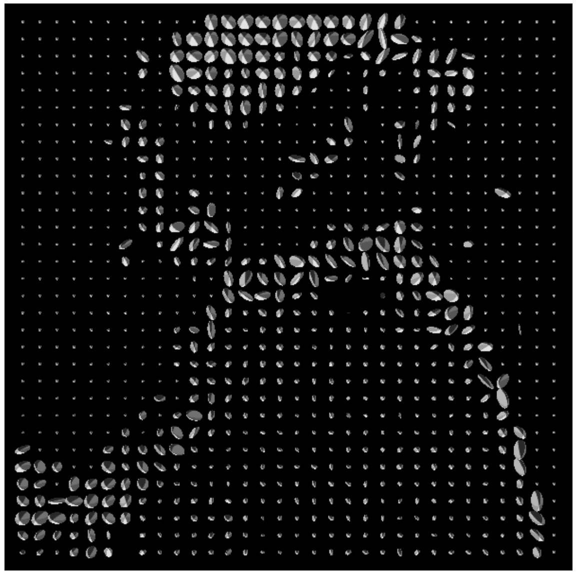
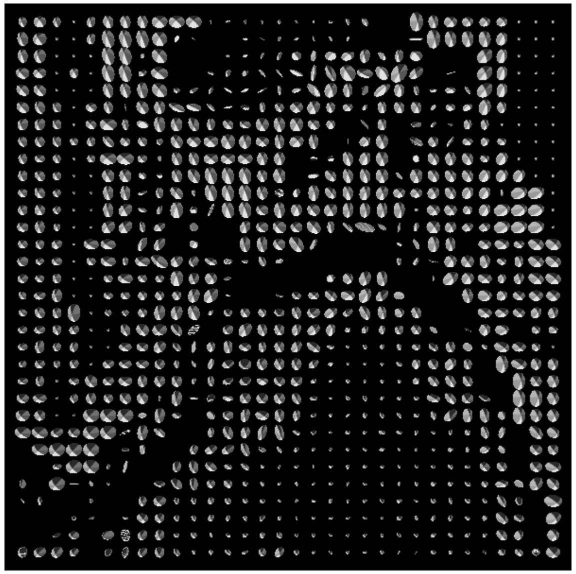
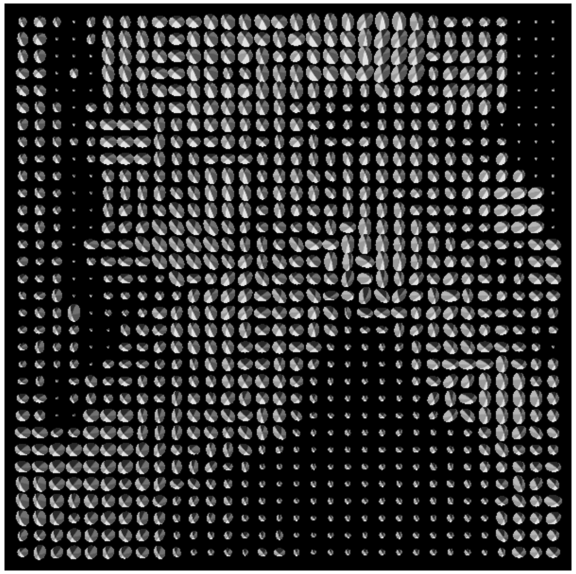
White Top Hat | Black Top Hat | Self-Dual Top Hat



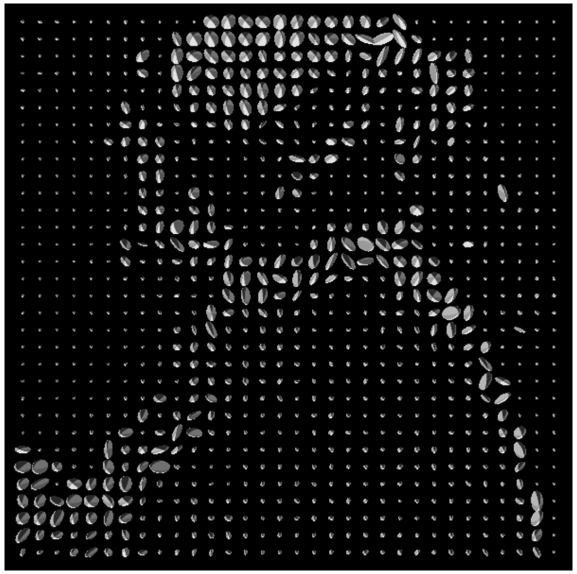
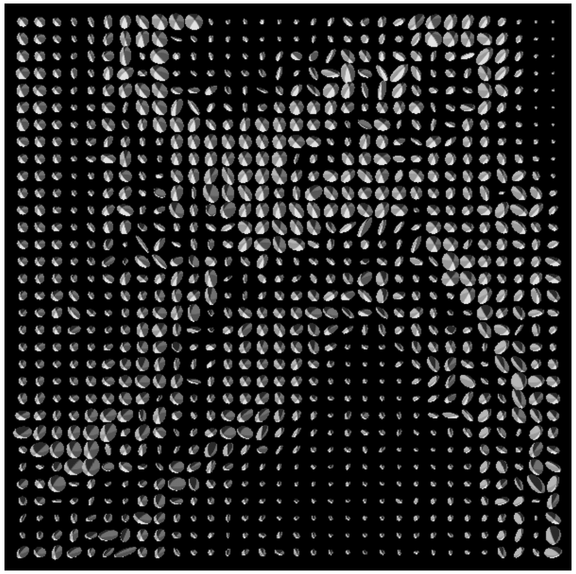
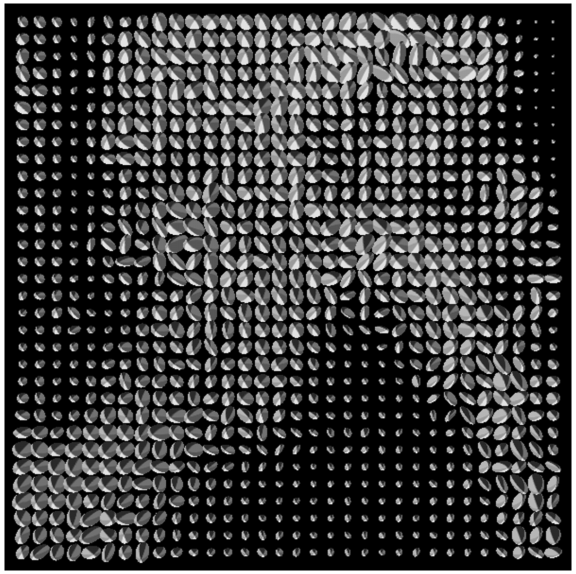
PDE based approach (stopping time 2)

Experiments: Ordering vs PDE V

Ordering based approach (ball-shaped structuring element BSE($\sqrt{2}$))

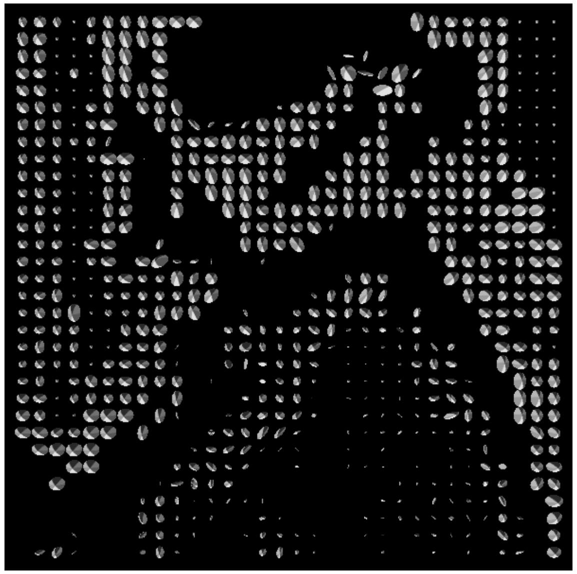
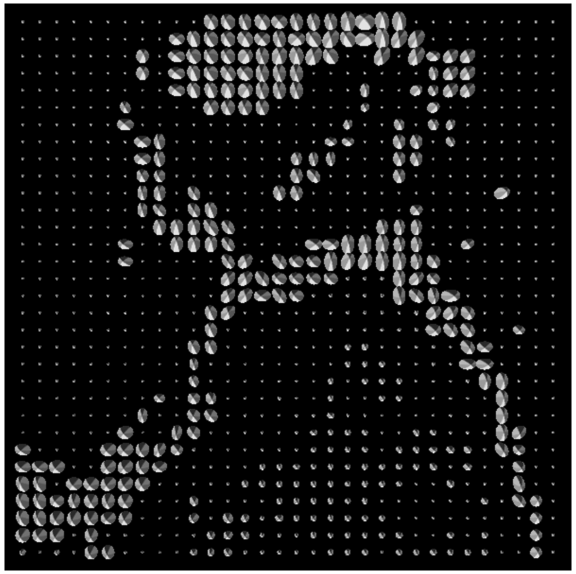
Internal Gradient | External Gradient | Beucher Gradient

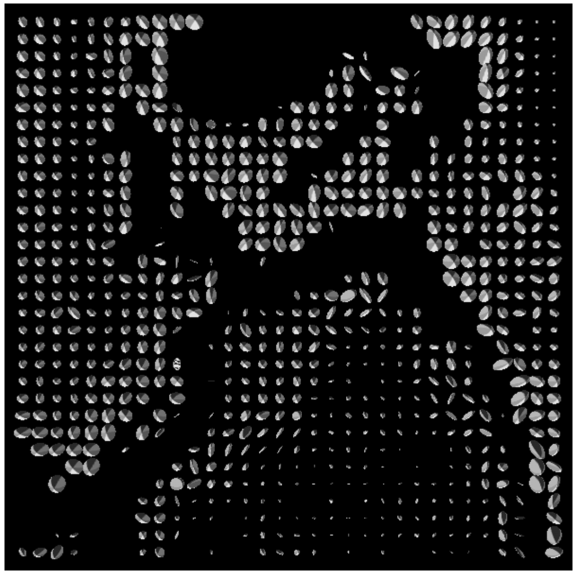
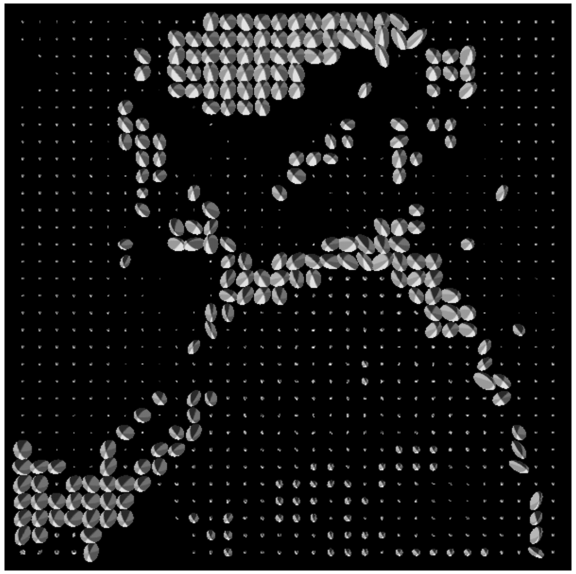
PDE based approach (stopping time 2)

Experiments: Ordering vs PDE VI

Ordering based approach (ball-shaped structuring element BSE($\sqrt{2}$))

Morphological Laplacian — Shock Filter

PDE based approach (stopping time 2)

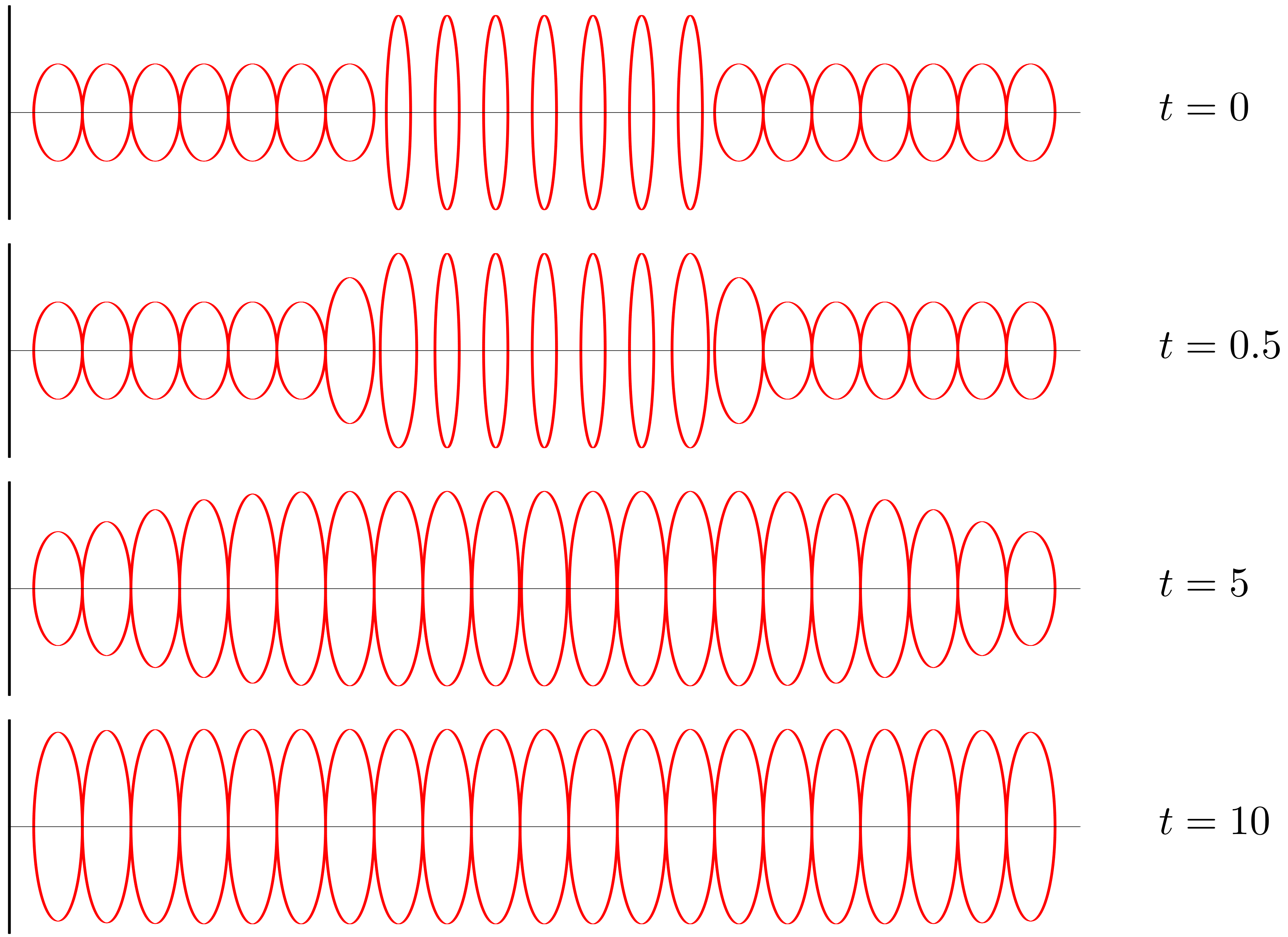
Concluding Remarks

- Two novel approaches to mathematical morphology for matrix fields:
  - A novel notion for the supremum and infimum of a set of matrices based on the **Loewner ordering**
  - A truely matrix-valued counterpart for nonlinear morphological **PDEs**
- **Numerical schemes** for scalar PDEs can be transfered to symmetric matrices.
- The properties of the proposed concepts allow for the application of
  - basic morphological operations as well as
  - morphological **derivatives**

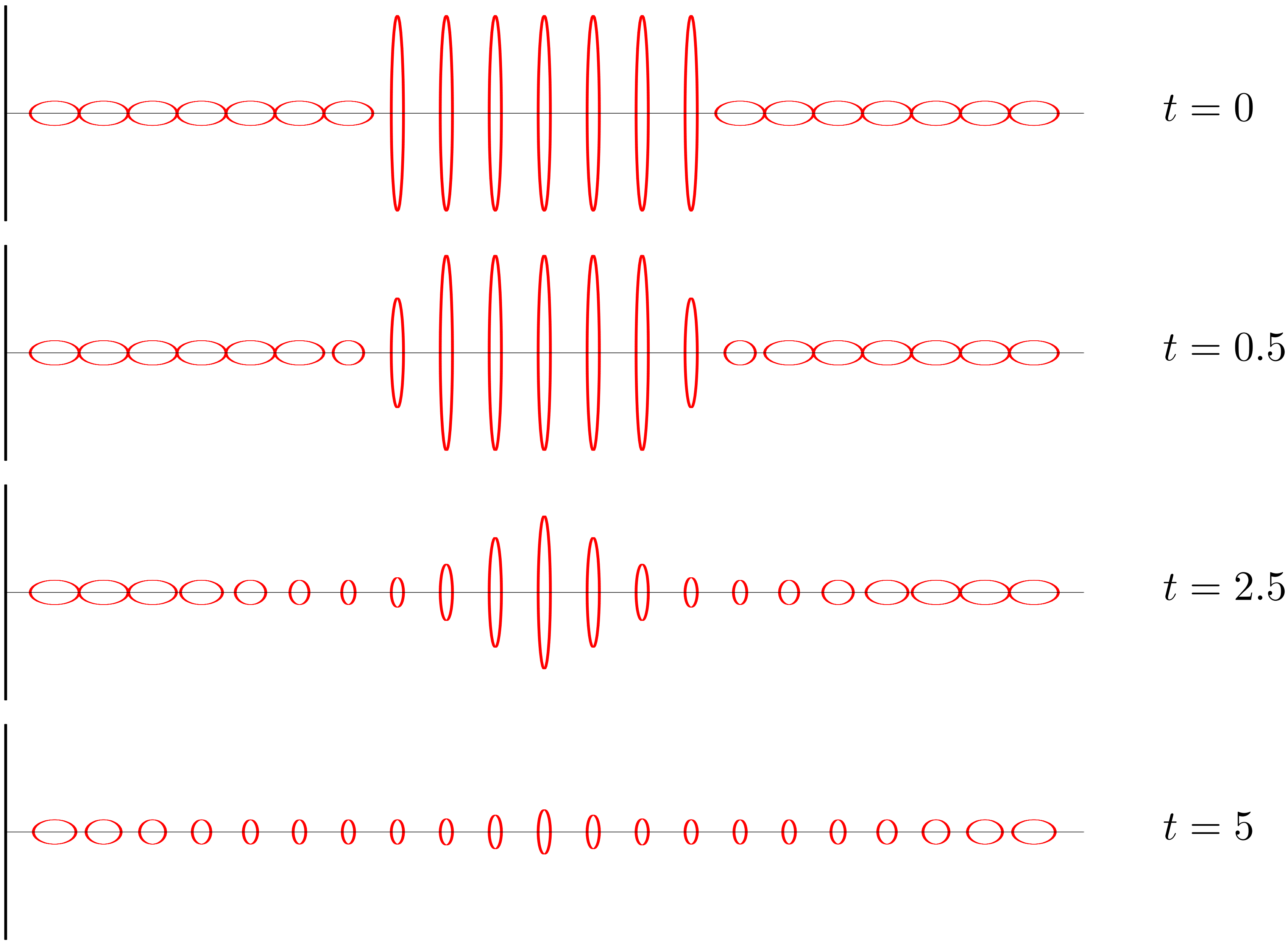
  to matrix-valued data
- However, matrix data are "high dimensional" data and some scalar concepts might **not** be directly transferable (discontinuity, ordering, oscillation,...)
- **Ongoing research** concentrates on the development of more sophisticated operations for matrix fields based on the above notions.

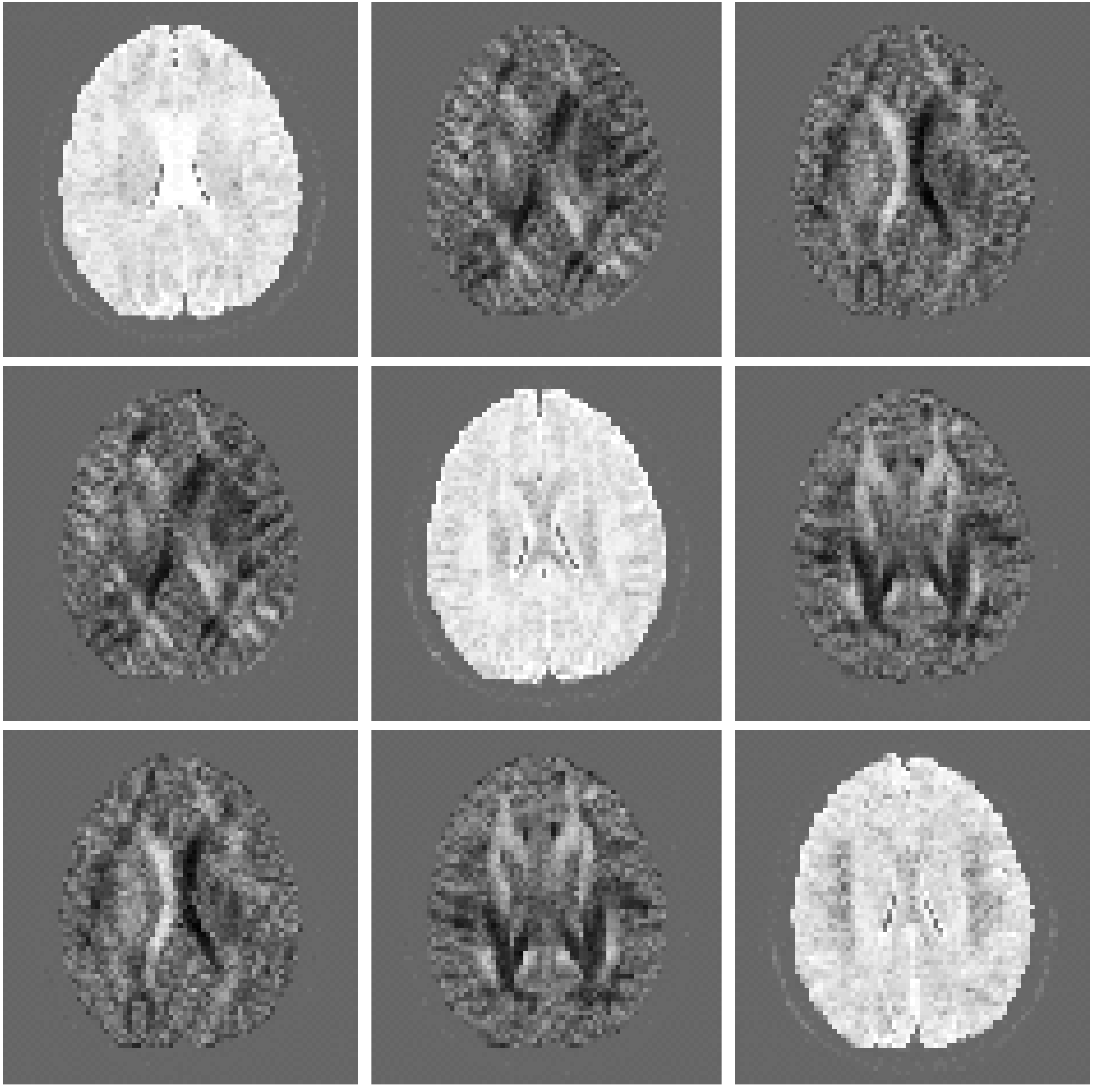
Thank you very much for your attention!

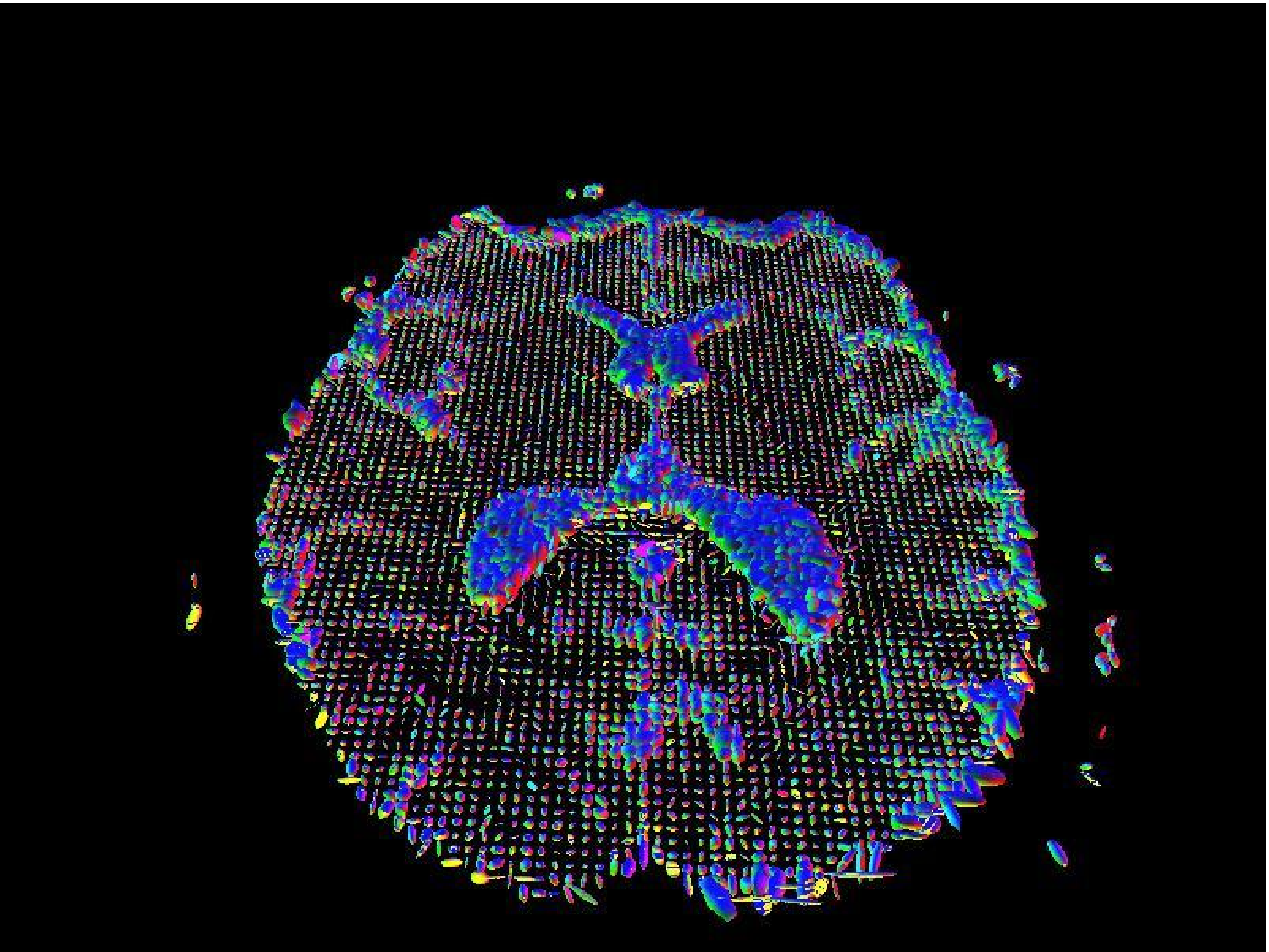
Experiments in 1D: PDE-driven dilation

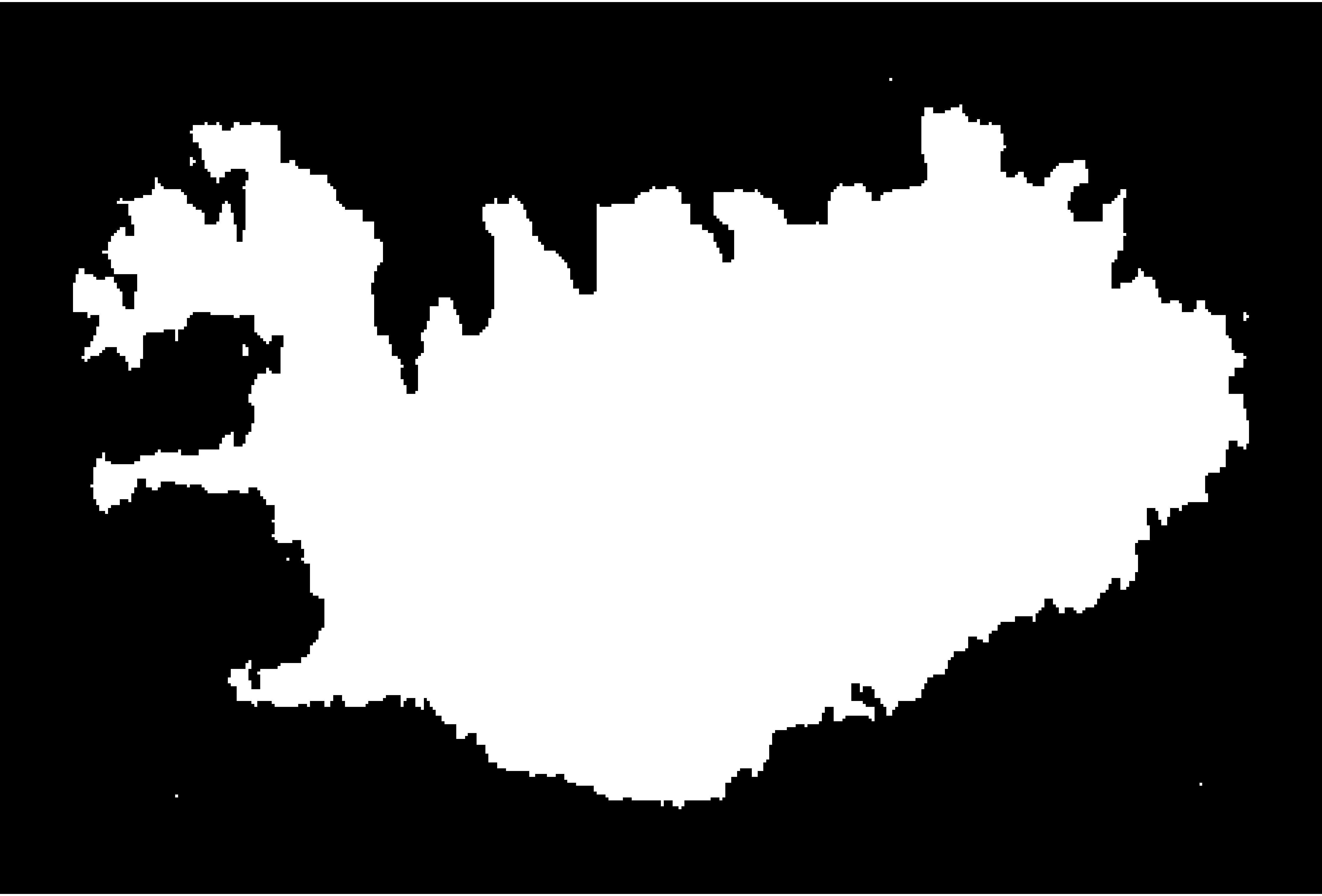


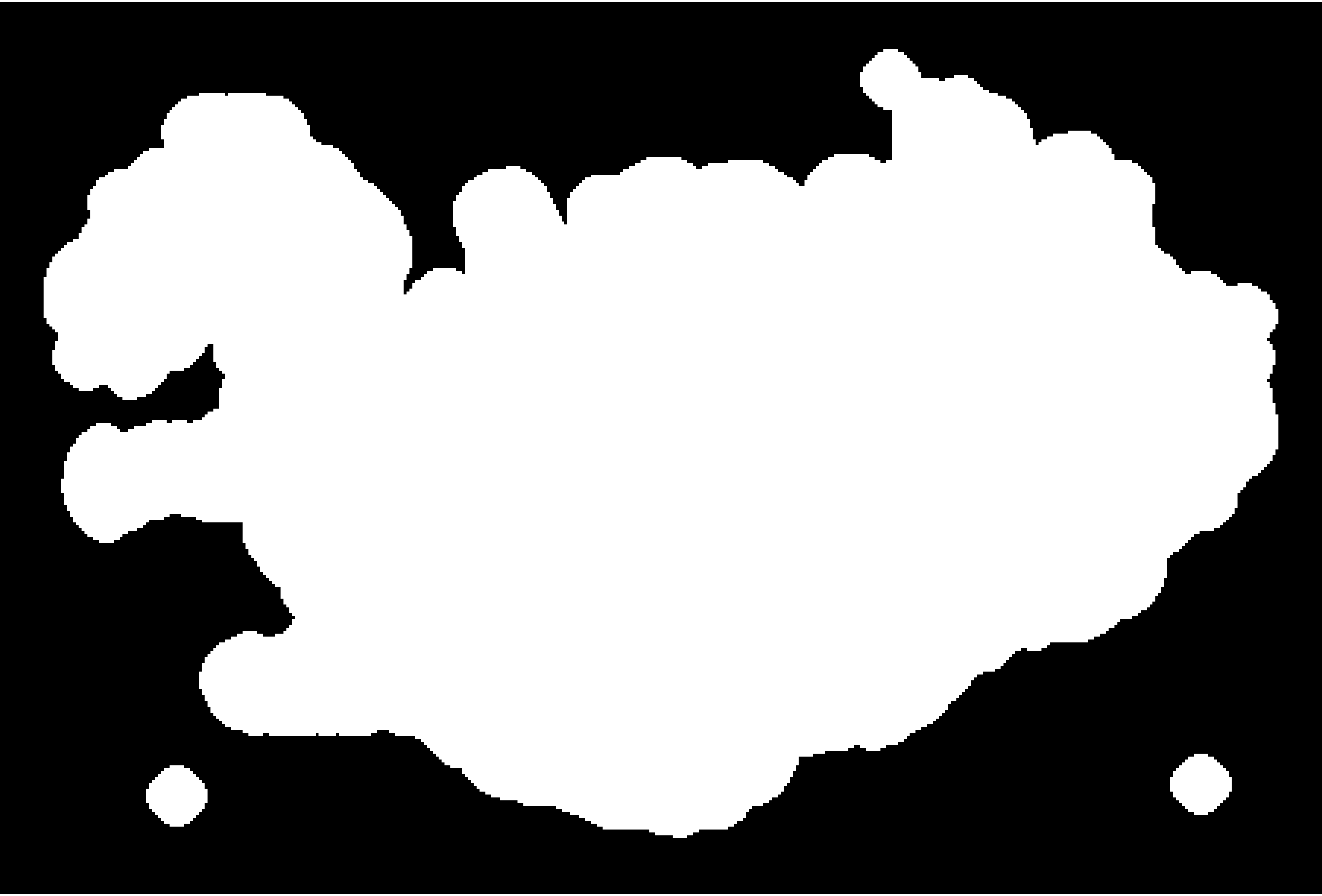
Experiments in 1D: PDE-driven erosion

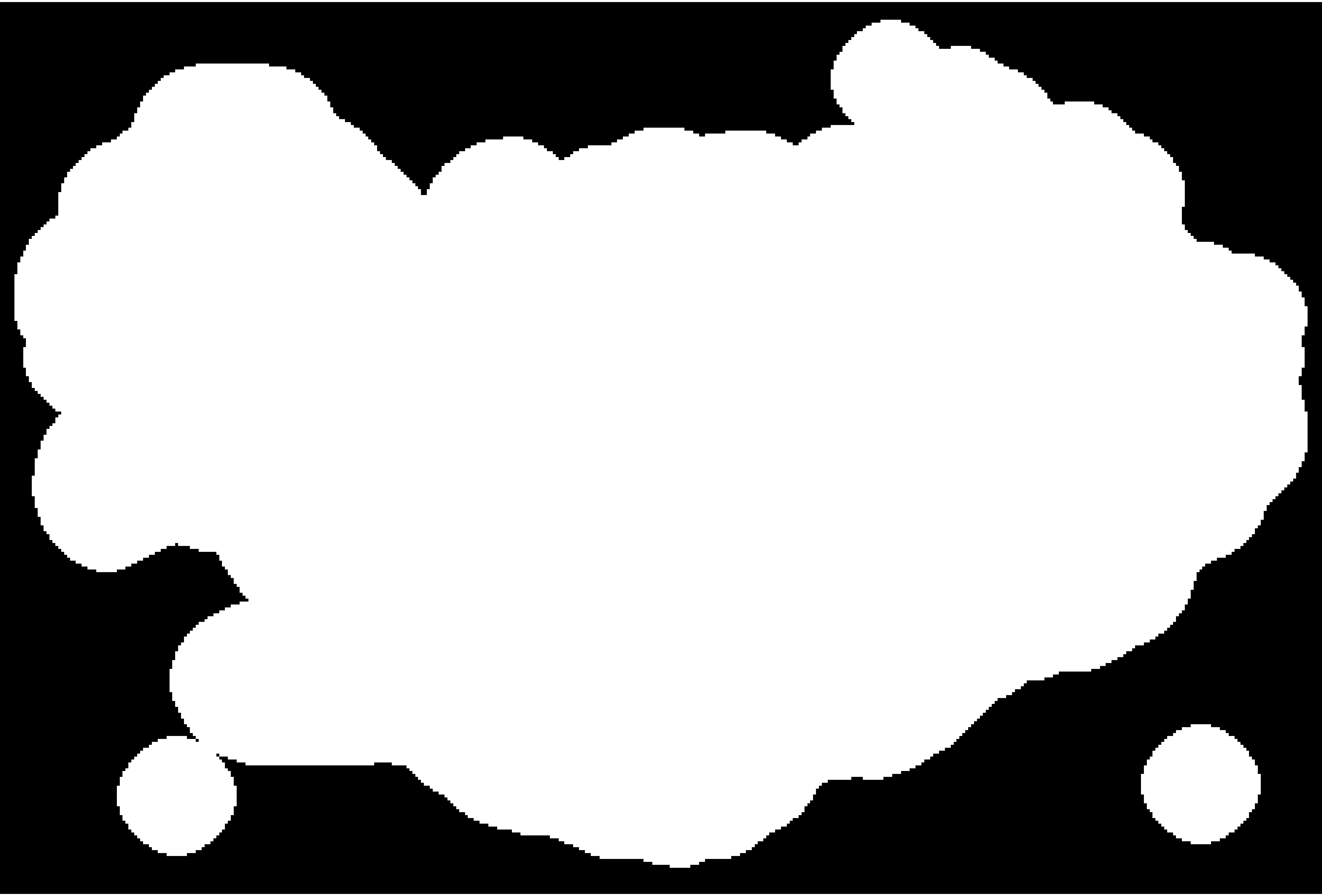


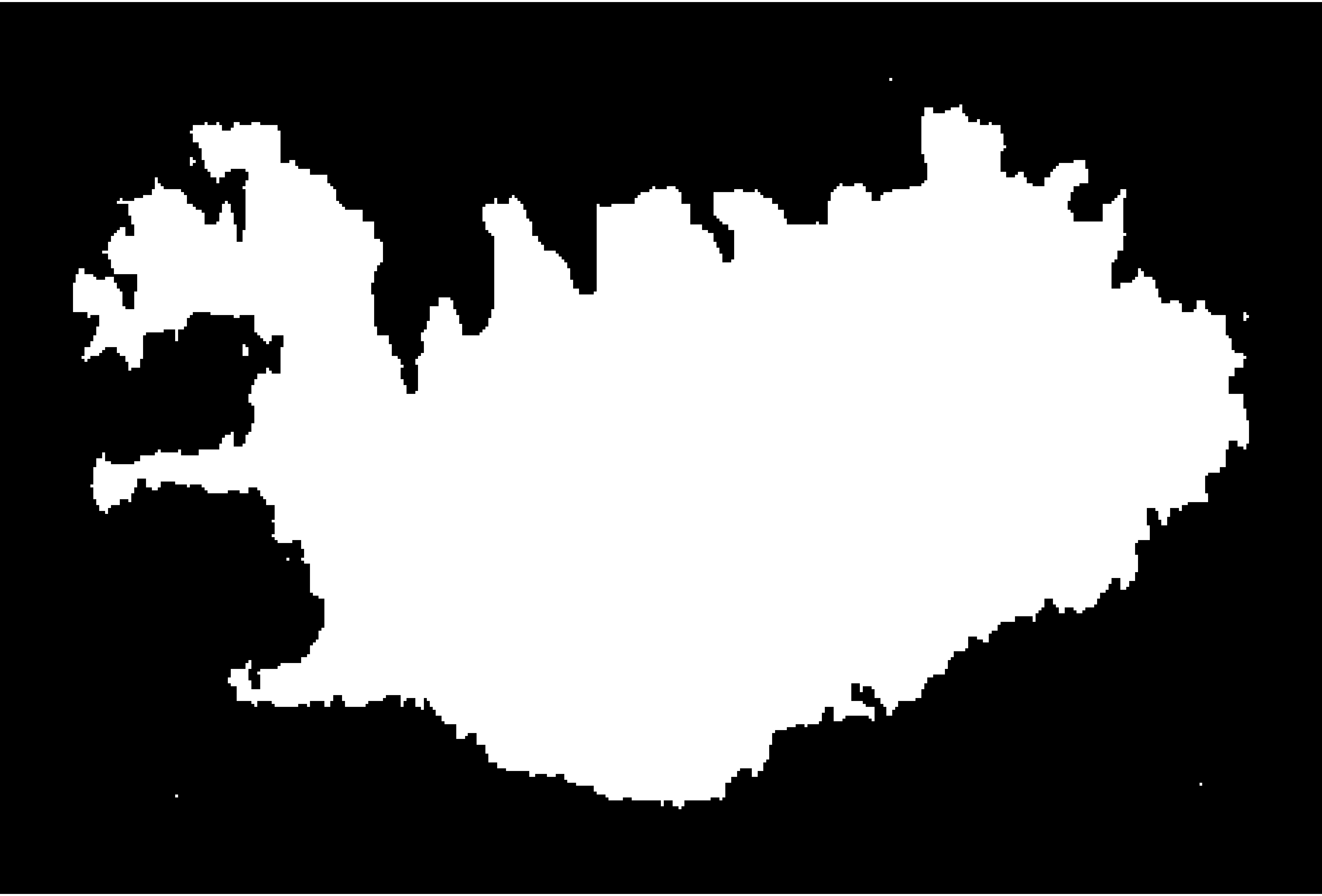


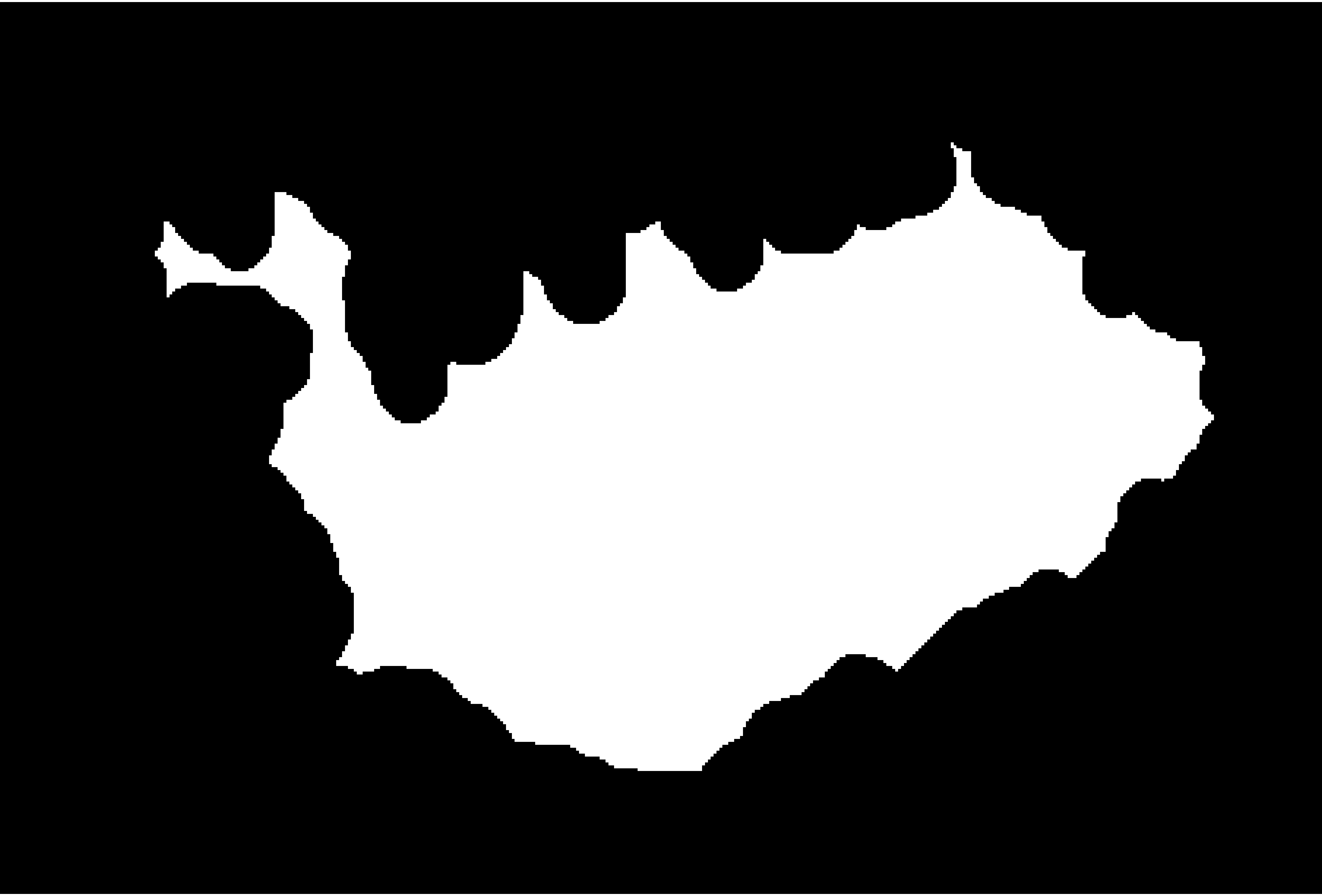


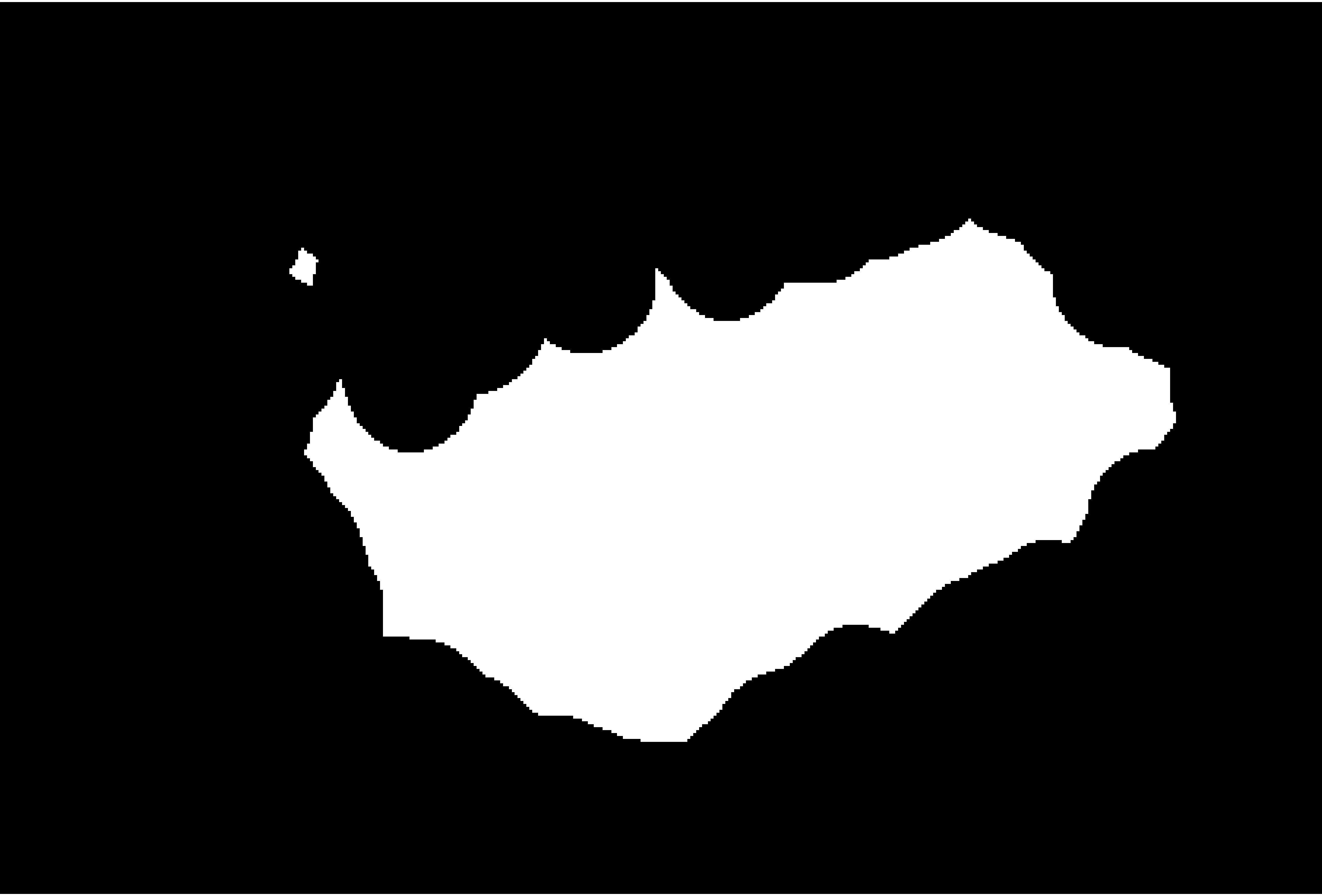


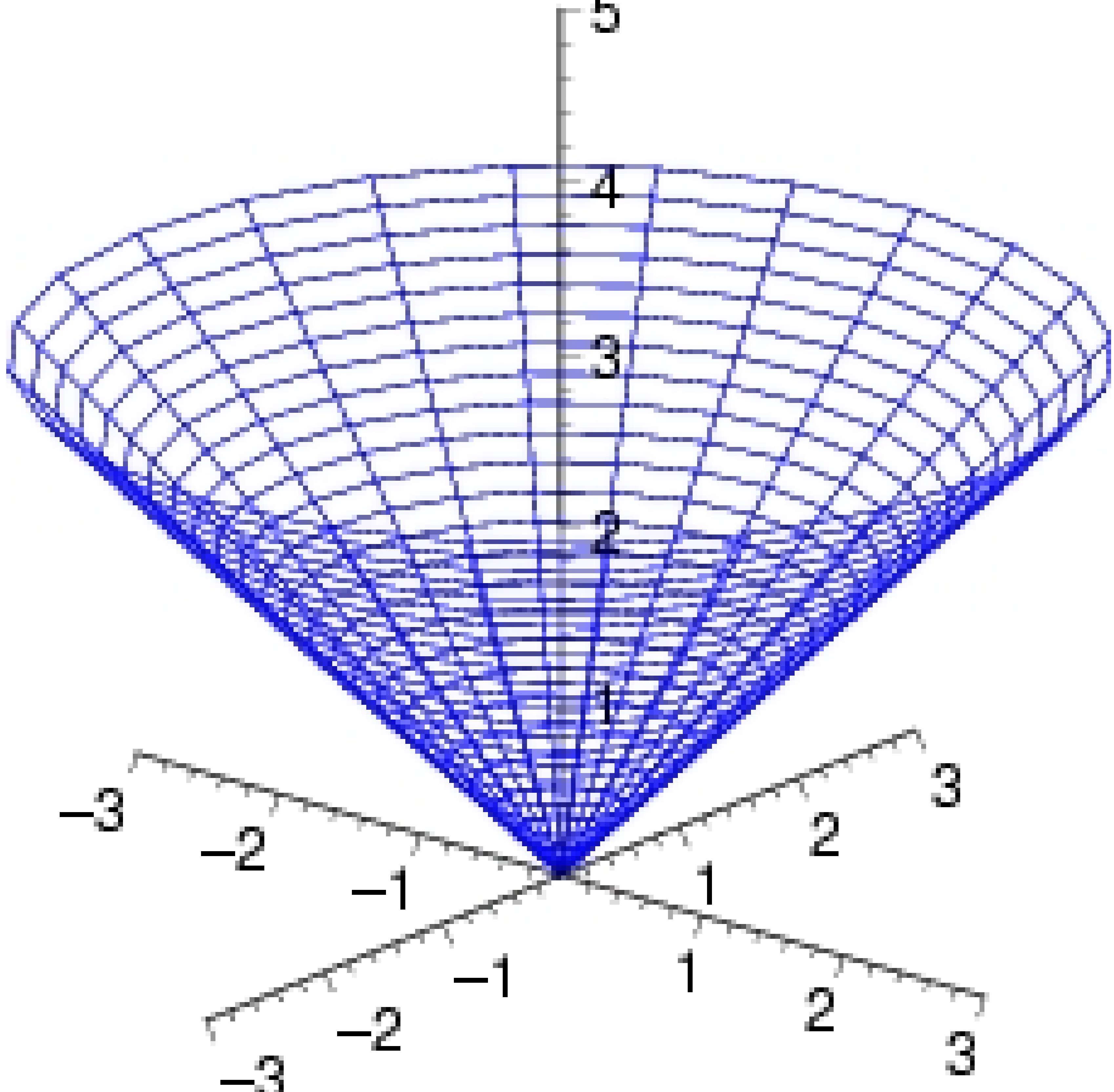








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2
1
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2
3
−1
−2
−3
1
2
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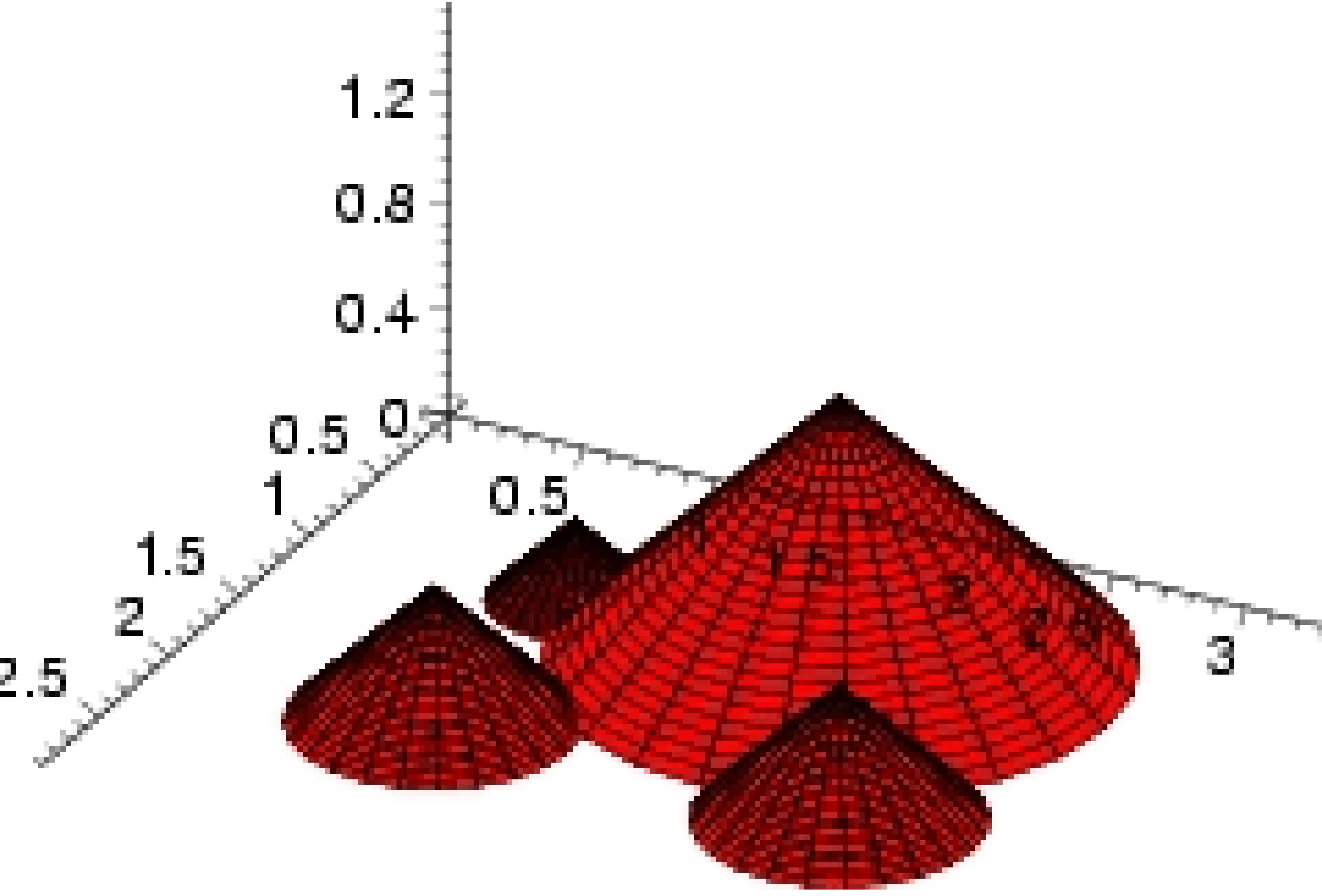

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0.4
0
0.5
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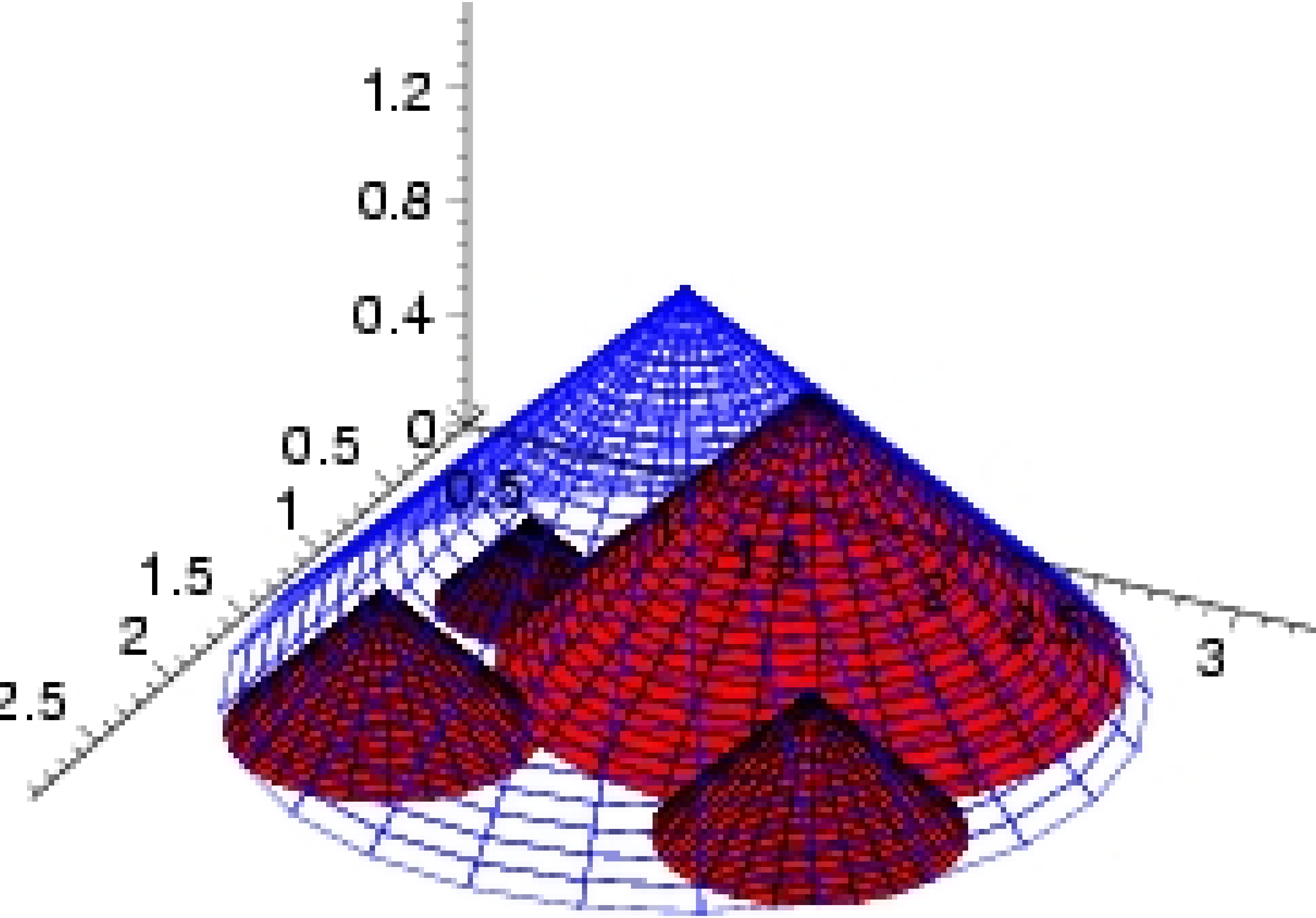
1.5
2
2.5
0.5
3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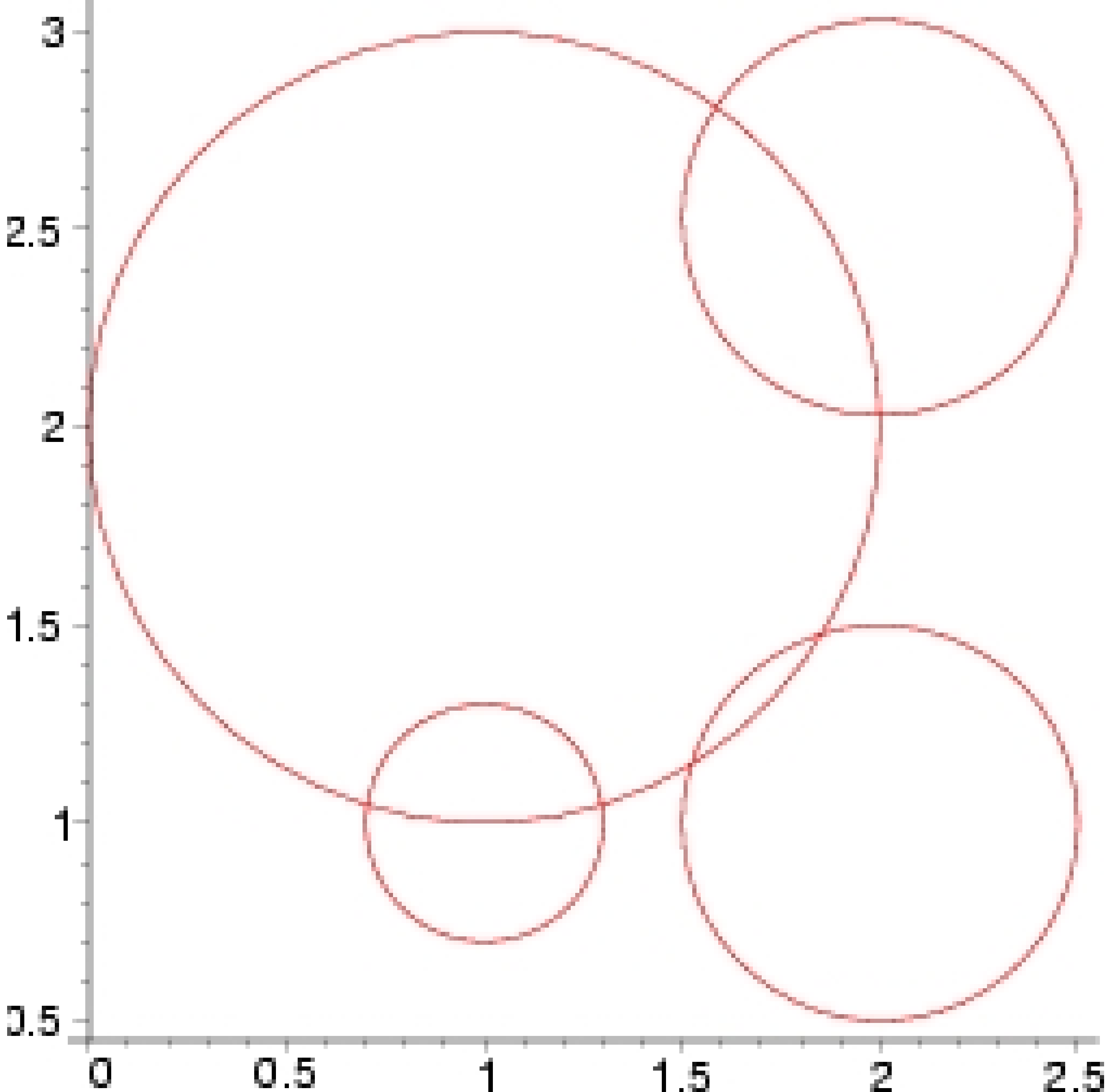

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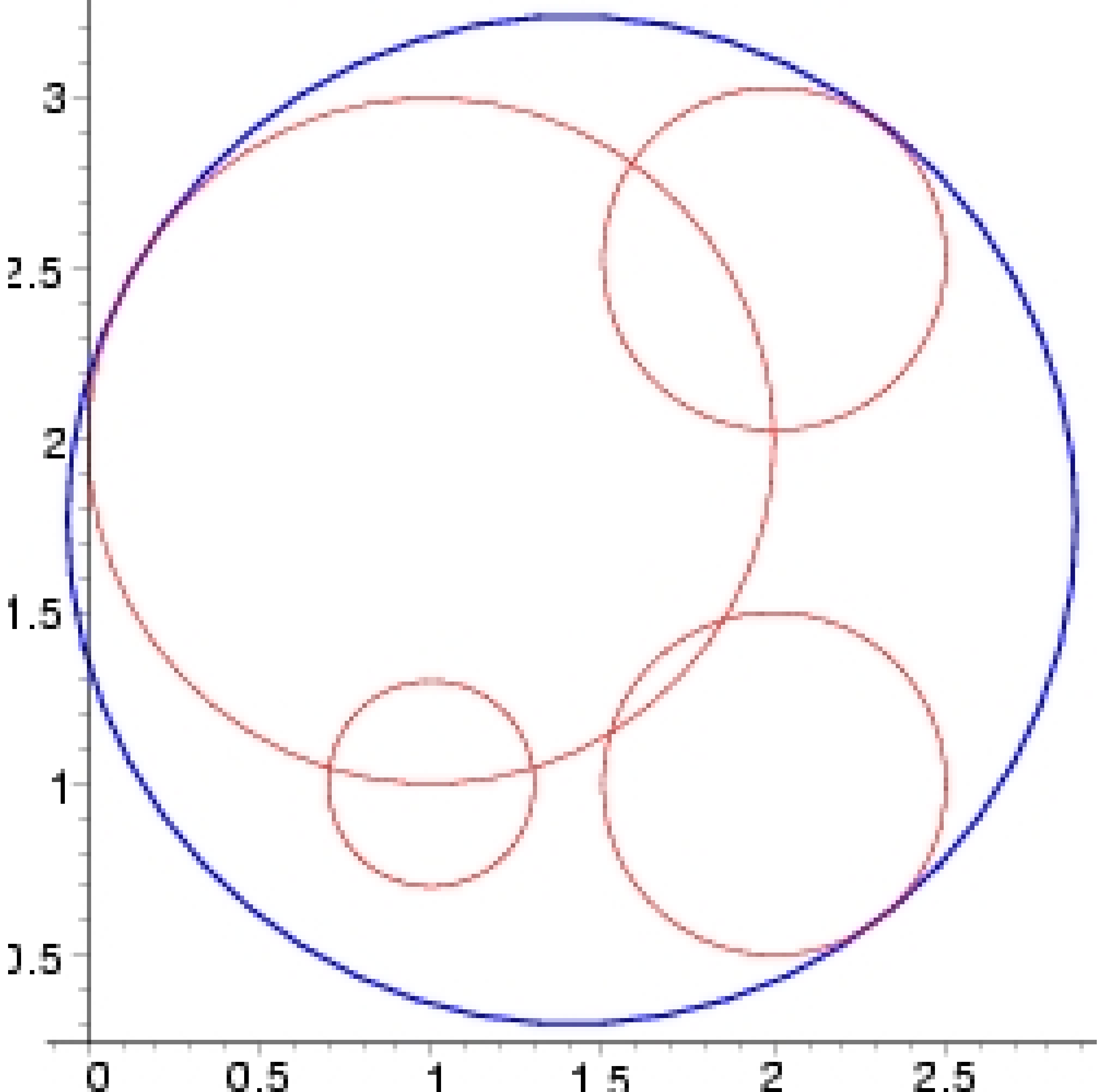
2.5



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